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Conditional Least Squares Estimation in stochastic nonlinear regression models: asymptotic properties and examples

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Goal

Model:
Let $\{Y_n\}$ be a discrete time stochastic process on $(\Omega, \mathcal{F}, P_\theta)$, $Y_n \in \mathbb{R}$

$$\begin{aligned} Y_n &= f_n(\theta) + \eta_n, \quad \theta \in \mathbb{R}^p, \; p < \infty \\ f_n(\theta) &= E_\theta(Y_n|\mathcal{F}_{n-1}) \quad \Longleftrightarrow E_\theta(\eta_n|\mathcal{F}_{n-1}) = 0; \;\; \sup_n E_\theta(\eta_n^2|\mathcal{F}_{n-1}) \overset{a.s.}{<} \infty \end{aligned}$$

$\{Y_n\}$ and $\{f_n(\theta)\}$ are observed except θ, θ unknown, $\theta \in \Theta$ open set

Conditional Least Squares Estimator of θ:
$\widehat{\theta}_n = \arg\min_{\theta\in\Theta} S_n(\theta), \quad S_n(\theta) = \sum_{k=1}^n (Y_k - f_k(\theta))^2$

Why this estimator ?:

1. $f_n(\theta) \overset{def.}{=} E_\theta(Y_n|\mathcal{F}_{n-1}) = \arg\min_{f:\mathcal{F}_{n-1}-meas.} E_\theta((Y_n - f)^2|\mathcal{F}_{n-1})$

2. depends only on the first two moments
$\Longrightarrow$ robustness, easy to compute, equivalent to the MLE if η_n is normally distributed

Strong consistency: $\lim_n \widehat{\theta}_n \overset{a.s.}{=} \theta_o \Longrightarrow$ accuracy

Asymptotic distribution: $\lim_n \Phi_n(\widehat{\theta}_n - \theta_o)$ exists in distribution $\Longrightarrow$ test, confidence interval
Examples: $\Phi_n = \sqrt{n}$, $\Phi_n = \sqrt{m^n}$

Examples of $Y_n = f_n(\theta) + \eta_n$

1. $f_n(\theta) = f(x_1, ..., x_d|\theta)$, $x_1, .., x_d$ deterministic covariates: classical regression
   Example: $f_n(\theta) = m + \nu n^{-\alpha}$, $\theta = (m, \nu, \alpha)$, $\alpha > 0$

2. $f_n(\theta) = f(x_{n,1}, ..., x_{n,d}|\theta)$, $x_{n,1}, ..., x_{n,d}$ deterministic vector of 0 and 1: ANOVA

3. $f_n(\theta) = f(X_1, ..., X_d|\theta)$, $X_1, .., X_d$ stochastic covariates: stochastic regression

4. $f_n(\theta) = f(Y_{n-1}, ..., Y_{n-d}|\theta)$: autoregressive processes, threshold models,. . .

5. $f_n(\theta) = f(Y_{n-1}, ..., Y_{n-d}, \eta_{n-1}, ..., \eta_{n-q}|\theta)$: ARMA processes

6. $f_n(\theta) = f(Y_{n-1}, ..., Y_{n-d}, E_n, E_{n-1}, ...|\theta)$: autoregressive processes with random environment

7. Discrete time branching processes:

- *Bienaymé-Galton-Watson process*:

$$
\begin{aligned}
N_n &= \sum_{i=1}^{N_{n-1}} X_{n,i}, \ \{X_{n,i}\}_i \text{ i.i.d. } (m, \sigma^2) \\
N_n &= mN_{n-1} + \sum_{i=1}^{N_{n-1}} (X_{n,i} - m) \\
Y_n &= N_n N_{n-1}^{-1/2} = mN_{n-1}^{1/2} + \eta_n, \ \ \eta_n = [\sum_{i=1}^{N_{n-1}} (X_{n,i} - m)][N_{n-1}]^{-1/2}
\end{aligned}
$$

- *Size-dependent branching processes (Klebaner, 1984,...)*

$$
\begin{aligned}
N_n &= \sum_{i=1}^{N_{n-1}} X_{n,i}, \ \{X_{n,i}\} \text{ i.i.d.} (m_\theta(N_{n-1}), \sigma^2(N_{n-1})), \ \lim_N m_\theta(N) = m, \ \ \sigma^2(N) = O(N^\beta), \beta < 1 \\
Y_n &= N_n N_{n-1}^{-(1+\beta)/2} = m_\theta(N_{n-1}) N_{n-1}^{(1-\beta)/2} + \eta_n
\end{aligned}
$$

- *Regenerative branching processes* (Bulgarian Academy of Sciences, Sofia (Yanev,...)), *bisexual branching processes* (Extremadura team, Spain (Molina,...)), *branching processes with random environment* (Steklov Institute, Moscou (Vatutin, Dyakonova)) . . .

8. Multivariate stochastic regression models $Z_k \in \mathbb{R}^d$ with $E(Z_n|\mathcal{F}_{n-1}) = f_n(\theta)$

$$\begin{aligned}
\widehat{\theta}_n &= \arg\min_{\theta\in\Theta} S_n(\theta) \\
S_n(\theta) &= \sum_{k=1}^{n}(Z_k - f_k(\theta))^t \Sigma_k^{-1}(Z_k - f_k(\theta)) \\
&= \sum_{k=1}^{n}(Z_k - f_k(\theta))^t U_k \Lambda_k^{-1} U_k^{-1}(Z_k - f_k(\theta)) \\
&= \sum_{k=1}^{n}[\Lambda_k^{-1/2} U_k^{-1}(Z_k - f_k(\theta))]^t [\Lambda_k^{-1/2} U_k^{-1}(Z_k - f_k(\theta))] \\
&= \sum_{j=1}^{d}\sum_{k=1}^{n}(Y_{k,j} - E_\theta(Y_{k,j}|\mathcal{F}_{k-1}))^2, \quad Y_k = \Lambda_k^{-1/2} U_k^{-1} Z_k
\end{aligned}$$

Consistency: state of the art

Model: $Y_k = f_k(\theta) + \eta_k$
Estimator: $\widehat{\theta}_n = \arg\min_{\theta\in\Theta} S_n(\theta), \quad S_n(\theta) = \sum_{k=1}^n (Y_k - f_k(\theta))^2$

Results depend on the linearity (direct proofs) or nonlinearity of $f_n(\theta)$ in θ
and if $f_n(.)$ is deterministic with $\{\eta_n\}$ independent, or stochastic with $\{\eta_n\}$ martingale differences

Crucial quantity:

$$D_n(\theta,\theta') \;=\; \sum_{k=1}^n (f_k(\theta) - f_k(\theta'))^2 \overset{\{f_k(\theta)=\theta^t W_k\}_k}{=} (\theta-\theta')^t [\sum_{k=1}^n W_k W_k^t](\theta-\theta') = \sum_{j=1}^p (\widetilde{\theta}_j - \widetilde{\theta}'_j)^2 \lambda_{j,n}$$

Identifiability

$$\forall \theta' \neq \theta,\ \{f_n(\theta)\}_n \overset{a.s.}{\neq} \{f_n(\theta')\}_n \qquad \Longleftrightarrow \qquad \forall \theta' \neq \theta,\ \lim_n D_n(\theta,\theta') \overset{a.s.}{\neq} 0$$

$$\overset{\{f_k(\theta)=\theta^t W_k\}_k}{\Longleftrightarrow} \quad \lim_n \lambda_{min}\{\sum_{k=1}^n W_k W_k^t\} > 0$$

Model: $Y_k = f_k(\theta) + \eta_k$
Estimator: $\widehat{\theta}_n = \arg\min_{\theta\in\Theta} S_n(\theta), \quad S_n(\theta) = \sum_{k=1}^n (Y_k - f_k(\theta))^2$
Identifiability criterion: $D_n(\theta,\theta') = \sum_{k=1}^n (f_k(\theta) - f_k(\theta'))^2$

References

1. **Jennrich (1969)**: nonlinear deterministic $f_n(\theta)$, $\{\eta_n\}_n$ i.i.d..
Assume: $\forall\theta \neq \theta', \ \lim_n D_n(\theta,\theta')n^{-1} = D(\theta,\theta')$ *and* $D(\theta,\theta') = 0 \iff \theta = \theta'$
(strong identifiability with rate n independent of the parameter).
Then $\lim_n \widehat{\theta}_n \overset{a.s.}{=} \theta_o$.

