Gravitational collapse

Jeans' instability

Jeans' instability of a self-gravitating, thermally supported interstellar cloud is thought to be responsible for the collapse of parts of the cloud larger than a scale size that goes unstable, eventually fragmenting and forming stars.

The dynamics of gas in the cloud is controlled by two fluid equations:

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v}\right] = -\vec{\nabla}P - \rho\nabla\Phi \quad \text{(Force Balance)} \tag{1}$$

and

$$\frac{\partial\rho}{\partial t} + \vec{v}.(\vec{\nabla}\rho) = -\rho\vec{\nabla}.\vec{v} \quad \text{(Continuity)} \tag{2}$$

where $\vec{v}$ is the gas velocity field, ρ is the mass density, P is the gas pressure and Φ is the local gravitational potential. Using the sound speed c_s one may write

$$\vec{\nabla}P = c_s^2\vec{\nabla}\rho$$

We now split the velocity and density into two parts, spatially uniform (subscript 0) and spatially varying (subscript 1):

$$\vec{v} = \vec{v}_0 + \vec{v}_1$$

$$\rho = \rho_0 + \rho_1$$

We also assume that the uniform components are stationary, i.e.

$$\frac{\partial\vec{v}_0}{\partial t} = \frac{\partial\rho_0}{\partial t} = 0$$

We can then write the linear equations in spatially varying quantities as

$$\frac{\partial\vec{v}_1}{\partial t} + \vec{v}_0.\vec{\nabla}\vec{v}_1 = -\vec{\nabla}\Phi_1 - c_s^2\vec{\nabla}\left(\frac{\rho_1}{\rho_0}\right) \tag{3}$$

$$\frac{\partial\rho_1}{\partial t} + \vec{v}_0.\vec{\nabla}\rho_1 = -\rho_0\vec{\nabla}.\vec{v}_1 \tag{4}$$

where we have kept as gravitational potential only that produced by the spatially varying part of the density distribution, since the gravitational force produced by a spatially uniform, infinite density distribution vanishes.

Transforming to a frame in which $\vec{v}_0 = 0$, we have

$$\frac{\partial \vec{v}_1}{\partial t} = -\vec{\nabla}\Phi_1 - \frac{c_s^2}{\rho_0}\vec{\nabla}\rho_1 \tag{5}$$

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \vec{\nabla}.\vec{v}_1 \tag{6}$$

Taking spatial derivative of eq. 5 and temporal derivative of eq. 6 we find

$$\vec{\nabla}\left(\frac{\partial \vec{v}_1}{\partial t}\right) = -\nabla^2\Phi_1 - \frac{c_s^2}{\rho_0}\nabla^2\rho_1 \tag{7}$$

and

$$-\frac{1}{\rho_0}\frac{\partial^2 \rho_1}{\partial t^2} = \frac{\partial}{\partial t}(\vec{\nabla}.\vec{v}_1) \tag{8}$$

Recognising that the LHS of eq. 7 and the RHS of eq. 8 are the same, we can write

$$\frac{\partial^2 \rho_1}{\partial t^2} = c_s^2\nabla^2\rho_1 + (4\pi G\rho_0)\rho_1 \tag{9}$$

where we have used Poisson's equation to write

$$\nabla^2\Phi_1 = 4\pi G\rho_1. \tag{10}$$

If we now write a Fourier component of the spatially varying density as

$$\rho_1 = A\exp\{i(\vec{k}.\vec{r} + \omega t)\} \tag{11}$$

We find

$$\omega^2 = c_s^2 k^2 - 4\pi G\rho_0 \equiv c_s^2(k^2 - k_J^2) \tag{12}$$

where

$$k_J^2 = \frac{4\pi G\rho_0}{c_s^2} = \frac{4\pi G\rho_0 m_p \mu}{k_B T} \tag{13}$$

Here m_p is the proton mass, μ is the mean molecular weight, k_B is the Boltzmann constant and T is the temperature of the gas. For $k < k_J$ the value of ω^2 is negative and hence the disturbance grows exponentially. k_J defines a minimum mass scale:

$$M_J = \left(\frac{2\pi}{k_J}\right)^3 \rho_0 = \left[\frac{\pi k_B T}{G\mu m_p}\right]^{3/2} \frac{1}{\rho_0^{1/2}} \tag{14}$$

This is called the Jeans' Mass. Perturbations of size larger than this in a gas cloud would grow, become self-gravitating and collapse.

