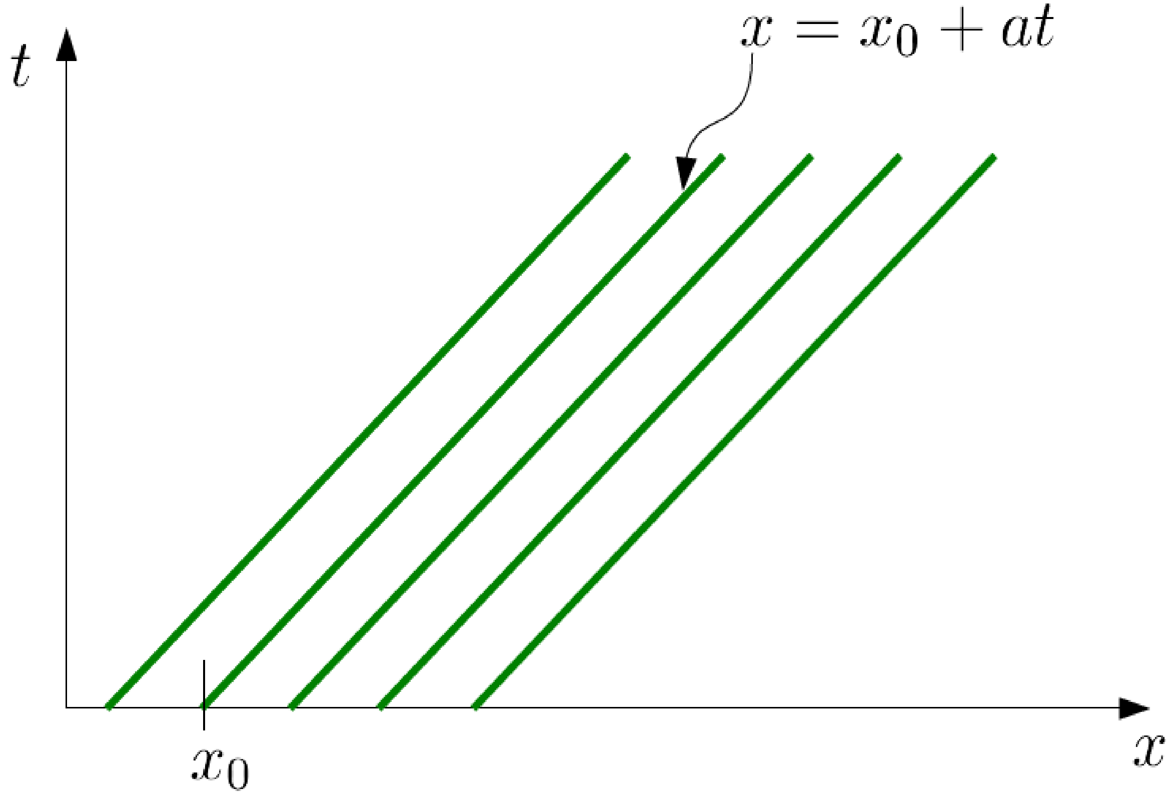
3-Linear System of Advection Equations

System of Equations: theory

- Recall the linear scalar advection equation: $\frac{\partial q}{\partial t} + a\frac{\partial q}{\partial x} = 0$
- The solutions are constant along lines $\frac{dx}{dt} = a$ (characteristic curves)



System of Equations: theory

- We turn our attention to the system of equations

$$\frac{\partial \mathbf{q}}{\partial t} + A \cdot \frac{\partial \mathbf{q}}{\partial x} = 0$$

- Where $\mathbf{q} = \{q_1, q_2, ...q_m\}$ is the vector of unknowns. A is a $m \times m$ constant matrix.
- For example, for *m=3* one has

$$\frac{\partial q_1}{\partial t} + A_{11}\frac{\partial q_1}{\partial x} + A_{12}\frac{\partial q_2}{\partial x} + A_{13}\frac{\partial q_3}{\partial x} = 0$$

$$\frac{\partial q_2}{\partial t} + A_{21}\frac{\partial q_1}{\partial x} + A_{22}\frac{\partial q_2}{\partial x} + A_{23}\frac{\partial q_3}{\partial x} = 0$$

$$\frac{\partial q_3}{\partial t} + A_{31}\frac{\partial q_1}{\partial x} + A_{32}\frac{\partial q_2}{\partial x} + A_{33}\frac{\partial q_3}{\partial x} = 0$$

