

## Shock Waves

A shock wave is a surface of discontinuity moving through a medium at a speed larger than the speed of sound upstream. The change in the fluid properties upon passing the shock can be investigated using simple conservation laws.

In a fluid flow, conservation laws are usually expressed in the form of continuity equations. Instead of the usual differential form, the continuity equations can also be expressed as relations between quantities on two sides of an arbitrary stationary surface: the mass flux, the momentum flux and the energy flux must be continuous through the surface. For a surface normal to the local fluid velocity  $\vec{v}$ , these relations can be expressed as:

$$\begin{aligned} [\rho v] &= 0 \quad (\text{mass conservation}) \\ [P + \rho v^2] &= 0 \quad (\text{momentum conservation}) \\ [v(u + P + \rho v^2/2)] &= 0 \quad (\text{energy conservation}) \end{aligned}$$

Here  $v = |\vec{v}|$ ,  $P$  is the local pressure and  $u$  is the internal energy per unit volume.  $u + P$  is the enthalpy, or *heat function* per unit volume of the fluid. The square brackets represent the difference between the enclosed quantity evaluated on two sides of the stationary surface. For an adiabatic index (ratio of specific heats)  $\gamma$ ,  $u = P/(\gamma - 1)$ , and hence the energy relation can be written as

$$[v(\gamma P/(\gamma - 1) + \rho v^2/2)] = 0$$

Using the mass conservation equation, this can be further simplified to

$$[\gamma P/(\gamma - 1)\rho + v^2/2] = 0$$

We can apply these relations to the quantities upstream and downstream of a shock wave, once we move to a reference frame in which the surface of discontinuity is at rest (fig. 1). In this frame, the upstream fluid approaches the discontinuity at a speed  $v_1$  and the downstream fluid leaves the shock with a speed  $v_2$ . Let  $\rho_1$  and  $\rho_2$  be the densities on the two sides respectively, and  $P_1$  and  $P_2$  the corresponding pressures. In this frame, we can then write

$$\rho_1 v_1 = \rho_2 v_2$$

$$P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2$$

$$\frac{\gamma}{\gamma - 1} \frac{P_1}{\rho_1} + \frac{1}{2} v_1^2 = \frac{\gamma}{\gamma - 1} \frac{P_2}{\rho_2} + \frac{1}{2} v_2^2$$

These are known as the shock *jump conditions* or *Rankine-Hugoniot* relations. Note that in the rest frame of the unshocked fluid,  $v_1$  is the speed of propagation of the shock, and  $v_1 - v_2$  is the speed with which the shocked fluid is seen to move forward.

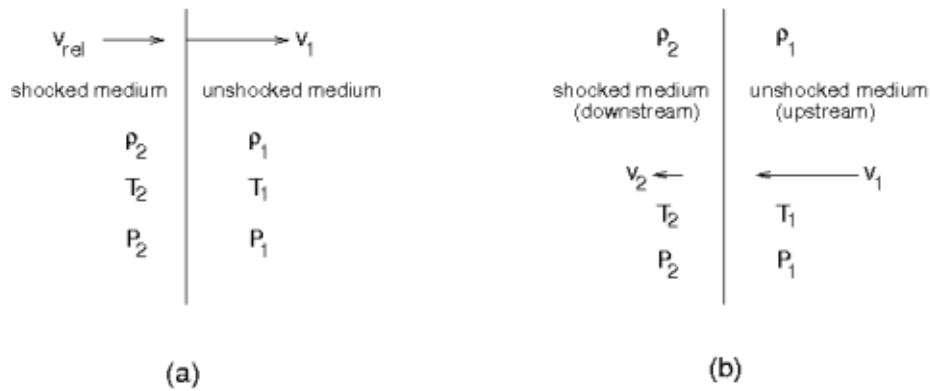


Figure 1: A shock wave discontinuity in (a) the reference frame of the unshocked medium and (b) in a reference frame where the surface of discontinuity is at rest. The shock advances into the unshocked fluid at speed  $v_1$ . In the rest frame of the shock, the upstream medium approaches it at speed  $v_1$ . The shocked fluid moves away from the discontinuity at speed  $v_2 = \rho_1 v_1 / \rho_2$ . The shocked fluid therefore approaches the unshocked fluid at a speed  $v_{\text{rel}} = v_1 - v_2$ .