Example: $f_n(m,\nu) = m + \nu n^{-\alpha}$, $\theta = (m,\nu)$, $\alpha > 0$
For $m = m'$, $D_n(\theta,\theta') = (\nu - \nu')\sum_{k=1}^n k^{-2\alpha} = O(n^{1-2\alpha}1_{\{2\alpha<1\}} + \ln(n)1_{\{2\alpha=1\}})$,
for $\nu = \nu'$, $D_n(\theta,\theta') = (m - m')^2 n \Longrightarrow$ condition not checked

2. **Lai, Robbins and Wei (1978, 1979)**: linear deterministic $f_n(\theta) = \theta^t W_n$, $\{\eta_n\}_n$ i.i.d.
$\widehat{\theta}_n = \sum_{k=1}^n Y_k W_k^t [W_k W_k^t]^{-1}$,
$\lim_n \widehat{\theta}_n \overset{a.s.}{=} \theta_o \iff \lim_n \lambda_{min}\{\sum_{k=1}^n W_k W_k^t\} \overset{a.s.}{=} \infty$ *(strong identifiability:* $\lim_n D_n(\theta,\theta') \overset{a.s.}{=} \infty$, $\theta \neq \theta'$*)*

Necessary and sufficient condition!!!!

Example: $f_n(m,\nu) = m + \nu n^{-\alpha}$, $\theta = (m,\nu)$, $\alpha > 0$
$D_n(\theta,\theta') \geq O(n^{1-2\alpha}1_{\{2\alpha<1\}} + \ln(n)1_{\{2\alpha=1\}}) \Longrightarrow$ condition checked for $2\alpha \leq 1$

Identifiability criterion: $D_n(\theta,\theta') = \sum_{k=1}^{n}(f_k(\theta) - f_k(\theta'))^2$

4. **Wu (1981)**: nonlinear deterministic case, $\{\eta_n\}_n$ i.i.d.

Assume $\forall \lambda \neq \theta_o$*,* $\exists$ *a ball* $B(\lambda)$*:*

$a.\ \forall \theta \neq \theta_o,\ \lim_n D_n(\theta,\theta_o) \overset{a.s.}{=} \infty$ ***(strong identifiability)***

$b.\ \overline{\lim}_n [[\sum_{k=1}^{n} \sup_{\theta\in B(\lambda)}(f_k(\theta) - f_k(\theta_o))^2]^{(1+c)/2}][\inf_{\theta\in B(\lambda)} \sum_{k=1}^{n}(f_k(\theta) - f_k(\theta_o))^2]^{-1} \overset{a.s.}{<} \infty$*,* $\quad c > 0$

(rate of identifiability)

$c.\ \sup_{\theta_1\in B(\lambda),\theta_2\in B(\lambda)} |f_k(\theta_1) - f_k(\theta_2)|[||\theta_1 - \theta_2||]^{-1} \overset{a.s.}{\leq} M_k(B(\lambda))$ ***(Lipschitz)***

Then $\lim_n \widehat{\theta}_n \overset{a.s.}{=} \theta_o$*.*

Example: $f_n(m,\nu) = m + \nu n^{-\alpha}$, $\theta = (m,\nu)$, $\alpha > 0$

The numerator in condition b is $O(n^{(1+c)/2})$ while the denominator is $O(n^{1-2\alpha}1_{\{2\alpha<1\}} + \ln(n)1_{\{2\alpha=1\}})$

$\Longrightarrow$ condition b checked for $2\alpha < 1$

5. **Lai and Wei (1982)**: linear stochastic case $f_n(\theta) = \theta^t W_n$

Assume

$a.\ \lim_n \lambda_{min}\{\sum_{k=1}^n W_k W_k^t\} \overset{a.s.}{=} \infty$ *(strong identifiability)*

$b.\ \lim_n [\ln(\lambda_{max}\{\sum_{k=1}^n W_k W_k^t\})]^\rho [\lambda_{min}\{\sum_{k=1}^n W_k W_k^t\}]^{-1} \overset{a.s.}{=} 0$, *for some* $\rho > 1$ *(rate of identifiability)*

Then $\lim_n \widehat{\theta}_n \overset{a.s.}{=} \theta_o$

Remark: b is checked for $p = 1$, and is hardly stronger than a, for $p > 1$

Example: $f_n(m, \nu) = m + \nu n^{-\alpha} = (m, \nu)(1, n^{-\alpha})^t$, $\alpha > 0$, $\theta = (m, \nu)$

$\lambda_{min}\{\sum_{k=1}^n W_k W_k^t\} = \sum_{k=1}^n k^{-2\alpha} = O(n^{1-2\alpha} 1_{\{2\alpha<1\}} + \ln(n) 1_{\{2\alpha=1\}})$, $\lambda_{max}\{\sum_{k=1}^n W_k W_k^t\} = n$

- $2\alpha \leq 1 \Longleftrightarrow$ strong identifiability of (m, ν)
- $2\alpha < 1 \Longrightarrow b$ is checked
- $2\alpha = 1 \Longrightarrow b$ is not checked ($\lim_n (\ln n)^{\rho-1} = \infty$, for $\rho > 1$)

Example: $N_n = \sum_{i=1}^{N_{n-1}} X_{n,i}$, $\{X_{n,i}\}$ i.i.d. with $m_\theta(N) = m + \nu N^{-\alpha}$, $m > 1$, $\alpha > 0$, $\sigma^2(N) = O(N^\beta)$, $\theta = (m, \nu)$

$\Longrightarrow f_n(m, \nu) = m N_{n-1}^{(1-\beta)/2} + \nu N_{n-1}^{(1-(2\alpha+\beta))/2} = (m, \nu)(N_{n-1}^{(1-\beta)/2}, N_{n-1}^{(1-(2\alpha+\beta))/2})^t$,

- $2\alpha + \beta \leq 1 \Longleftrightarrow$ strong identifiability of (m, ν) on the nonextinction set
- $2\alpha + \beta = 1 \Longrightarrow b$ not checked ($\lim_n [\ln m^{n(1-\beta)}]^\rho n^{-1} = \infty$, for $\rho > 1$)

Identifiability criterion: $D_n(\theta, \theta') = \sum_{k=1}^{n} (f_k(\theta) - f_k(\theta'))^2$