System of Equations: theory

- The system of PDEs is hyperbolic if A is diagonalizable with real eigenvalues, $\lambda^1 \leq \lambda^2 \leq ... \leq \lambda^m$ and a complete set of linearly independent eigenvectors $\mathbf{r}^k$ such that

$$A \cdot \mathbf{r}^k = \lambda^k \mathbf{r}^k \quad \text{for} \quad k = 1, 2, ..., m$$

- For convenience we define the following matrices:

$$R = \left(\mathbf{r^1} | \mathbf{r^2} | ... | \mathbf{r^m}\right), \quad L = R^{-1} = \begin{pmatrix} \mathbf{l}^1 \\ \hline \mathbf{l}^2 \\ \hline \vdots \\ \hline \mathbf{l}^m \end{pmatrix}$$

- So that the *columns* of *R* contains the "right" eigenvectors and the *rows* of *L* contains the "left" eigenvectors.

System of Equations: theory

- With these definitions one can verify that the following matrix multiplications hold:

$$A \cdot R = R \cdot \Lambda \,, \quad L \cdot A = \Lambda \cdot L \,, \quad L \cdot R = R \cdot L = 1$$

- Here Λ is a diagonal matrix containing the eigenvalues:

$$\Lambda = L \cdot A \cdot R = \begin{pmatrix} \lambda^1 & & & \\ & \lambda^2 & & \\ & & \ddots & \\ & & & \lambda^m \end{pmatrix}$$

System of Equations: theory

- The linear system of equations can be reduced to a set of decoupled scalar linear advection eqations.

- Multiply the original system of PDE's by *L* on the left:

$$L \cdot (\mathbf{q}_t + A \cdot \mathbf{q}_x) = L \cdot \mathbf{q}_t + L \cdot A \cdot R \cdot L \cdot \mathbf{q}_x = 0$$

- Define the *characteristic variables* $\mathbf{w} \equiv L \cdot \mathbf{q}$ so that

$$\boxed{\mathbf{w}_t + \Lambda \cdot \mathbf{w}_x = 0}$$

- Since Λ is diagonal, these equations do not couple anymore.

System of Equations: theory

- In this form, the system decouples into *m* independent advection equations for the characteristic variables:

$$\mathbf{w}_t + \Lambda \cdot \mathbf{w}_x = 0 \;\Rightarrow\; w_t^k + \lambda^k w_x^k = 0$$

with $w^k = \mathbf{l}^k \cdot \mathbf{q}$ being the *k*-th (k=1,2,…,m) characteristic variable.

- When *m=3* one has, for instance,

$$\frac{\partial w^1}{\partial t} + \lambda^1 \frac{\partial w^1}{\partial x} = 0$$

$$\frac{\partial w^2}{\partial t} + \lambda^2 \frac{\partial w^2}{\partial x} = 0$$

$$\frac{\partial w^3}{\partial t} + \lambda^3 \frac{\partial w^3}{\partial x} = 0$$

System of Equations: theory

- The *m* advection equations can be solved independently by applying the standard solution techniques developed for the scalar equation. Thus for the k-th characteristic one finds:

$$w^k(x,t) = w^k(x - \lambda^k t, 0)$$

  i.e., the initial profile of w^k "shifts" with uniform velocity λ^k

- Given the initial profile $w^k(x - \lambda^k t, 0) = \mathbf{l}^k \cdot \mathbf{q}(x - \lambda^k t, 0)$

  this is the *exact analytical solution* for the *k*-th characteristic.

- The characteristics are thus constant along the characteristic curves $dx/dt = \lambda^k$

System of Equations: theory

- Once the solutions in characteristic form are known, we can solve the original system via the inverse transformation