We will be interested in very strong shocks, for which the upstream medium can be approximated as a cold ( $T \sim 0$ ) fluid. The upstream pressure can therefore be neglected in comparison with all other quantities appearing in the jump conditions. Setting  $P_1 = 0$  in the above equations, and defining the *Compression Ratio*

$$R \equiv \frac{v_1}{v_2}$$

one obtains from the above:

$$\begin{aligned}\frac{\rho_2}{\rho_1} &= R \\ \frac{P_2}{\rho_2} &= v_1^2 \frac{R-1}{R^2} \\ 1 &= \frac{2\gamma}{\gamma-1} \frac{R-1}{R^2} + \frac{1}{R^2}\end{aligned}$$

The third equation is a quadratic in  $R$  and can be easily solved to yield

$$R = \frac{\gamma+1}{\gamma-1}$$

where we have ignored the trivial solution  $R = 1$  which is valid in the case of no discontinuity in flow. For a monatomic gas with three degrees of freedom,  $\gamma = 5/3$  and hence  $R = 4$ . This shows that on crossing a strong shock the density of the fluid jumps by a factor of 4. Noting that  $P_2/\rho_2 = kT_2/(\mu m_p)$ , one can estimate the postshock temperature:

$$T_2 = \frac{\mu m_p}{k} v_1^2 \frac{R-1}{R^2}$$

for  $R = 4$ ,

$$T_2 = \frac{3}{16} \frac{\mu m_p}{k} v_1^2$$

which gives  $T_2 \sim 10^7$  K for  $v = 1000$  km/s. Clearly, gas shocked in a young supernova remnant, where the expansion speed of the blast wave is several thousand km/s, is hot enough to produce X-rays. X-ray spectroscopy of this hot gas is now the subject of detailed study by contemporary X-ray satellites Chandra and XMM-Newton. One of the important results emerging from such studies is the composition of the matter ejected in the supernova explosion.

As a supernova remnant (SNR) expands, it sweeps up more and more matter. The kinetic and thermal energy is shared with all the swept-up matter and hence the expansion slows down. When the swept-up matter dominates the dynamics and radiation losses from the material are small, the dynamics of the blast wave is governed by only two dimensional

parameters: the energy of the blast wave  $E_0$  and the external density  $\rho$ . In such a situation the expansion would have a self-similar solution, with the radius of the blast wave proportional to the only possible combination of these parameters that yields a dimension of length:

$$R \propto \left(\frac{E_0}{\rho}\right)^{1/5} t^{2/5}$$

This is known as the ‘‘Sedov-Taylor’’ expansion law, originally derived to explain the behaviour of the expanding blast waves of atmospheric nuclear detonations. The speed of the blast wave then falls as

$$v \propto t^{-3/5}$$

This power-law behaviour is applicable only at times  $t > t_0$  when the swept-up mass  $4\pi R^3 \rho / 3$  exceeds the originally ejected mass  $M_{\text{ej}}$  in the explosion. Until  $t = t_0$  the expansion is practically uniform, at a constant speed equal to the initial speed  $v_0$ . The value of  $t_0$  is then given by

$$t_0 = \left(\frac{3M_{\text{ej}}}{4\pi\rho}\right)^{1/3} \frac{1}{v_0}$$

which evaluates to about 250 y for  $M_{\text{ej}} = 2M_{\odot}$ ,  $v_0 = 1000$  km/s and a typical interstellar density of 1 atom per  $\text{cm}^3$ . We have then the scaling law

$$v = v_0 \left(\frac{t}{t_0}\right)^{-3/5}$$

and the evolution of the postshock temperature, for typical parameters adopted above,

$$T_2 \approx 10^7 \text{ K} \left(\frac{t}{250 \text{ y}}\right)^{-6/5}$$

As the temperature falls to  $T < 10^4$  K, line radiation becomes copious, causing a serious drain of blast wave energy. The blast wave stagnates and eventually disperses into the interstellar medium. The typical lifetime of a supernova remnant can be estimated from the above, as the time required for the postshock temperature to fall to  $\sim 10^4$  K. This works out to be a few times  $10^5$  y.