6. **Lai (1994)**: nonlinear stochastic case, complex conditions

7. **Skouras (2000)**: nonlinear stochastic case.
Assume $\forall \lambda \neq \theta_o$, $\exists$ *a ball* $B(\lambda)$ *and* $r_\lambda \in]1, 2[$*:*

$a.$ $\lim_n \inf_{\theta \in B(\lambda)} D_n(\theta, \theta_o) \stackrel{a.s.}{=} \infty$ ***(strong identifiability)***

$b.$ $\exists g_k(.)$ *and* $h(.)$*:* $\forall \theta_1$, θ_2 *in* $B(\lambda)$, $|f_k(\theta_1) - f_k(\theta_2)| \stackrel{a.s.}{\leq} h(||\theta_1 - \theta_2||) g_k(\lambda), \ \lim_{y \downarrow 0} h(y) = 0$ ***(Lipschitz)***

$c.$ $\sum_{k=1}^{n} \sup_{\theta \in B(\lambda)} (f_k(\theta) - f_k(\theta_o))^2 \stackrel{a.s.}{=} O([\inf_{\theta \in B(\lambda)} D_n(\theta, \theta_o)]^{r_\lambda})$ ***(rate)***

$d.$ $[\sum_{k=1}^{n} g_k(\lambda)][\inf_{\theta \in B(\lambda)} D_n(\theta_o, \theta)]^{-1} \stackrel{a.s.}{=} O(1)$ ***(rate)***

$\Longrightarrow \lim_n \widehat{\theta}_n \stackrel{a.s.}{=} \theta_o$.

Remark: d not checked for some transient phenomena:
Example: $f_n(\theta) = m + \theta n^{-\alpha}$, $\alpha > 0$, $D_n(\theta_o, \theta) = (\theta_o - \theta)^2 \sum_{k=1}^{n} k^{-2\alpha}$, $g_k(\lambda) = k^{-\alpha}$
$2\alpha \leq 1 \Longrightarrow a$ checked. But d not checked since, for $\alpha > 0$, $\lim_n [\sum_{k=1}^{n} k^{-\alpha}][\sum_{k=1}^{n} k^{-2\alpha}]^{-1} = \infty$
Lai and Wei condition checked for $2\alpha < 1$

Example: $N_n = \sum_{i=1}^{N_{n-1}} X_{n,i}$ with $m_\theta(N) = m + \nu N^{-\alpha}$, $\alpha > 0$, $m > 1$, $\sigma^2(N) = O(N^\beta)$, $\theta = (m, \nu)$
$\Longrightarrow f_n(\theta) = m N_{n-1}^{(1-\beta)/2} + \nu N_{n-1}^{(1-(2\alpha+\beta))/2}$
Then $g_n(\lambda) = N_{n-1}^{(1-\beta)/2}$, $\inf_{\theta \in B(\lambda)} D_n(\theta_o, \theta) = O(\sum_{k=1}^{n} N_{k-1}^{(1-(2\alpha+\beta))}$

a checked for $2\alpha + \beta \leq 1$, d checked for $2\alpha + \beta \leq (1+\beta)/2$, c checked for $2\alpha + \beta < (1+\beta)/2$
$\Longrightarrow$ Lai and Wei ($2\alpha + \beta < 1$) better than Skouras here when $\beta < 1$

8. **Jacob, Lalam, and Yanev N. (2005)**: allows transient phenomena but needs some identifiability rate.

Conclusion

Linear deterministic (or stochastic with $p = 1$) model:
$\lim_n \widehat{\theta}_n \overset{a.s.}{=} \theta_o \Longleftrightarrow \forall \delta > 0, \lim_n \inf_{\theta \in B_\delta^c(\theta_o)} D_n(\theta_o, \theta) = \infty$

Nonlinear stochastic model:
$\lim_n \widehat{\theta}_n \overset{a.s.}{=} \theta_o \Longleftarrow$ strong identifiability, various rates of identifiability, smoothness (Lipschitz)

Is it possible to weaken these conditions?????

Jacob (2007) submitted

Model: $Y_n = f_n(\theta_o) + \eta_n$, $E(\eta_n|\mathcal{F}_{n-1}) = 0$, $\sup_n E(\eta_n^2|\mathcal{F}_{n-1}) < \infty$
Estimator: $\widehat{\theta}_n = \arg\min_{\theta\in\Theta} S_n(\theta), \quad S_n(\theta) = \sum_{k=1}^{n}(Y_k - f_k(\theta))^2$, $\theta \in \Theta$ open set in $\mathbb{R}^p$
Identifiability criterion: $D_n(\theta_o, \theta) = \sum_{k=1}^{n}(f_k(\theta_o) - f_k(\theta))^2$

Proposition. *Assume that there exists Ω_∞ with $P(\Omega_\infty) > 0$ such that on Ω_∞*
a. $\forall\delta > 0, \quad \lim_n \inf_{\theta\in B_\delta^c(\theta_o)} D_n(\theta_o, \theta) \overset{a.s.}{=} \infty$ *(strong identifiability)*

b. $\forall k, \forall\theta_1, \theta_2 \, : \, |f_k(\theta_1) - f_k(\theta_2)| \overset{a.s.}{\leq} h(||\theta_1 - \theta_2||)\, g_k$, *where* g_k *is* $\mathcal{F}_{k-1}$*-measurable,* $\lim_{x\downarrow 0} h(x) = 0$ *(Lipschitz)*

Then $\lim_n \widehat{\theta}_n \overset{a.s.}{=} \theta_o$ *on* Ω_∞

Estimator $\widehat{\theta}_n = \arg\min_{\theta \in \Theta} S_n(\theta), \quad S_n(\theta) = \sum_{k=1}^{n} (Y_k - f_k(\theta))^2$
Identifiability criterion: $D_n(\theta_o, \theta) = \sum_{k=1}^{n} (f_k(\theta_o) - f_k(\theta))^2$

Proof

1. Use Wu's lemma (1981):

For all $\delta > 0$, $\underline{\lim}_n \inf_{\theta \in B_\delta^c(\theta_o)} (S_n(\theta) - S_n(\theta_o)) \overset{a.s.(P.)}{>} 0 \Longrightarrow \lim_n \widehat{\theta}_n \overset{a.s.(P)}{=} \theta_o$

2. Use Wu's decomposition (1981) based on

$$\begin{aligned} Y_k - f_k(\theta) &= (Y_k - f_k(\theta_o)) + (f_k(\theta_o) - f_k(\theta)) \\ &\overset{notation}{=} \eta_k + d_k(\theta_o, \theta) \\ \Longrightarrow S_n(\theta) - S_n(\theta_o) &= D_n(\theta_o, \theta)[1 + 2\frac{\sum_{k=1}^{n} \eta_k d_k(\theta_o, \theta)}{D_n(\theta_o, \theta)}] \\ \Longrightarrow \inf_{\theta \in B_\delta^c(\theta_o)} S_n(\theta) - S_n(\theta_o) &\geq \inf_{\theta \in B_\delta^c(\theta_o)} D_n(\theta_o, \theta)[1 - 2 \sup_{\theta \in B_\delta^c(\theta_o)} |\frac{\sum_{k=1}^{n} \eta_k d_k(\theta_o, \theta)}{D_n(\theta_o, \theta)}|] \end{aligned}$$

3. Prove that $\lim_n \sup_{\theta \in B_\delta^c(\theta_o)} |\sum_{k=1}^{n} \eta_k d_k(\theta_o, \theta)[D_n(\theta_o, \theta)]^{-1}| \overset{a.s.}{=} 0$ (main result)
(use the properties of the submartingale $\sup_{\theta \in B_\delta^c(\theta_o)} |\sum_{k=1}^{n} \eta_k d_k(\theta_o, \theta)[D_k(\theta_o, \theta)]^{-1}|$, Hall and Heyde,19

Consistency in a model with a negligible nuisance part

$Y_n = f_n(\theta,\nu) + \eta_n,\ \ f_n(\theta,\nu) = f_n^{(1)}(\theta) + f_n^{(2)}(\theta,\nu)$, $\nu = \{\nu_n\}$: nuisance parameter, $\nu \in \mathbb{R}^q$, $q \leq \infty$

Estimator: $\widehat{\theta}_{\nu,n} = \arg\min_{\theta\in\Theta} S_{\nu,n}(\theta)$, $S_{\nu,n}(\theta) = \sum_{k=1}^{n}(Y_k - f_k(\theta,\nu))^2$.