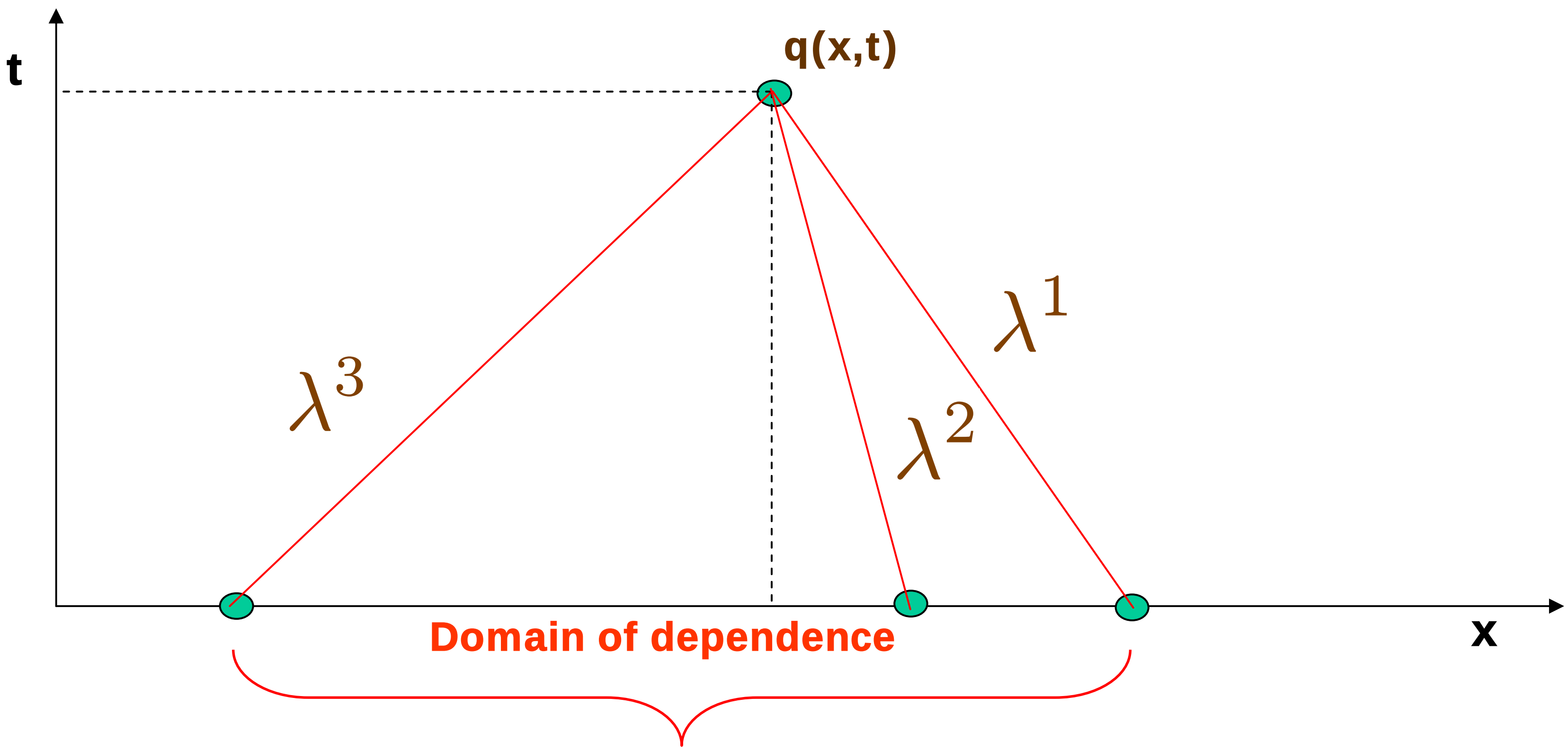
$$\mathbf{q}(x,t) = R \cdot \mathbf{w}(x,t) = \sum_{k=1}^{k=m} w^k(x,t)\mathbf{r}^k = \sum_{k=1}^{k=m} w^k(x-\lambda^k t, 0)\mathbf{r}^k$$

- The characteristic variables are thus the coefficients of the right eigenvector expansion of *q*.
- The solution to the linear system reduces to a linear combination of *m* waves traveling with velocities λ^k .
- Expressing everything in terms of the original variables *q*,

$$\boxed{\mathbf{q}(x,t) = \sum_{k=1}^{k=m} \mathbf{l}^k \cdot \mathbf{q}(x-\lambda^k t, 0)\mathbf{r}^k}$$

