2-Numerical Methods for the Advection Equation

\[
\frac{\partial q}{\partial t} + a \frac{\partial q}{\partial x} = 0
\]
The Advection Equation: Theory

- 1st order partial differential equation (PDE) in (x,t):

\[
\frac{\partial q(x, t)}{\partial t} + a(x, t) \frac{\partial q(x, t)}{\partial x} = 0
\]

- Hyperbolic PDE: information propagates across the domain at finite speed \( \rightarrow \) method of characteristics

- Characteristic are the solutions of the equation

\[
\frac{dx}{dt} = a(x, t)
\]

- So that, along each characteristic, the solution satisfies

\[
\frac{dq}{dt} = \frac{\partial q}{\partial t} + \frac{dx}{dt} \frac{\partial q}{\partial x} = 0
\]
The Advection Equation: Theory

\[ \frac{dq}{dt} = \frac{\partial q}{\partial t} + \frac{dx}{dt} \frac{\partial q}{\partial x} = 0, \quad \text{with} \quad \frac{dx}{dt} = a \]

- The solution is constant along the characteristic curves. The solution at the point \((x,t)\) is found by tracing the characteristic back to some initial point \((x,0)\).

- This defines the physical domain of dependence.
The Advection Equation: Theory

If \( a \) is constant: characteristics are straight parallel lines and the solution to the PDE is a uniform translation of the initial profile:

\[
q(x, t) = \phi(x - at)
\]

where \( \phi(x) = q(x, 0) \) is the initial condition
Numerical Methods for the Linear Advection Equation

\[ \frac{\partial q}{\partial t} + a \frac{\partial q}{\partial x} = 0 \]

- 2 popular methods for performing discretization:
  - Finite Differences
  - Finite Volume

- For some problems, the resulting discretizations look identical, but they are distinct approaches.
- We begin using finite-difference as it will allow us to quickly learn some important ideas
A finite-difference method stores the solution at specific points in space and time.

Associated with each grid point is a function value,

$$q_i = q(x_i)$$

We replace the derivatives in our PDEs with differences between neighboring points.
Linear Advection Equation: Finite Volumes

- In a finite volume discretization, the unknown is the average value of the function:

\[ \langle q \rangle_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x) \, dx \]

where \( x_{i-1/2} \) is the position of the left edge zone \( i \)

- Solving out conservation laws involves computing fluxes through the boundaries of these control volumes.
Linear Advection Equation:

- We start with the linear advection equation

\[
\frac{\partial q(x, t)}{\partial t} + a \frac{\partial q(x, t)}{\partial x} = 0
\]

- With initial conditions (i.c.)

\[ q(x, 0) = q_0(x) \]

- And boundary conditions (b.c.)

\[
\begin{cases} 
q(0, t) = q_l(t) \\
q(L, t) = q_r(t) 
\end{cases}
\]

- Actually, only one b.c. is needed since this is a 1st order equation. Which boundary depends on the sign of \(a\).
Linear Advection Equation:

- We use a finite difference mesh:

- We discretize the function $q(x,t)$ by storing its value at each point in the finite-difference grid

\[ q_i^n = q(x_i, t^n) \]

- Subscript “$i$” $\to$ grid location
- Superscript “$n$” $\to$ time level
- In addition to discretizing in space, we introduce time discretization. Thus $\Delta t^n = t^{n+1} - t^n$
Linear Advection Equation:

- We need to approximate the derivatives in our PDE:

\[
\frac{\partial q(x, t)}{\partial t} + a \frac{\partial q(x, t)}{\partial x} = 0
\]

- In time, we use fwd derivative:

\[
\frac{\partial q(x, t)}{\partial t} \approx \frac{q_{i+1}^n - q_i^n}{\Delta t}
\]

since we want to use information from the previous time level.

- In space, we use centered derivative, since it is more accurate:

\[
\frac{\partial q(x, t)}{\partial x} \approx \frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x}
\]
Linear Advection Equation:

- Putting all together:
  \[ \frac{q_{i}^{n+1} - q_{i}^{n}}{\Delta t} + a \left( \frac{q_{i+1}^{n} - q_{i-1}^{n}}{2\Delta x} \right) = 0 \]

- and solving with respect to \( q_{i}^{n+1} \):
  \[ q_{i}^{n+1} = q_{i}^{n} - C \left( \frac{q_{i+1}^{n} - q_{i-1}^{n}}{2} \right) \]

  where \( C = a \frac{\Delta t}{\Delta x} \) is called the Courant number or the Courant-Friedrichs-Lewy (CFL) number.

- We call this method **FTCS** for forward in time, center in space.

- The value at the new time level depends only on quantities at the old time step \( \rightarrow \) explicit method.
Linear Advection Equation:

- At $t = 0$, we prescribe a square pulse:

- and prescribe periodic b.c.
Linear Advection Equation:

- After one period, the solution looks like:

- Oops!! Something isn’t right… WHY??
Linear Advection Equation: stability analysis

Let’s perform an analysis of FTCS by expressing the solution as a Fourier series. Since the equation is linear, we only need to examine the behavior of a single mode. Consider a trial solution of the form:

\[ q_i^n = A^n e^{Ii\theta}, \quad I = (-1)^{1/2}, \quad \theta = k\Delta x \]

This is a spatial Fourier expansion. Plugging in the difference formula:

\[ q_i^{n+1} = q_i^n - \frac{C}{2} (q_{i+1}^n - q_{i-1}^n) \rightarrow A^{n+1} = A^n - \frac{C}{2} A^n (e^{I\theta} - q^{-I\theta}) \]
Defining the amplification factor $\frac{A^{n+1}}{A^n}$ one obtains

$$\frac{A^{n+1}}{A^n} = 1 - \frac{C}{2} (e^{i\theta} - e^{-i\theta}) = 1 - IC \sin \theta$$

A method is well-behaved or stable if $\left|\frac{A^{n+1}}{A^n}\right| \leq 1$

But for FTCS one gets $\left|\frac{A^{n+1}}{A^n}\right| = 1 + C^2 \sin^2 \theta \geq 1$

Indepdently of the CFL number all Fourier modes increase in magnitude as time advances.