Notations

$D_n^{(1)}(\theta_o,\theta) = \sum_{k=1}^{n}[f_k^{(1)}(\theta_o) - f_k^{(1)}(\theta)]^2$, $D_n^{(2)}(\nu_o,\nu|\theta_o) = \sum_{k=1}^{n}[f_k^{(2)}(\theta_o,\nu_o) - f_k^{(2)}(\theta_o,\nu)]^2$.

Proposition. *Assume that there exists* Ω_∞ *with* $P(\Omega_\infty) > 0$ *such that on* Ω_∞*,*

a. $\forall k$, $f_k(\theta,\nu)$ *is Lipschitz in* θ

b. *for all* $\delta > 0$, $\lim_n \inf_{\theta\in B_\delta^c(\theta_o)} D_n^{(1)}(\theta_o,\theta) \overset{a.s.}{=} \infty$ *(strong identifiability)*

3. $\overline{\lim}_n D_n^{(2)}(\nu_o,\nu|\theta_o)[\inf_{\theta\in B_\delta^c(\theta_o)} D_n^{(1)}(\theta_o,\theta)]^{-1} \overset{a.s.}{=} 0$.

Then $\lim_n \widehat{\theta}_n \overset{a.s.}{=} \theta_o$ *on* Ω_∞*.*

Asymptotic distribution

$$A_0:\ \forall\delta>0, \lim_n \inf_{\theta\in B_\delta^c(\theta_o)} \sum_{k=1}^{n}(f_k(\theta_o)-f_k(\theta))^2 \overset{a.s.}{=} \infty$$

$$A_1:\ \lim_{n:\theta_n\to\theta_0} \sup_{k\leq n} \left|\left[\frac{\partial f_k}{\partial\theta_i}(\theta_n)\frac{\partial f_k}{\partial\theta_l}(\theta_n)\right]\left[\frac{\partial f_k}{\partial\theta_i}(\theta_0)\frac{\partial f_k}{\partial\theta_l}(\theta_0)\right]^{-1}-1\right| \overset{a.s.}{=} 0$$

$$A_2:\ \lim_n\left[\sum_{k=1}^{n}\frac{\partial f_k}{\partial\theta}(\theta_o)\frac{\partial f_k}{\partial\theta^t}(\theta_o)\right]^{-1}[l,j] \overset{a.s.}{=} 0,\ \ \overline{\lim}_n \inf_{\theta\in\Theta}\sum_{k=1}^{n}\left[\frac{\partial^2 f_k}{\partial\theta_i\partial\theta_l}(\theta)\right]^2\left|\left[\sum_{k=1}^{n}\frac{\partial f_k}{\partial\theta}(\theta_o)\frac{\partial f_k}{\partial\theta^t}(\theta_o)\right]^{-1}[l,j]\right| \overset{a.s.}{<} \infty$$

$$A_3:\ \overline{\lim}_n \sup_{\theta\in\Theta}\sum_{k=1}^{n} g_k\left|\frac{\partial^2 f_k}{\partial\theta_i\partial\theta_l}(\theta)\right|\left|\left[\sum_{k=1}^{n}\frac{\partial f_k}{\partial\theta}(\theta_o)\frac{\partial f_k}{\partial\theta^t}(\theta_o)\right]^{-1}[l,j]\right| \overset{a.s.}{<} \infty.$$

Particular case: $f_n(\theta)=\theta^t W_n \Longrightarrow A_1$, A_{2_2}, A_3 checked

Proposition. *Assume the existence of* $\Omega_\infty \subset \Omega$, $P(\Omega_\infty) > 0$ *and such that on* Ω_∞*:*

1. $\frac{\partial^2 f_k}{\partial\theta_i \partial\theta_l}(\theta)$ *exists and is Lipschitz*

2. A_0, A_1, A_2 *are checked, and there exists* Ψ_n*:*

$$\lim_n \Psi_n[\sum_{k=1}^{n} \frac{\partial f_k}{\partial\theta}(\theta_o)\frac{\partial f_k}{\partial\theta^t}(\theta_o)]^{-1} \sum_{k=1}^{n} \eta_k \frac{\partial f_k}{\partial\theta}(\theta_o) \text{ exists in distribution}$$

Then on Ω_∞

$$\lim_n \Psi_n(\widehat{\theta}_n - \theta_0) \stackrel{d}{=} \lim_n \Psi_n[\sum_{k=1}^{n} \frac{\partial f_k}{\partial\theta}(\theta_o)\frac{\partial f_k}{\partial\theta^t}(\theta_o)]^{-1} \sum_{k=1}^{n} \eta_k \frac{\partial f_k}{\partial\theta}(\theta_o)$$

$$\lim_n \Psi_n [\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_o)\frac{\partial f_k}{\partial \theta^t}(\theta_o)]^{-1} \sum_{k=1}^n \eta_k \frac{\partial f_k}{\partial \theta}(\theta_o) \text{ exists in distribution } ?$$

Deterministic regression

Choose $\Psi_n = [\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_o)\frac{\partial f_k}{\partial \theta^t}(\theta_o)]^{1/2}$

$\Longrightarrow \lim_n \Psi_n(\widehat{\theta}_n - \theta_o) \stackrel{d}{=} \lim_n [\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_o)\frac{\partial f_k}{\partial \theta^t}(\theta_o)]^{-1/2} \sum_{k=1}^n \eta_k \frac{\partial f_k}{\partial \theta}(\theta_o)$

Use a classical CLT for martingale triangular arrays

Proposition(Dacunha-Castelle and Duflo, 1993)

Let M_k^n *be a multidimensional martingale triangular array. Assume*

$a. \lim_n < M >_n^n \stackrel{P}{=} \Gamma$

$b. \forall \xi > 0, \lim_n \sum_{k=1}^n E(||M_k^n - M_{k-1}^n||^2 1_{||M_k^n - M_{k-1}^n|| \geq \xi} | \mathcal{F}_{k-1}) \stackrel{P}{=} 0$ ***(Lindeberg)***

Then $\lim_n M_n^n \stackrel{d}{=} \mathcal{N}(0, \gamma)$

$M_k^n = [\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_o)\frac{\partial f_k}{\partial \theta^t}(\theta_o)]^{-1/2} \sum_{l=1}^k \eta_l \frac{\partial f_l}{\partial \theta}(\theta_o)$