System of Equations: theory

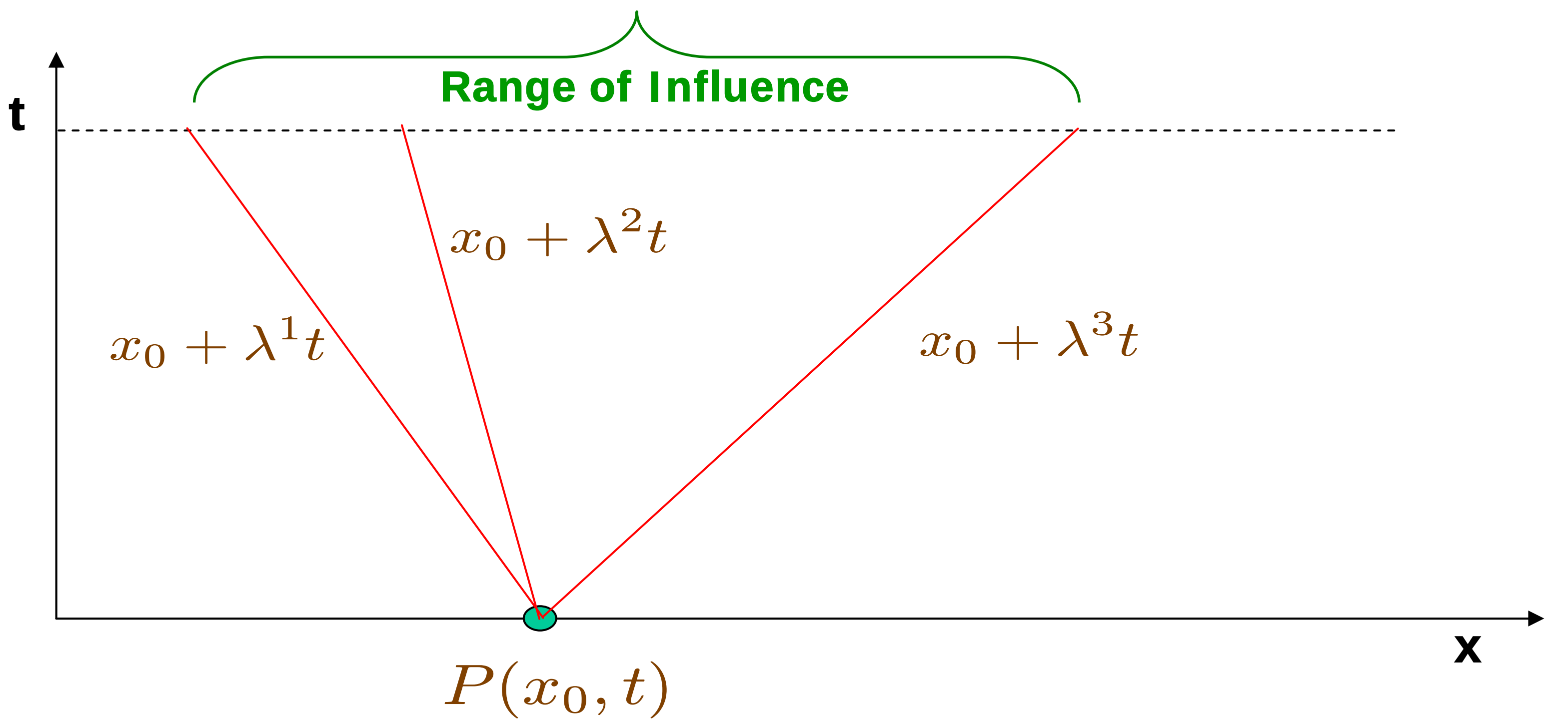
- As for the scalar equation, we can define the *domain of dependence* by tracing back *ALL* characteristic lines:



- Notice that characteristics are straight lines only for a linear system. In general, for a nonlinear systems, they are curves.

System of Equations: theory

- The concept of *domain of dependence* can be reversed by looking at the *range of influence*: the range of points influenced by the information at some point in the past *P(x,0)*



System of Equations: Numerics

- The numerical solution can now be easily found by applying the same arguments used for scalar advection case.

- We suppose the solution is known at time level n ($\rightarrow q^n$) and we wish to compute the solution at the next time step $n+1$ ($\rightarrow q^{n+1}$?).

- Our numerical scheme can be derived by working in the characteristic space, where we have developed a stable numerical method.

- Thus, we need the eigenvalue and eigenvector decomposition of the original matrix A.

System of Equations: Numerics

1) Start from the characteristic variables: $w_i^{k,n} = \mathbf{l}^k \cdot \mathbf{q}_i^n$

2) Solve indipendently each *k*:

$$w_i^{k,n+1} = w_i^{k,n} - \frac{\Delta t}{\Delta x}\left(H_{i+1/2}^{k,n} - H_{i-1/2}^{k,n}\right)$$

where

$$H_{i+1/2}^{k} = \frac{\lambda^k}{2}\left(w_i^{k,n} + w_i^{k,n}\right) - \frac{|\lambda^k|}{2}\left(w_{i+1}^{k,n} - w_i^{k,n}\right)$$

is the flux function in the characteristic fields, exactly as for the scalar advection case.