## Shock Acceleration of Cosmic Rays

Shock waves in astrophysical situations are thought to be the main contributors to the acceleration process of the very high energy electrons and nuclei that are observed as the so-called “Cosmic Rays”. The mechanism, called *Diffusive Shock Acceleration*, proceeds as follows.

Astrophysical shocks are mostly “collisionless”. The momentum exchange between the shocked matter and newly swept-up matter is normally accomplished by the local magnetic field. Magnetic fields are ubiquitous in astrophysics, at any point there are relatively large scale components along with irregularities of small scale. Small-scale, tangled fields may have a substantial strength and act as “magnetic mirrors” for even high energy particles. In postshock regions, magnetic fields can be amplified by turbulent motions.

Let us consider a non-relativistic shock, with speed  $v_{\text{sh}}$ , and relativistic particles, with energy  $E = pc$  ( $p$ =momentum), present in a given region. The speed of these particles are much higher than the shock speed, and they hardly notice the shock as a discontinuity. However, they do scatter from the magnetic irregularities on both sides of the shock.

Scattering with a magnetic irregularity is energy-conserving in the rest frame of the irregularity. Repeated scatterings of this kind with many such irregularities in a medium randomize the momentum of the particle, and establish an isotropic distribution in the rest frame of the medium. Around the shock, then, we have such isotropic distribution of the high-energy particles established, separately, both upstream and downstream of the shock. These two distributions then approach each other with a speed

$$v_{\text{rel}} = v_{\text{sh}} - \frac{v_{\text{sh}}}{R} = \frac{R-1}{R} v_{\text{sh}}$$

where  $R$  is the compression ratio.

In either distribution of the high energy particles, the average velocity of the whole distribution of particles is equal to the velocity of the local medium,

but the speed of random motion is very nearly  $c$ , as the particles are relativistic. Since this is much larger than the shock speed, these particles can easily cross the shock front from either side, namely, from upstream to downstream and vice versa. Since the two media approach each other at  $v_{\text{rel}}$ , upon crossing the shock the particles will find their momentum isotropized by scattering with magnetic irregularities approaching at speed  $v_{\text{rel}}$  with respect to the rest frame of the previous distribution. This process increases the average energy of the particles, with a particle gaining energy every time a shock crossing occurs.

If the particle has an energy  $E$  in the local rest frame before shock crossing, in the frame of the medium encountered after shock crossing its energy is seen to be, by Lorentz transformation,

$$E' = \gamma_{\text{rel}}(E + v_{\text{rel}}p_x)$$

where  $p_x$  is the  $x$ -component of the momentum and the  $x$ -axis is chosen to be along  $v_{\text{rel}}$ . We note that the Lorentz factor  $\gamma_{\text{rel}} \equiv (1 - v_{\text{rel}}^2/c^2)^{-1/2}$  is very close to 1 for the non-relativistic shocks under discussion. Writing  $p_x = p \cos \theta = E \cos \theta/c$ , we find that the gain in energy in each shock crossing is

$$\Delta E = E' - E = E \frac{v_{\text{rel}}}{c} \cos \theta$$

i.e.

$$\frac{\Delta E}{E} = \frac{v_{\text{rel}}}{c} \cos \theta = \frac{R-1}{R} \frac{v_{\text{sh}}}{c} \cos \theta$$

Noting that the flux of particles arriving at an angle  $\theta$  is proportional to  $c \cos \theta \sin \theta d\theta$ , we find a flux-weighted fractional energy gain per shock crossing:

$$\left\langle \frac{\Delta E}{E} \right\rangle = \frac{R-1}{R} \frac{v_{\text{sh}}}{c} \frac{\int_0^{\pi/2} \cos^2 \theta \sin \theta d\theta}{\int_0^{\pi/2} \cos \theta \sin \theta d\theta} = \frac{2}{3} \frac{R-1}{R} \frac{v_{\text{sh}}}{c}$$