$< M >_n^n = [\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_o)\frac{\partial f_k}{\partial \theta^t}(\theta_o)]^{-1} \sum_{k=1}^n var(\eta_k^2 | \mathcal{F}_{k-1}) \frac{\partial f_k}{\partial \theta}(\theta_o)\frac{\partial f_k}{\partial \theta^t}(\theta_o)$

Stochastic regression

Assume $\lim_n \frac{\partial f_n}{\partial \theta_i}(\theta_o)/a_{n,i} \stackrel{a.s.}{=} W_i$, W_i random or nonrandom

Choose $\Psi_n = [\sum_{k=1}^n a_k a_k^t]^{-1/2}[\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_o)\frac{\partial f_k}{\partial \theta^t}(\theta_o)]$

$\Longrightarrow \lim_n \Psi_n(\widehat{\theta}_n - \theta_o) \stackrel{d}{=} \lim_n [\sum_{k=1}^n a_k a_k^t]^{-1/2} \sum_{k=1}^n \eta_k \frac{\partial f_k}{\partial \theta}(\theta_o)$

Use a classical CLT for martingale triangular arrays if W deterministic, or if W random, use a generalized CLT for martingale arrays (van Zanten, 2000), or CLT for random sums (branching processes) (Billingsley, 1968, , Rahimov, 1995, 2008?)

$\lim_n \Psi_n(\widehat{\theta}_n - \theta_0) \stackrel{d}{=} \lim_n \Psi_n[\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_o)\frac{\partial f_k}{\partial \theta^t}(\theta_o)]^{-1} \sum_{k=1}^n \eta_k \frac{\partial f_k}{\partial \theta}(\theta_o)$?

Proof. Taylor's expansion at first order:

$$\frac{\partial S_n}{\partial \theta}(\widehat{\theta}_n) = 0 = \frac{\partial S_n}{\partial \theta}(\theta_o) + \frac{\partial^2 S_n}{\partial \theta \partial \theta^t}(\theta_{*n})(\widehat{\theta}_n - \theta_0) \Longrightarrow \Psi_n(\widehat{\theta}_n - \theta_0) = -\Psi_n[\frac{\partial^2 S_n}{\partial \theta \partial \theta^t}(\theta_{*n})]^{-1}\frac{\partial S_n}{\partial \theta}(\theta_o),$$

Derivatives of S_n

$$\frac{1}{2}\frac{\partial S_n}{\partial \theta}(\theta_o) = -\sum_{k=1}^n \eta_k \frac{\partial f_k}{\partial \theta}(\theta_o)$$

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 S_n}{\partial \theta \partial \theta^t}(\theta_{*n}) &= \sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_{*n})\frac{\partial f_k}{\partial \theta^t}(\theta_{*n}) - \sum_{k=1}^n \eta_k \frac{\partial^2 f_k}{\partial \theta \partial \theta^t}(\theta_{*n}) + \sum_{k=1}^n (f_k(\theta_o) - f_k(\theta_{*n}))\frac{\partial^2 f_k}{\partial \theta \partial \theta^t}(\theta_{*n}) \\ &= [I - [\sum_{k=1}^n \eta_k \frac{\partial^2 f_k}{\partial \theta \partial \theta^t}(\theta_{*n}) + \sum_{k=1}^n (f_k(\theta_o) - f_k(\theta_{*n}))\frac{\partial^2 f_k}{\partial \theta \partial \theta^t}(\theta_{*n})][\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_{*n})\frac{\partial f_k}{\partial \theta^t}(\theta_{*n})]^{-1}] \times \\ &\quad [\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_{*n})\frac{\partial f_k}{\partial \theta^t}(\theta_{*n})]\end{aligned}$$

$\Longrightarrow \frac{\partial^2 S_n}{\partial \theta \partial \theta^t}(\theta_{*n})$ may be replaced by $2\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_0)\frac{\partial f_k}{\partial \theta^t}(\theta_0)$ under A_1, A_2, A_3

$A_1 : \lim_{n:\theta_n \to \theta_0} \sup_{k \le n} |[\frac{\partial f_k}{\partial \theta_i}(\theta_n)\frac{\partial f_k}{\partial \theta_l}(\theta_n)][\frac{\partial f_k}{\partial \theta_i}(\theta_0)\frac{\partial f_k}{\partial \theta_l}(\theta_0)]^{-1} - 1| \overset{a.s.}{=} 0$ (uniform continuity) $\Longrightarrow$ not always checked

Example: $N_n = \sum_{i=1}^{N_{n-1}} X_{n,i}$ with $m_\theta(N) = 1 + \theta^N$
$\Longrightarrow f_n(\theta) = (1 + \theta^{\ln(N_{n-1})})N_{n-1}$, $\frac{\partial f_n}{\partial m} = N_{n-1}\ln(N_{n-1})\theta^{\ln(N_{n-1})-1} \Longrightarrow \lim_{n:\theta_n \to \theta_o} \sup_{k \le n}[\theta_n/\theta_o]^{\ln(N_{k-1})} =?$

If A_1 is not checked, use the Taylor's expansion at second order of $\frac{\partial S_n}{\partial \theta}(\widehat{\theta}_n)$ at θ_0
Define A_4, A_5:

$$A_4 : \lim_n [\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_0)\frac{\partial f_k}{\partial \theta^t}(\theta_0)]^{-1}[l,j] \overset{a.s.}{=} 0,\ \ \overline{\lim}_n \inf_\theta \sum_{k=1}^n [\frac{\partial^3 f_k}{\partial \theta_{l'}\theta_j \partial \theta_l}(\theta)]^2 [\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_0)\frac{\partial f_k}{\partial \theta^t}(\theta_0)]^{-1}[l,j] \overset{a.s.}{=} \infty$$

$$A_5 : \sup_l \lim_n \sup_\theta || [\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_0)\frac{\partial f_k}{\partial \theta^t}(\theta_0)]^{-1}[\frac{\partial^3 \widetilde{S}_n}{\partial \theta \partial \theta^t \partial \theta_l}(\theta) ||_\infty \overset{a.s.}{=} 0$$

$$\frac{\partial^3 \widetilde{S}_n}{\partial \theta_i \partial \theta_j \partial \theta_l}(\theta) = 2[\sum_{k=1}^n \frac{\partial^2 f_k}{\partial \theta_j \partial \theta_l}(\theta)\frac{\partial f_k}{\partial \theta_i}(\theta) + \sum_{k=1}^n \frac{\partial^2 f_k}{\partial \theta_i \partial \theta_l}(\theta)\frac{\partial f_k}{\partial \theta_j}(\theta) + \sum_{k=1}^n \frac{\partial^2 f_k}{\partial \theta_i \partial \theta_j}(\theta)\frac{\partial f_k}{\partial \theta_l}(\theta)]$$

Proposition. *Assume that there exists* $\Omega_\infty \subset \Omega$, $P(\Omega_\infty) > 0$ *and such that, on* Ω_∞*:*

1. $\frac{\partial^2 f_k}{\partial\theta_i\partial\theta_j}(\theta_0)$ *and* $\frac{\partial^3 f_k}{\partial\theta_i\partial\theta_j\partial\theta_l}(\theta)$ *are Lipschitz*