3) Transform back to the *q*-space: $\mathbf{q}_i^{n+1} = \sum_k w_i^{k,n+1}\mathbf{r}^k$

System of Equations: Numerics

- Doing the math, one ends up with the *conservative form*

$$\mathbf{q}_i^{n+1} = \sum_k w_i^{k,n+1} \mathbf{r}^k = \mathbf{q}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+1/2}^n - \mathbf{F}_{i-1/2}^n\right)$$

- With the flux function:

$$\mathbf{F}_{i+1/2}^n = A \cdot \frac{\mathbf{q}_i^n + \mathbf{q}_{i+1}^n}{2} - \frac{1}{2}\sum_k |\lambda^k|\mathbf{l}^n \cdot (\mathbf{q}_{i+1} - \mathbf{q}_i)\,\mathbf{r}^k$$

i.e., the *Godunov flux* for a linear system of advection equations.

- Proof: *left as exercise*!

Conservative & Integral Formulations

- The *conservative form* of the equations provides the link between the *differential* form of the equation,

$$\frac{\partial \mathbf{q}}{\partial t} + A \cdot \frac{\partial \mathbf{q}}{\partial x} = 0$$

and the *integral* form, obtained by integrating the equations over a time interval $\Delta t = t^{n+1} - t^n$ and cell size $\Delta x = x_{i+1/2} - x_{i-1/2}$

$$\int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \left(\frac{\partial \mathbf{q}}{\partial t} + A \cdot \frac{\partial \mathbf{q}}{\partial x} \right) dt dx = 0$$

Conservative & Integral Formulations

- Performing the spatial integration yields

$$\int_{t^n}^{t^{n+1}} \left[\Delta x \frac{d}{dt} \langle \mathbf{q}_i \rangle + A \cdot \left(\mathbf{q}_{i+1/2} - \mathbf{q}_{i-1/2} \right) \right] dt = 0$$

- With $\langle \mathbf{q} \rangle_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{q}(x,t)\, dx$ being a *spatial* average.

- A second integration in time gives

$$\Delta x \left(\langle \mathbf{q} \rangle_i^{n+1} - \langle \mathbf{q} \rangle_i^{n} \right) + \Delta t\, A \cdot \left(\tilde{\mathbf{q}}_{i+1/2}^{n} - \tilde{\mathbf{q}}_{i-1/2}^{n} \right) = 0$$

- With $\tilde{\mathbf{q}}_{i\pm1/2}^{n} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathbf{q}(x_{i\pm1/2}, t)\, dt$ being a *temporal* average

Conservative & Integral Formulations

- Rearranging terms yields

$$\langle \mathbf{q} \rangle_i^{n+1} = \langle \mathbf{q} \rangle_i^n - \frac{\Delta t}{\Delta x}\left(A \cdot \tilde{\mathbf{q}}_{i+1/2}^n - A \cdot \tilde{\mathbf{q}}_{i-1/2}^n \right)$$

Integral form

with *spatial* and *temporal* averages given by

$$\langle \mathbf{q} \rangle_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{q}(x,t)\, dx \ , \quad \tilde{\mathbf{q}}_{i\pm1/2}^n = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathbf{q}(x_{i\pm1/2}, t)\, dt$$

- We have derived an *EXACT* evolutionary equation for the spatial averages of *q*.
- This relation provides an *integral* representation of the original differential equation.
- The integral form does not make use of partial derivatives!

Conservative & Integral Formulations

- Comparing $\langle \mathbf{q} \rangle_i^{n+1} = \langle \mathbf{q} \rangle_i^n - \frac{\Delta t}{\Delta x}\left(A \cdot \tilde{\mathbf{q}}_{i+1/2}^n - A \cdot \tilde{\mathbf{q}}_{i-1/2}^n\right)$

  with $$\mathbf{q}_i^{n+1} = \mathbf{q}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+1/2}^n - \mathbf{F}_{i-1/2}^n\right)$$

  one notices that our 1st order discretization of the original differential equation looks very similar to the integral form, provided that:

  1- q_i^n and q_i^{n+1} are re-interpreted as *integral averages*.

  2- $\mathbf{F}_{i\pm1/2}^n$ are re-interpreted as *time averages* of point values located on the interfaces (i+1/2) and (i-1/2).