This method is unconditional unstable!!.
Linear Advection Equation:

- Let’s try a different approach. Consider the backward derivative:

\[
\frac{\partial q(x, t)}{\partial x} \approx \frac{q^n_i - q^n_{i-1}}{\Delta x}
\]

- Let’s apply the von Neumann stability analysis on the resulting discretized equation:

\[
\frac{q^{n+1}_i - q^n_i}{\Delta t} + a \left( \frac{q^n_i - q^n_{i-1}}{\Delta x} \right) = 0 \quad \text{with} \quad q^n_i = A^n e^{i\theta}
\]

- Solving for the amplification factor gives

\[
\frac{A^{n+1}}{A^n} = 1 - C + C \cos \theta - I \sin \theta
\]
Linear Advection Equation:

- Taking the norm,
  \[ \left| \frac{A^{n+1}}{A^n} \right| = 1 - 2C(1 - C)(1 - \cos \theta) \]

- Recall that for stability one needs
  \[ \left| \frac{A^{n+1}}{A^n} \right| \leq 1 \]

- But \( 1 - \cos \theta \geq 0 \) so the stability condition is met when
  \[ 2C(1 - C) \geq 0 \]

- Recalling the definition \( C = a \frac{\Delta t}{\Delta x} \), one has for \( a > 0 \)

\[ 0 \leq a \frac{\Delta t}{\Delta x} \leq 1 \]

Condition for stability
Linear Advection Equation:

- Since the advection speed $a$ is a parameter of the equation, $\Delta x$ is fixed from the grid, this is a constraint on the time step:

  $$\Delta t \leq \frac{\Delta x}{\alpha}$$

- $\Delta t$ cannot be arbitrarily large.

- In the case of nonlinear equations, the speed can vary in the domain and the maximum of $a$ should be considered.
Linear Advection Equation:

- Repeating the argument for the fwd derivative,

\[
\frac{q_{i}^{n+1} - q_{i}^{n}}{\Delta t} + a \left( \frac{q_{i+1}^{n} - q_{i}^{n}}{\Delta x} \right) = 0 \quad \text{with} \quad q_{i}^{n} = A^{n} e^{i\theta}
\]

- Gives

\[
\left| \frac{A^{n+1}}{A^{n}} \right| = 1 + 2C(1 - C)(1 - \cos \theta)
\]

- If \( a > 0 \), the method will always be unstable.
- However, if \( a \) is negative, then this method is stable and the previous is unstable.
Linear Advection Equation:
What Have We Learned?

- The stable method is the one with the difference that makes use of the grid point where information is coming from.

- This type of discretization goes under the name “upwind”:

  - For $a > 0$ we want
    \[ q_i^{n+1} = q_i^n - \frac{a \Delta t}{\Delta x} \left( q_i^n - q_{i-1}^n \right) \]
  
  - The $a < 0$ we want
    \[ q_i^{n+1} = q_i^n - \frac{a \Delta t}{\Delta x} \left( q_{i+1}^n - q_{i}^n \right) \]

- This is the **first-order Godunov Method**.
Linear Advection Equation:

- After one period, the solution looks like:

- Much better now…
- But we still see some smearing…
A discretized P.D.E gives the exact solution to an equivalent equation with a diffusion term:

\[ \frac{\partial q}{\partial t} + a \frac{\partial q}{\partial x} = 0, \quad a > 0 \]

Consider \[ q^{n+1}_i - q^n_i \]

discretize w/ upwind \[ \frac{q^n_{i+1} - q^n_i}{\Delta t} + a \frac{q^n_i - q^n_{i-1}}{\Delta x} = 0 \]

do Taylor expansion on \( q^{n+1}_i \) and \( q^n_{i-1} \)

The solution to the discretized equation is also the solution of

\[ \frac{\partial q}{\partial t} + a \frac{\partial q}{\partial x} = \frac{a \Delta x}{2} \left( 1 - a \frac{\Delta t}{\Delta x} \right) \frac{\partial^2 q}{\partial x^2} + H.O.T. \]
Linear Advection Equation:

Time: 0.00, First order
Linear Advection Equation: Conservative Form

- Godunov method can be cast in conservative form, i.e.

\[ q_{i}^{n+1} = q_{i}^{n} - \frac{\Delta t}{\Delta x} \left( F_{i+1/2}^{n} - F_{i-1/2}^{n} \right) \]

by defining the “flux” function

\[ F_{i+1/2}^{n} = \frac{a}{2} (q_{i+1}^{n} + q_{i}^{n}) - \frac{|a|}{2} (q_{i+1}^{n} - q_{i}^{n}) \]

- In fact for \( a > 0 \), one has

\[ q_{i}^{n+1} = q_{i}^{n} - \frac{a \Delta t}{\Delta x} (q_{i}^{n} - q_{i-1}^{n}) \]

- and for \( a < 0 \)

\[ q_{i}^{n+1} = q_{i}^{n} - \frac{a \Delta t}{\Delta x} (q_{i+1}^{n} - q_{i}^{n}) \]
C Implementation

- Look → advection.c