2. A_0, A_2, A_4, A_5 *and the existence of* Ψ_n *such that*

$$\lim_n \Psi_n[\sum_{k=1}^{n} \frac{\partial f_k}{\partial\theta}(\theta_0)\frac{\partial f_k}{\partial\theta^t}(\theta_0)]^{-1}\sum_{k=1}^{n}\eta_k\frac{\partial f_k}{\partial\theta}(\theta_0) \text{ exists in distribution}$$

Then

$$\lim_n \Psi_n(\widehat{\theta}_n - \theta_0) \overset{d}{=} \lim_n \Psi_n[\sum_{k=1}^{n} \frac{\partial f_k}{\partial\theta}(\theta_0)\frac{\partial f_k}{\partial\theta^t}(\theta_0)]^{-1}\sum_{k=1}^{n}\eta_k\frac{\partial f_k}{\partial\theta}(\theta_0) \text{ on } \Omega_\infty$$

Example: Polymerase Chain Reaction (PCR)

Technology of amplification of a population of DNA fragments through successive replication cycles *in vitro*
Goal: estimation of N_0 (initial population size) from the amplified populations $\{N_n\}$
Applications: gene expression (biotechnology), virus quantification (medicine), GMO detection (food industry, environment),...

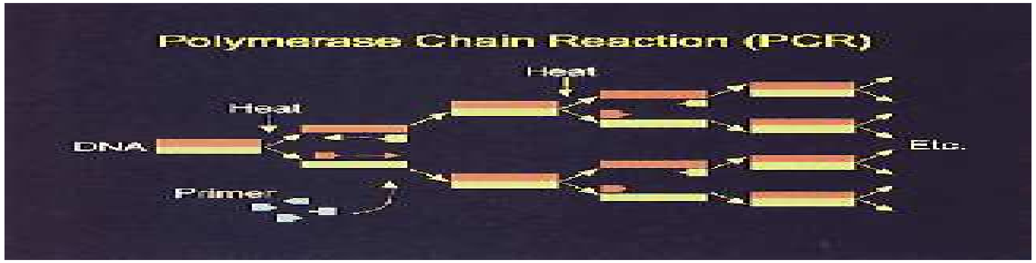

Figure 1: The three steps of a replication cycle: heating, annealing, DNA synthesis

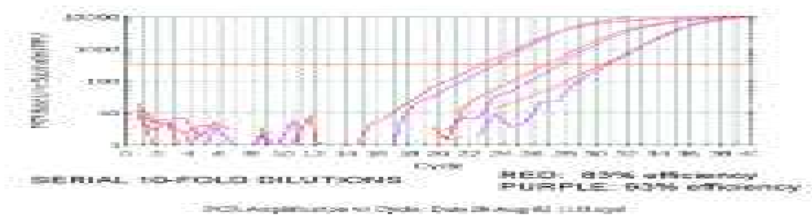

Figure 2: Exponential phase, saturation phase (linear phase, plateau phase)

Exponential phase

No saturation $\Longrightarrow$ Bienaymé-Galton-Watson process
(Peccoud and Jacob, 1996, Jacob and Peccoud, 1998)

$N_n = \sum_{i=1}^{N_{n-1}} X_{n,i}$ with $P(X_{n,i} = 2|N_{n-1}) = p$, $P(X_{n,i} = 1|N_{n-1}) = 1 - p$

$\theta = m = 1 + p$, $f_n(m) = mN_{n-1}^{1/2}$, $\frac{\partial f_n}{\partial m}(m) = N_{n-1}^{1/2}$

Consistency of $\widehat{m}_n$

- $|f_n(m_1) - f_n(m_2)| \leq ||m_1 - m_2||N_{n-1}^{1/2}$ (linear model $\Longrightarrow$ Lipschitz)
- $\inf_{m \in B_\delta^c(m_o)} D_n(m_o, m) = (m_o - m)^2 \sum_{k=1}^{n} N_{k-1} \xrightarrow{a.s.} \infty$ on the nonextinction set Ω_∞

$\Longrightarrow \lim_n \widehat{m}_n \stackrel{a.s.}{=} m_o$ on Ω_∞; $P(\Omega_\infty) > 0$ for $m_o > 1$

Remark (direct proof). Harris estimator:

$$\widehat{m}_n = \frac{\sum_{k=1}^{n} N_k}{\sum_{k=1}^{n} N_{k-1}} = \frac{\sum_{k=1}^{n} (N_k m_o^{-k}) m_o^k}{\sum_{k=1}^{n} m_o^k} \frac{m_o \sum_{k=1}^{n} m_o^{k-1}}{\sum_{k=1}^{n} (N_{k-1} m_o^{-(k-1)}) m_o^{k-1}}$$

Use $\lim_n N_n m_o^{-n} \stackrel{a.s.}{=} W$ and Toeplitz lemma $\Longrightarrow \lim_n \widehat{m}_n \stackrel{a.s.}{=} m_o$

Asymptotic distribution of $\widehat{m}_n$

$$\Psi_n \text{ ? } : \lim_n \Psi_n(\widehat{m}_n - m_o) \overset{d}{=} \lim_n \Psi_n[\sum_{k=1}^{n} N_{n-1}]^{-1} \sum_{k=1}^{n} \eta_k N_{k-1}^{1/2}$$

$\lim_n N_n m^{-n} \overset{a.s.}{=} W \Longrightarrow \Psi_n = [\sum_{k=1}^n m_o^{k-1}]^{-1/2}[\sum_{k=1}^n N_{k-1}]$

Write $\eta_k N_{k-1}^{1/2} = \sum_{i=1}^{N_{k-1}}(X_{k,i} - m_o) = \sum_{i=1}^{N_{k-1}}(X_{k,i} - m_o) = \sum_{j=1}^{\sum_{k=1}^n N_{k-1}}(X_{k,j} - m_o) \Longrightarrow$ CLT for random sums

$$\Psi_n(\widehat{m}_n - m_o) = \Psi_n[\sum_{k=1}^{n} N_{n-1}]^{-1} \sum_{k=1}^{n} \sum_{i=1}^{N_{k-1}} (X_{k,i} - m_o) = [\sum_{k=1}^{n} m_o^{k-1}]^{-1/2}[\sum_{j=1}^{\sum_{k=1}^n N_{k-1}} (X_{k,j} - m_o)] \qquad (1)$$

$$\Longleftrightarrow [\sum_{k=1}^{n} m_o^{k-1}]^{1/2}(\widehat{m}_n - m_o) = ([\sum_{k=1}^{n} m_o^{k-1}]^{-1}[\sum_{k=1}^{n} N_{k-1}])^{-1} \times [\sum_{k=1}^{n} m_o^{k-1}]^{-1/2} \sum_{j=1}^{\sum_{k=1}^n N_{k-1}} (X_{k,j} - m_o)]$$

$$\Longrightarrow \lim_n [\sum_{k=1}^{n} m_o^{k-1}]^{1/2}(\widehat{m}_n - m_o) \overset{d}{=} W^{-1}U,\ U \sim \mathcal{N}(0, \sigma^2), U \text{ and } W \text{ independent.}$$

Remark(direct proof). If we use directly the expression of $\widehat{m}_n - m_o \Longrightarrow$ (1)