Implosion and Explosion

Catastrophic gravitational collapse occurs in the cores of massive stars at the end of their evolution. The nuclear burning proceeds until Fe is synthesised at the core, which cannot burn further as the peak of the binding energy has been reached. This is a degenerate white-dwarf-like configuration whose mass continues to grow as ashes are added from the nuclear burning shell around it. Eventually the Chandrasekhar limit is exceeded and collapse occurs. As collapse proceeds, the Fe nuclei are first photodissociated, and then electrons are captured by protons to produce neutron-rich matter. The loss of electrons means the loss of degeneracy pressure, the main support against gravity at this stage. As a result the collapse accelerates and in hydrodynamic time scale of a few seconds produces a very compact configuration, made primarily of neutrons. As the neutrons are squeezed together at densities higher than nuclear density ($\rho_{\rm nuc} = 2.8 \times 10^{14}$ g/cm^3) the mutual repulsion between neutrons will halt the collapse, and the core will bounce back to an equilibrium configuration, which is now a neutron star. The bounce will send a shock wave through the surrounding envelope, making the envelope explode in a type II supernova. The gravitational binding energy released in the collapse of the core is $\sim 10^{53}$ erg, about 1% of which goes into the kinetic energy of the expanding envelope. Neutrinos carry the rest of the energy away.

The expanding ejecta, with the kinetic energy of 10^{51} erg, is heated by the shock, as well as the decay of radioactive elements synthesised and ejected. It therefore shines brightly. The total energy emitted in radiation amounts to $\sim 10^{49}$ erg. Very massive stars would be able to grow cores too massive to be supported as neutron stars, the reason for this being the additional radiation pressure support in the pre-collapse core. Such cores will collapse to black holes. A spinning black hole produced this way will swallow the inner parts of the envelope through a dense accretion disk, and eject a small fraction of matter in a jet along the spin axis. With large amount of energy imparted to this small amount of matter, the material in the jet would move at relativistic speeds. Viewed along the jet, this will be a copious source of high energy radiation. This model is believed to explain the Gamma-Ray Burst sources. The rest of the envelope in this case will get eventually expelled in a supernova-like explosion (often referred to as a "hypernova").

de Laval nozzle: Jets

Let us consider a one-dimensional flow, and assume that gravity can be ignored. This is described by

$$\frac{dv}{dt} = v\frac{dv}{dx} = -\frac{1}{\rho}\frac{dP}{dx} = -\frac{c_s^2}{\rho}\frac{d\rho}{dx}$$

which gives

$$d\ln\rho = -\frac{v^2}{c_s^2}d\ln v$$

If A is the cross sectional area of the flow then $\rho v A$ = constant. Thus

$$d\ln v = -\frac{d\ln A}{1-(v^2/c_s^2)}$$

which shows that if $v^2 < c_s^2$ then decreasing cross sectional area leads to an increase in Mach number, while for a supersonic flow an *increasing* cross sectional area increases the Mach number. So if the flow has a throat, with converging subsonic approach and diverging supersonic exit, highly supersonic jet flows can be produced. This idea has been applied to explain the formation of powerful jets seen in active galaxies - light jet matter forcing its way through interstellar medium and converging, but as the interstellar density drops away from the galactic centre, the flow cross section would expand and supersonic jets may be produced.

Today we know that magnetic fields play a more important role in the production of jets. This involves magnetic fields anchored to an accretion disk, which we discuss below.

Accretion

There are many situations that lead to the accretion of matter from the immediate surroundings onto a compact object. A compact star may capture some of the stellar wind from a binary companion, or exert sufficient tidal force on the companion to strip matter from it and produce a flow dircted towards itself (Roche Lobe Overflow). If the compact object is a stellar mass black hole or a neutron star, they show up as X-ray binaries. Accreting White Dwarfs undergo nova explosions, and are called "Cataclysmic Variables". Supermassive black holes at centres of galaxies can be fed matter from the surrounding interstellar medium or tidally stripped stars. Large accretion rates on such objects lead to the generation of high luminosity at the galactic nucleus, as well as production of powerful jets. These are known as Active Galactic Nuclei (AGNs).