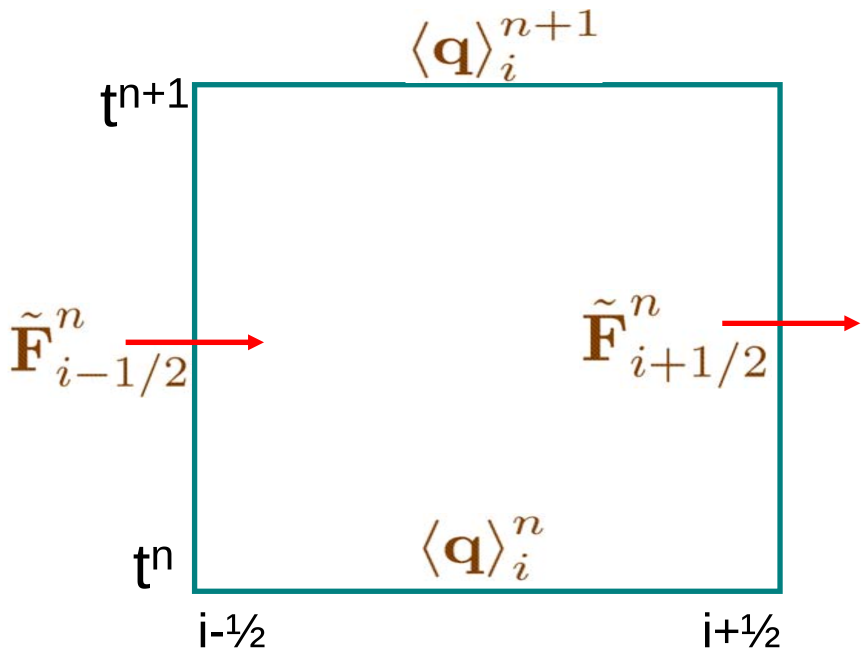
- This is the *FINITE VOLUME FORMULATION*.

Finite Volume Formulation

- Writing in a more general form,

$$\langle \mathbf{q} \rangle_i^{n+1} = \langle \mathbf{q} \rangle_i^{n} - \frac{\Delta t}{\Delta x}\left(\tilde{\mathbf{F}}^n_{i+\frac{1}{2}} - \tilde{\mathbf{F}}^n_{i-\frac{1}{2}}\right), \quad \begin{cases} \langle \mathbf{q} \rangle_i^{n} = \dfrac{1}{\Delta x}\displaystyle\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{q}(x,t^n)dx \\ \tilde{\mathbf{F}}^n_{i+\frac{1}{2}} = \dfrac{1}{\Delta t}\displaystyle\int_{t^n}^{t^{n+1}} \mathbf{F}(\mathbf{u}(x_{i+\frac{1}{2}},t))dt \end{cases}$$

- The Finite Volume Formulation is appropriate for the treatment of discontinuities. It relates the rate of change of some physical quantity to its fluxes through the region boundary.
- Discontinuities are confined to the edges of the cell.



Finite Volume Formulation: The Riemann Problem

- The previous relation is exact.
- However, since the solution is known only at t^n, some kind of approximation is required in order to evaluate the flux through the boundary:

$$\tilde{\mathbf{F}}^n_{i+\frac{1}{2}} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{F}(\mathbf{u}(x_{i+\frac{1}{2}}, t))dt$$

- This achieved by solving the so-called "*Riemann Problem*", i.e., the evolution of an inital discontinuity separating two constant states. The Riemann problem is defined by the initial condition:

$$\mathbf{q}(x,0) = \begin{cases} \mathbf{q}_L & \text{for} \quad x < 0 \\ \mathbf{q}_R & \text{for} \quad x > 0 \end{cases}$$

The Riemann Problem

- If *q* is initially discontinuous, one or more characteristic variables will also have a discontinuity. Indeed, at *t = 0*,

$$w^k(x,0) = \mathbf{l}^k \cdot \mathbf{q}(x,0) = \begin{cases} w_L^k = \mathbf{l}^k \cdot \mathbf{q}_L & \text{if} \quad x < 0 \\ w_R^k = \mathbf{l}^k \cdot \mathbf{q}_R & \text{if} \quad x > 0 \end{cases}$$

- From the analytical solution (which still retains its validity),

$$w^k(x,t) = \begin{cases} w_L^k & \text{if} \quad x - \lambda^k t < 0 \\ w_R^k & \text{if} \quad x - \lambda^k t > 0 \end{cases}$$

- The initial discontinuity is decomposed in several characteristics “jumps”, each propagating unchanged at the speed λ^k.