Exponential and saturation phases

Size-dependent branching process with Schnell and Mendoza model of replication (1997)

$$N_n = \sum_{i=1}^{N_{n-1}} X_{n,i}, \qquad P(X_{n,i} = 2|N_{n-1}) = \frac{K}{K + N_{n-1}}, \ K: \text{ Michaelis-Menten constant}$$

$$P(X_{n,i} = 1|N_{n-1}) = 1 - P(X_{n,i} = 2|N_{n-1})$$

$\{N_n\}$ is a near-critical process, $\lim_n N_n n^{-1} \stackrel{a.s.}{=} K$ (Jagers and Klebaner, 2003)
$Y_n = N_n$, $f_n(K) = (1 + K(K + N_{n-1})^{-1})N_{n-1}$, $\lim_n var(\eta_n|\mathcal{F}_{n-1}) = K$

Asymptotic properties of $\widehat{K}_n$ (Lalam, Jacob and Jagers, 2004)
$D_n(K_o, K) = O(n) \Longrightarrow \lim_n \widehat{K}_n \stackrel{a.s.}{=} K_o$
$\lim_n \frac{f_n}{\partial K} = 1 \Longrightarrow \lim_n \sqrt{n}(\widehat{K}_n - K_o) \stackrel{d}{=} \mathcal{N}(0, K)$ (CLT for martingale triangular arrays or CLT for random sums)

Generalized Schnell-Mendoza model taking into account a saturation threshold $S \geq N_0$
(Lalam, Jacob, Jagers, 2004)

$$P(X_{n,i} = 2|N_{n-1}) = (\frac{K}{K + N_{S,n-1}})(\frac{1 + \exp(-C(N_{S,n-1}S^{-1} - 1))}{2}), \;\; N_{S,n-1} = S1_{\{N_{n-1}<S\}} + N_{n-1}1_{\{N_{n-1}\geq S\}}$$

Particular case: $C = 0$ and $S = N_0 \Longrightarrow$ Schnell-Mendoza model

Assume $C \neq 0$.
$\{N_n\}$ is a near-critical process, $\lim_n N_n n^{-1} \overset{a.s.}{=} K/2$, $Y_n = N_n$ (or $Y_n = N_n \frac{N_{n-1}}{N_{S,n-1}}$)

$$\begin{aligned} f_n(\theta) &= [1 + (\frac{K}{K + N_{S,n-1}})(\frac{1 + \exp(-C(N_{S,n-1}S^{-1} - 1))}{2})]N_{n-1} \\ &= N_{n-1} + \frac{KN_{n-1}}{2(K + N_{S,n-1})} + \frac{KN_{n-1}\exp(-C(N_{S,n-1}S^{-1} - 1))}{2(K + N_{S,n-1})} \\ &= explosive + permanent + transient \end{aligned}$$

Asymptotic properties of $\widehat{\theta}_n$, $\theta = (K, C, S^{-1})$

$\lim_n D_n(\theta_o, \theta) < \infty \Longrightarrow$ no consistency of $\widehat{\theta}_n$

But write $f_n(\theta) = (1 + \frac{K}{2(K+N_{S,n-1})})N_{n-1} + \frac{KN_{n-1}}{2(K+N_{S,n-1})}\exp(-C(N_{S,n-1}S^{-1} - 1)) \overset{notat.}{=} f_n^{(1)}(\theta) + f_n^{(2)}(\theta)$

$\Longrightarrow \lim_n \widehat{K}_{\nu,n} \overset{a.s.}{=} K_o$, $\lim_n \sqrt{n/4}(\widehat{K}_n - K) \overset{d}{=} \mathcal{N}(0, K/2)$

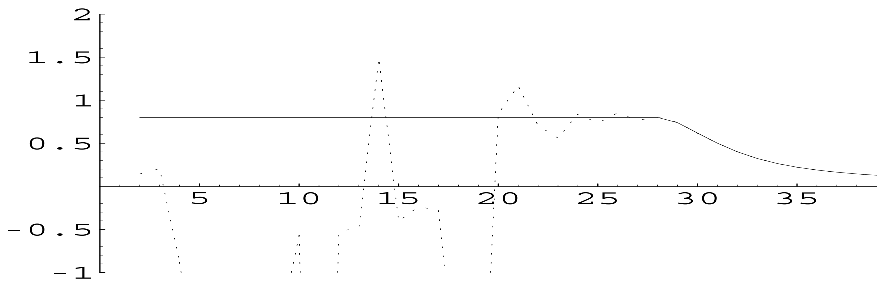

Figure 3: Probability of replication calculated from a trajectory of $N_n + \varepsilon_n$ simulated with $K = 4.00311.10^{10}$, $S = 10^{10}$, $C = 0$ ($p = K(K+S)^{-1} = 0.800125$). In dashed line: $\overline{p}(X_{k-1}) = X_k X_{k-1}^{-1} - 1$ (empirical probability of replication). In continuous line: $\widehat{p}(X_{k-1})$ (estimated probability of replication)

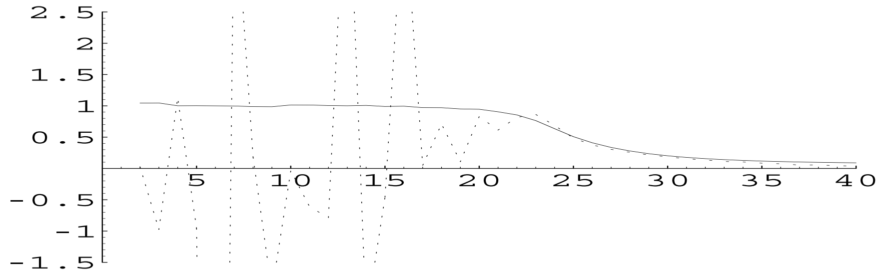

Figure 4: Well 21 of data set 1, $\widehat{K}_{h,n} = 0.22769$, $h = 23$, $n = 27$, $S = N_0$, $C = 0$ (Schnell-Mendoza model)

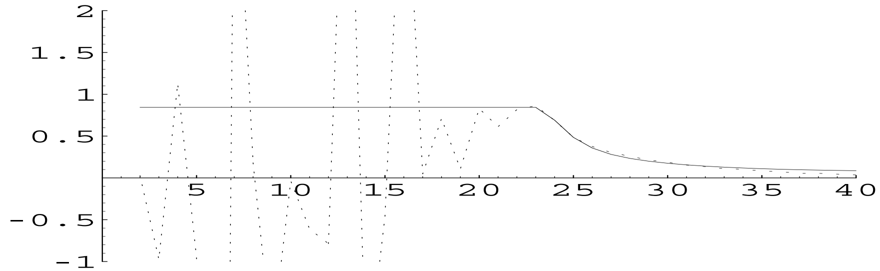

Figure 5: Well 21 of data set 1, $h = 21$, $n = 25$, $\widehat{n}_s = 23$, $\widehat{K}_{h,n,\nu} = 0.38055$, $\widehat{S}_{h,n} = 0.070553$, $\widehat{C}_{h,n} = 0.6$, $\widehat{p}_{h,n,\nu} = 0.843599$ (generalized model)

Generalized Schnell-Mendoza model with a non negligible transient part

$$P(X_{n,i}=2|N_{n-1}) \;=\; (\frac{K}{K+N_{S,n-1}})(\frac{1+S^{\alpha}N_{S,n-1}^{-\alpha}}{2}),\alpha>0,\;\; N_{S,n-1}=S1_{\{N_{n-1}<S\}}+N_{n-1}1_{\{N_{n-1}\geq S\}}$$