Gravitational capture of matter by a body from a passing flow in which it is immeresed is treated in the classical Bondi-Hoyle picture of accretion. gravitational acceleration by the immersed body bends the trajectory of the flowing matter, causing convergence behind the body (this effect is called gravitational focussing). The trajectories cross behind the object and matter collides at the crossings. In the collision one may assume that the velocity components opposing each other are fully dissipated (and corresponding energy radiated away), while the parallel component remains. Up to a certain distance from the body the remaining parallel component would be less than the escape velocity at that point, and matter will fall in. Matter on trajectories colliding beyond that distance will escape. Tracing these trajectories back to their initial impact parameter one may define a gravitational capture cross section for the body πr_a^2, where the 'accretion radius' r_a is given by

$$r_a = \frac{2GM}{(v_w^2 + c_s^2)}$$

where M is the mass of the accreting body, v_w is the speed of the wind and c_s is the sound velocity in the wind. For wind from hot massive stars usually $v_w \gg c_s$, and the sound speed in the above expression can be ignored.

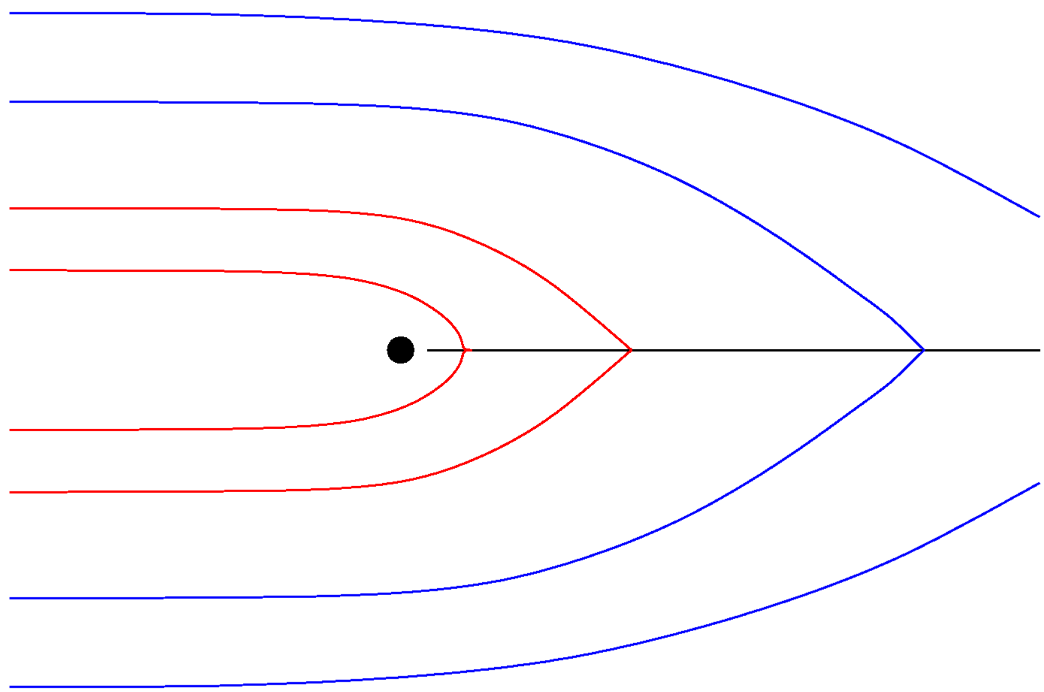

Figure 1: The geometry of Bondi-Hoyle accretion. Wind flows from left to right in the figure, past the accretor (black dot). Trajectories of wind matter are gravitationally focussed and made to collide on a line behind the accretor (horizontal black line). The velocity component perpendicular to this line is dissipated in the collision, while that parallel to this line remains. Matter on outer trajectories (blue) retains sufficient velocity to escape the gravity of the accretor while that on inner ones (red) would be captured.

If the flow past the body is not symmetric, then there is a net angular momentum in the captured matter. This is true also in case of the matter accreted in a Roche Lobe Overflow. The angular momentum will cause the matter to form a ring around the accretor. The ring will intersect the accretion stream and dissipation will ensue. Eventually through viscous dissipation matter will proceed to smaller and smaller orbits, angular momentum being transported outwards in the process. This forms an *accretion disk* around the accretor, which is encountered in a wide variety of accretion situations. At any radius R of the disk the matter rotates around the central mass at the local Keplerian speed $v_\phi = \sqrt{GM/R}$, i.e. the angular speed $\Omega = \sqrt{GM/R^3}$. As matter in inner orbits rotate faster than that in outer orbits, viscosity can make angular momentum flow outwards in the disk, and sustain an inward flow. If $v_r(R)$ is the radial inflow velocity at radius R and $\Sigma(R)$ is the surface mass density at that radius then by continuity of mass $2\pi R\Sigma(R)v_r(R) = \dot{M}$, the mass accretion rate. In a steady state the above product is constant at all radii. Normally this v_r is much smaller than the Keplerian speed v_ϕ at the same radius, and therefore the kinetic energy of matter is dominated by the Keplerian motion. It follows therefore that of the Gravitational potential energy released in the process of matter coming to radius R from far away, nearly half the energy remains in kinetic energy and the rest must have been radiated away.