Defining a ‘‘cycle’’ as two crossings of the shock, downstream to upstream and back to downstream, we find the fractional energy gain per cycle

$$\eta \equiv \left\langle \frac{\Delta E}{E} \right\rangle_{\text{cycle}} = \frac{4}{3} \frac{R-1}{R} \frac{v_{\text{sh}}}{c}$$

All particles which cross into the downstream region will, however, not be able to cross the shock back to upstream, since the downstream medium moves away from the shock front at a speed  $v_2 = v_{\text{sh}}/R$ . Since the average speed of the particles in the distribution is very close to  $c$ , the flux of particles crossing into the downstream region, according to standard kinetic theory, is  $nc/4$  where  $n$  is the number density of the particles. In the downstream medium, the particles are advected away from the shock front with a flux  $nv_2$ . Hence the probability of a particle escaping the acceleration zone in any cycle is given by

$$P_{\text{esc}} = \frac{nv_2}{nc/4} = \frac{4}{R} \frac{v_{\text{sh}}}{c}$$

Starting with  $N_0$  particles at energy  $E_0$ , after  $n$  cycles the energy of the particles will become

$$E_n = E_0(1 + \eta)^n$$

And the number of particles to survive  $n$  cycles would be

$$N_n = N_0(1 - P_{\text{esc}})^n$$

Some of these particles will go on for more cycles and gain more energy. The number  $N_n$  thus stands for the cumulative number of particles with energy  $E > E_n$ . Taking logarithm and dividing the above two expressions one can eliminate  $n$ , and the resulting distribution can be written as

$$N(> E) = E^{-x} \quad \text{where } x = -\frac{\ln(1 - P_{\text{esc}})}{\ln(1 + \eta)}$$

Since both  $P_{\text{esc}}$  and  $\eta$  are much smaller than unity, we could expand the logarithms and keep only the first order term, giving

$$x = \frac{P_{\text{esc}}}{\eta} = \frac{3}{R-1}$$

which depends only on the compression ratio of the shock.

The differential energy distribution of these particles can be written from the above as

$$N(E)dE = E^{-p}dE$$

where

$$p = 1 + x = \frac{R + 2}{R - 1}$$

for a non-relativistic strong shock  $R = 4$ , giving  $p = 2$ , close to the observed power-law energy spectrum of Cosmic Rays, as well as of non-thermal synchrotron-emitting ultrarelativistic electrons in a variety of astrophysical situations. Interstellar shocks generated by supernovae are thought to be the prime acceleration sites for Cosmic rays. The fact that supernova remnants can accelerate particles to very high energies and produce power-law energy distributions is evident from the strong non-thermal radio emission seen in them. Recently, the non-thermal component of the X-ray emission has also been discovered in supernova remnants, and indeed the ultrarelativistic particles responsible for this emission are seen to lie very close to the advancing shock.