Assume $S<\infty \Longrightarrow$ near-citical process, $\lim_n \frac{N_n}{n} \overset{a.s.}{=} \frac{K}{2}$, $f_n(\theta)=[1+(\frac{K}{K+N_{S,n-1}})(\frac{1+S^{\alpha}N_{S,n-1}^{-\alpha}}{2})]N_{n-1}$

Remark. If $\theta\supset\alpha$, the order of the nth derivatives increase with $n \Longrightarrow A_2$ not checked
Example: $\theta=\alpha$, K is known

$$\inf_{\alpha}\sum_{k=1}^{n}[\frac{\partial^2 f_k}{(\partial\alpha)^2}(\alpha)]^2[\sum_{k=1}^{n}[\frac{\partial f_k}{\partial\alpha}(\alpha_o)]^2]^{-1} \;=\; O(\inf_{\alpha}[\sum_{k=1}^{n}(\ln(k))^4k^{-2\alpha}][\sum_{k=1}^{n}(\ln(k))^2k^{-2\alpha}]^{-1})$$

$$\overset{2\alpha\leq 1}{=} \; O((\ln(n))^2)\rightarrow\infty$$

So assume that α is known and $\theta = (K, S^\alpha)$

Consistency of $\widehat{\theta}_n$

$\inf_\theta D_n(\theta,\theta) = O(n^{1-2\alpha} 1_{\{2\alpha<1\}} + \ln(n) 1_{\{2\alpha\le 1\}}) \Longrightarrow \lim_n \widehat{\theta}_n \overset{a.s.}{=} \theta_o$

Asymptotic distribution of $\widehat{\theta}_n$

$\Psi_n = \Phi_n^{-1}[\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_o)\frac{\partial f_k}{\partial \theta^t}(\theta_o)]$ with

$$\Phi_n^2 = O(\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_o)\frac{\partial f_k}{\partial \theta^t}(\theta_o)) = \frac{1}{4}\begin{pmatrix} n & Kn^{1-\alpha} \\ Kn^{1-\alpha} & K^2[n^{1-2\alpha}1_{2\alpha<1} + \ln(n)1_{2\alpha=1}] \end{pmatrix}$$

Then using CLT for martingale arrays (Dacunha-Castelle and Duflo, 1993),

$$\lim_n \Phi_n(\widehat{\theta}_n - \theta_o) \overset{d}{=} \mathcal{N}(0, (K/2)I),\ \ 2\alpha \le 1$$

Bienaymé-Galton-Watson regenerative process (Yanev N., Jacob, Lalam and Yanev N. (2005))

$N_n = 1_{\{N_{n-1} \neq 0\}} \sum_{i=1}^{N_{n-1}} X_{n,i} + 1_{\{N_{n-1}=0\}} I_n \delta_n^I$, $\{X_{n,i}\} \sim (m, \sigma^2)$, $\{I_n\} \sim (\lambda, b^2)$ given $\{\delta_n^I\}$, $m < 1$

$$\Longrightarrow Y_n = 1_{\{N_{n-1} \neq 0\}} N_n N_{n-1}^{-1/2} + 1_{\{N_{n-1}=0\}} N_n = 1_{\{N_{n-1} \neq 0\}} m N_{n-1}^{1/2} + 1_{\{N_{n-1}=0\}} \lambda \delta_n^I + \eta_n$$

Goal: estimate (m, λ), $\{N_n\}$ and $\{\delta_n^I\}$ observed

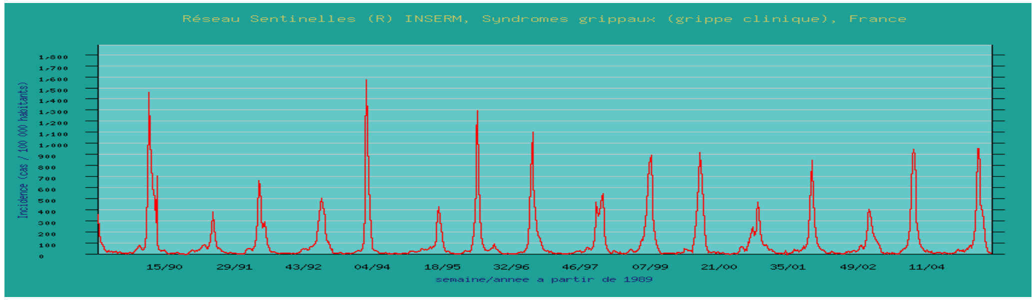

Figure 6: Example of regenerative process: Influenzia in France since 1989 (http://www.b3e.jussieu.fr/senti/)

$$\theta = (m, \lambda), \quad \widehat{m}_n = \frac{\sum_{k=1}^n N_k 1_{N_{k-1}\neq 0}}{\sum_{k=1}^n N_{k-1} 1_{N_{k-1}\neq 0}}; \quad \widehat{\lambda}_n = \frac{\sum_{k=1}^n I_k \delta_k^I 1_{\{N_{k-1}=0\}}}{\sum_{k=1}^n \delta_k^I 1_{\{N_{k-1}=0\}}}.$$

$D_n(\theta_o, \theta) = (m_o - m)^2 D_n(m) + (\lambda_o - \lambda)^2 D_n(\lambda)$
$D_n(m) = \sum_{k=1}^n N_{k-1} 1_{\{N_{k-1}\neq 0\}}$, $D_n(\lambda) = \sum_{k=1}^n \delta_k^I 1_{\{N_{k-1}=0\}}$

Proposition (Jacob, Lalam and Yanev, 2005)

On $\{\lim_n D_n(\lambda) = \infty\}$*,* $\lim_n(\widehat{m}_n, \widehat{\lambda}_n) = (m_o, \lambda_o)$ *and* $\lim_n \sqrt{n}(\widehat{\theta}_n - \theta_o) \overset{d}{=} \mathcal{N}(0, \Lambda)$, *where* Λ *is a diagonal matrix with* $(\sigma^2 s_*^{-1} E(\mathcal{T})$*,* $b^2 E(\mathcal{T}))$ *on the diagonal.*

$\mathcal{T}$: working period + resting period, $s_* = \lambda(1-m)^{-1}$

Remark. The independence comes from: $\sum_{k=1}^n \frac{\partial f_k}{\partial \theta}(\theta_o) \frac{\partial f_k}{\partial \theta^t}(\theta_o)$ is a diagonal matrix

The asymptotic independence of $\widehat{m}_n$ and $\widehat{\lambda}_n$ is difficult to prove directly using their explicit expressions

Conclusion

1. Indirect proofs $\Longrightarrow$ use the best LSE even if it has no explicit form $\Longrightarrow$ generalize the CLSE in branching processes

2. Main result: $\lim_n \sup_\theta [\sum_{k=1}^n \eta_k d_k(\theta)][\sum_{k=1}^n d_k^2(\theta)]^{-1} \stackrel{a.s.}{=} 0$ on $\{\lim_n \sum_{k=1}^n d_k^2(\theta) = \infty\}$
$\Longrightarrow$ consistency, asymptotic distribution

3. Strong identifiability ($\lim_n \inf_{\theta \in B_\delta^c(\theta)} \sum_{k=1}^n (f_k(\theta) - f_k(\theta'))^2 \stackrel{a.s.}{=} \infty$) is a necessary and sufficient condition for consistency

4. The model may contain a stationary part, a transient part and an explosive part

Thank you for your attention!