If ν is the coefficient of kinematic viscosity, then the viscous force per unit length around the circumference at any R is $\nu\Sigma(Rd\Omega/dR)$. So the viscous torque around the whole circumference is $\tau(R) = R(2\pi R)\nu\Sigma(Rd\Omega/dR)$.

Now consider a ring of material between R and $R + dR$. In unit time, material of amount $\dot{M}$ enters $R + dR$ with specific angular momentum $(R + dR)^2\Omega(R + dR)$ and leaves R with specific angular momentum $R^2\Omega(R)$. This loss of angular momentum takes place because of the action of net viscous torque $(d\tau/dR)dR$. Thus

$$\dot{M}\frac{d(R^2\Omega)}{dR} = -\frac{d}{dR}\left[\nu\Sigma 2\pi R^3\frac{d\Omega}{dR}\right]$$

Using Keplerian Ω and integrating, one finds

$$\nu\Sigma = \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]$$

where the boundary condition used is that the shear τ vanishes at an inner radius R_*. This inner radius could be the last stable orbit around a black hole, or the stellar surface in case of a white dwarf or a weakly magnetized neutron star, or approximately the Alfvén radius (distance at which the ram pressure of accreting matter equals the magnetic pressure) around a strongly magnetized accretor.

The viscous dissipation rate per unit area can then be computed:

$$D(R) = \nu\Sigma\left(R\frac{d\Omega}{dR}\right)^2 = \frac{3GM\dot{M}}{4\pi R^3}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]$$

This must be radiated away. The disk is nearly optically thick and hence the emitted radiation can be approximated to be a blackbody at the local temperature $T(R)$. Accounting for the two surfaces of the disk, $D(R) = 2\sigma T^4$. Therefore

$$T(R) = \left\{\frac{3GM\dot{M}}{8\pi R^3\sigma}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]\right\}^{1/4}$$

Magnetic fields anchored to the accretion disk can play a very important and interesting role. In figure 2 consider a magnetic field line anchored to the disk at P. The field line rotates with the keplerian angular speed at P. The disk being hot, matter will evaporate from the surface and move preferentially along magnetic field lines. One such blob, at R, will now be rotating faster than the local Keplerian speed and feel a net centrifugal acceleration outward. Such an effect can effectvely cause matter to leave the disk. Beyond the Alfvén distance, the field lines will twist and wrap around the rotation axis, resulting in a highly collimated matter outflow along the polar axes. This is now considered the most important mechanism for jet formation in accreting systems.

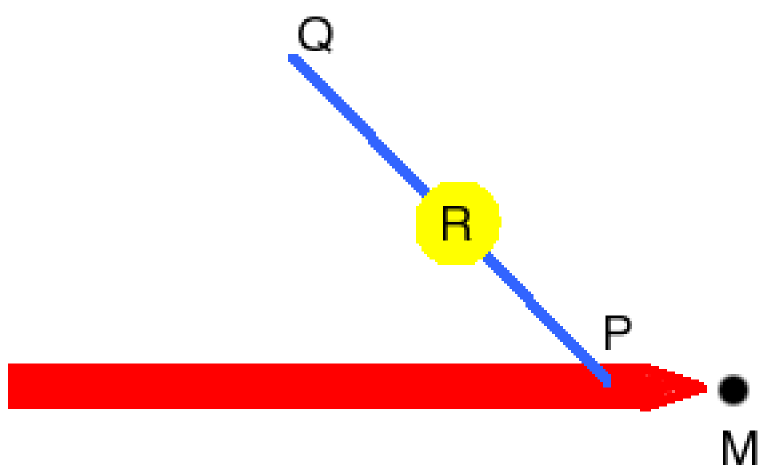

Figure 2: A gas blob R moving on a magnetic field line PQ anchored to an accretion disk (red) around a mass M can be centrifugally ejected from the disk and form part of a jet