System of Equations: Discontinuous data

- For the complete solution, we need to add the solutions to all the independent advection equations:

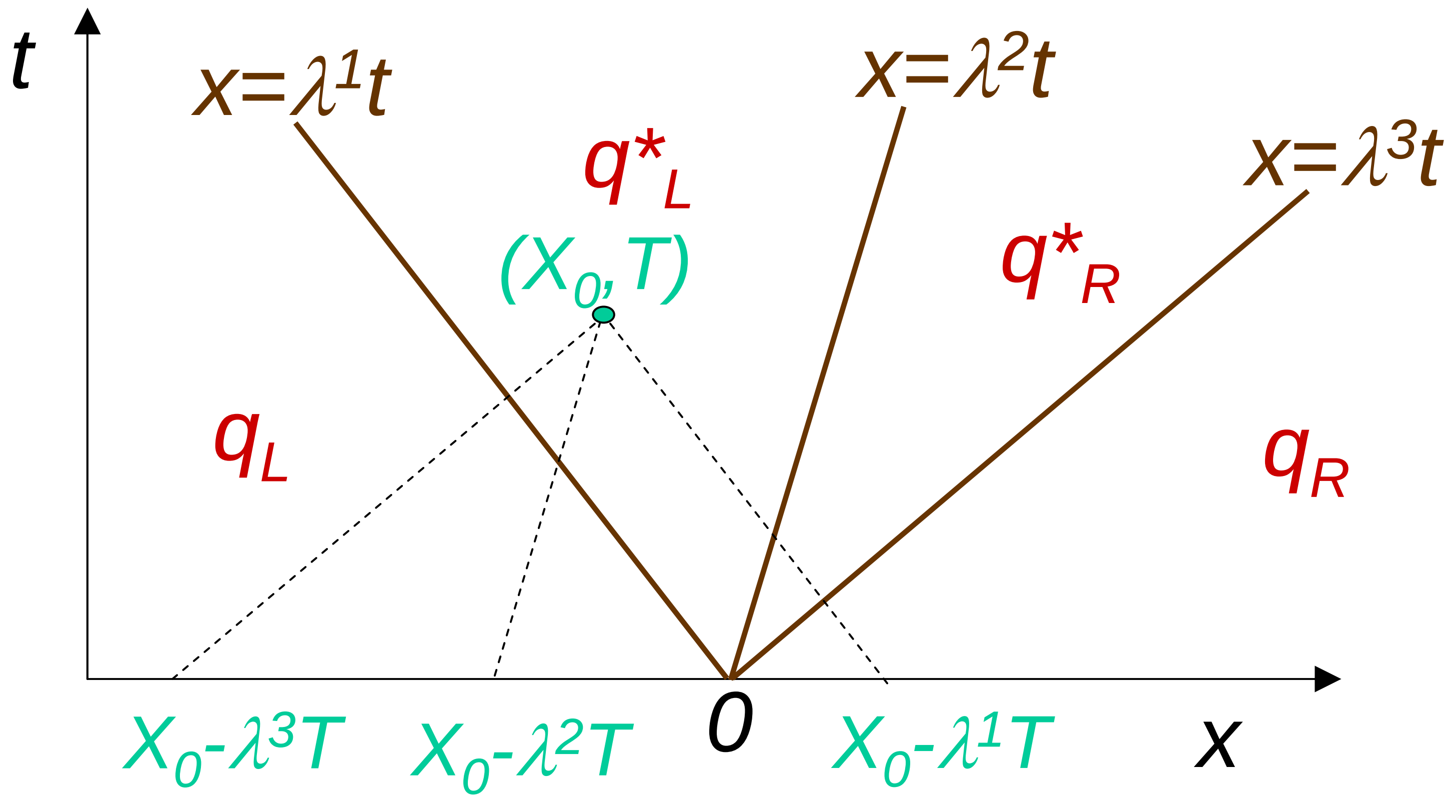
$$\mathbf{q}(x,t) = \sum_{k=1}^{k=m} w^k(x,t)\mathbf{r}^k = \sum_{k=1}^{k=m} w^k(x-\lambda^k t, 0)\mathbf{r}^k$$

- Using the previous solution for *w* one has

$$\mathbf{q}(x,t) = \sum_{k=1}^{k=K(x,t)} w_R^k \mathbf{r}^k + \sum_{k=K(x,t)+1}^{m} w_L^k \mathbf{r}^k$$

- Where *K(x,t)* is the maximum value of *k* such that $x-\lambda^k t>0$.