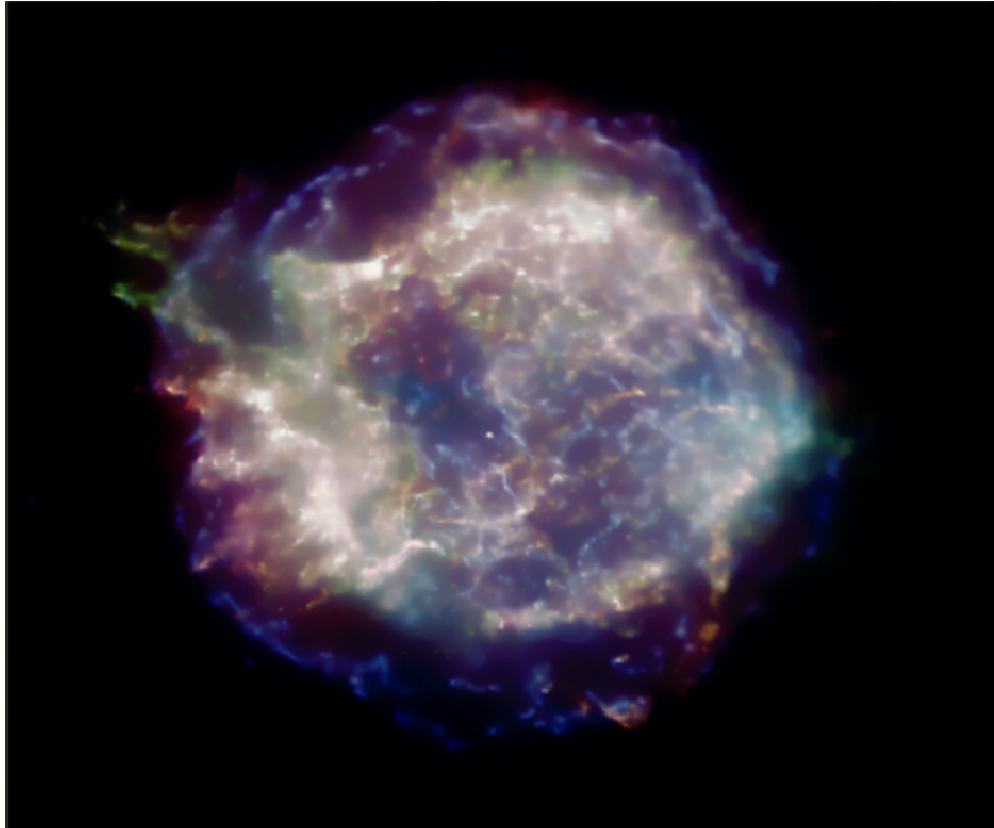


Figure 2: X-ray image of a 300 year-old supernova remnant Cas A taken with the Chandra observatory. The image is taken in three different X-ray energy bands and superposed in false colour. The outer red filaments are rich in Iron. (Credit: John Hughes et al, Rutgers University)



Figure 3: A large, 20,000 year-old supernova remnant Cygnus Loop imaged in optical light. The filaments produce strong emission in optical recombination lines such as  $H\alpha$  (red) and forbidden metal lines (green)

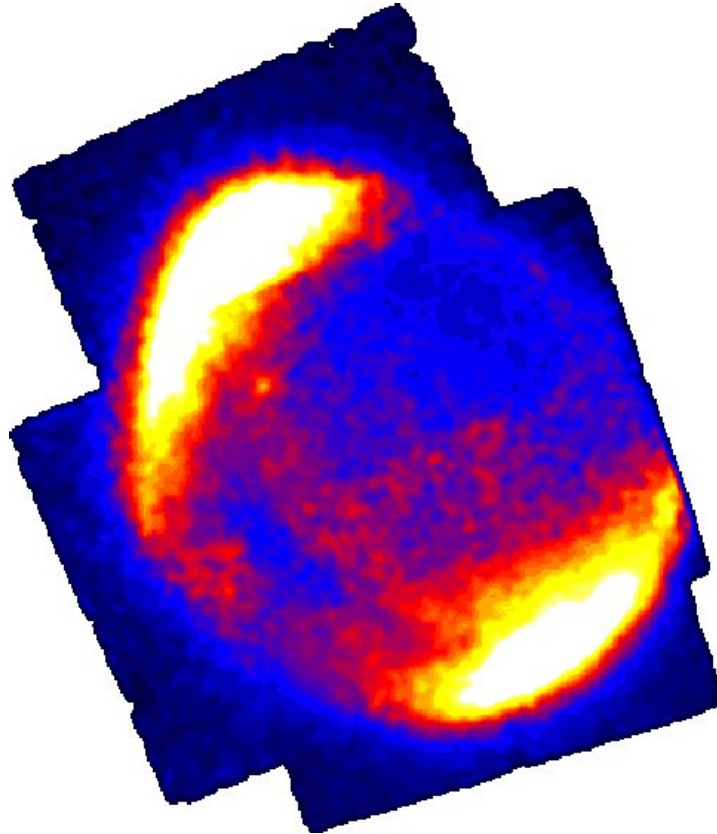


Figure 4: X-ray image of the remnant of a supernova seen to have exploded in the year 1006AD. The blue rims of the SNR in this false-colour image indicate non-thermal X-ray emission, locating the acceleration sites of the ultrarelativistic particles.