The Riemann Problem

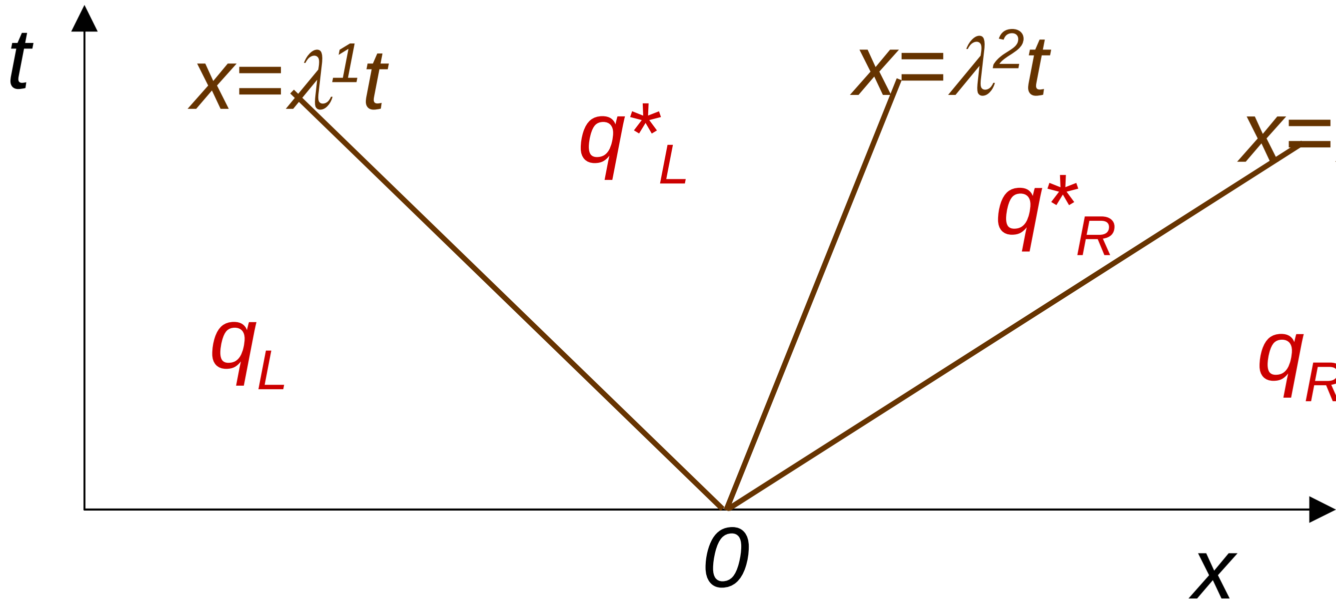


Point (X_0,T) is the right of the λ^1 characteristic emanating from the initial jump, but to the left of the other 2, so the solution is:

$$q(X_0,T) = w_R^1 \mathbf{r}^1 + w_L^2 \mathbf{r}^2 + w_L^3 \mathbf{r}^3$$

The Riemann Problem

- The 3 characteristics divide the domain into 4 regions:



- With-in each of these regions the solution is *constant*.
- Each time we cross a characteristic, the solution jumps by an amount proportional to the eigenvector associated with that characteristic.

The Riemann Problem

- Across the k-th characteristic the jump in q is given by

$$\left(w_R^k - w_L^k\right)\mathbf{r}^k = \alpha^k \mathbf{r}^k$$

- Note that this jump is also an eigenvector of the matrix A
- Solving the Riemann problem consits of taking the initial data (qL, qR) and decomposing the jump qR-qL into eigenvectors of A:

$$\mathbf{q}_R - \mathbf{q}_L = \alpha^1 \mathbf{r}^1 + \alpha^2 \mathbf{r}^2 + \cdots + \alpha^m \mathbf{r}^m$$

- This is equivalent to solving the system $R \cdot \alpha = \mathbf{q}_R - \mathbf{q}_L$
- Which has solution $\alpha = L \cdot (\mathbf{q}_R - \mathbf{q}_L)$, or in components,

$$\alpha^k = \mathbf{l}^k \cdot (\mathbf{q}_R - \mathbf{q}_L)$$