4-Nonlinear Advection Equation

Nonlinear Advection Equation

- We turn our attention the the scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

- Where *f(u)* is, in general, a nonlinear function of *u*.

- To gain some insights on the role played by nonlinear effects, we start by considering the inviscid Burger's equation:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

Nonlinear Advection Equation

- We can write Burger's equation also as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

- In this form, Burger's equation resembles the linear advection equation, with the only difference being that the velocity is no longer constant, but it is equal to the solution itself.
- The characteristic curve for this equation is

$$\frac{dx}{dt} = u(x,t) \quad \Longrightarrow \quad \frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = 0$$

- which tells us that u is constant along the curve $dx/dt=u(x,t)$.
- Along these curves the PDE becomes an ODE.

Nonlinear Advection Equation

- A quantity that remains constant along a characteristic curve is called a *Riemann invariant.*

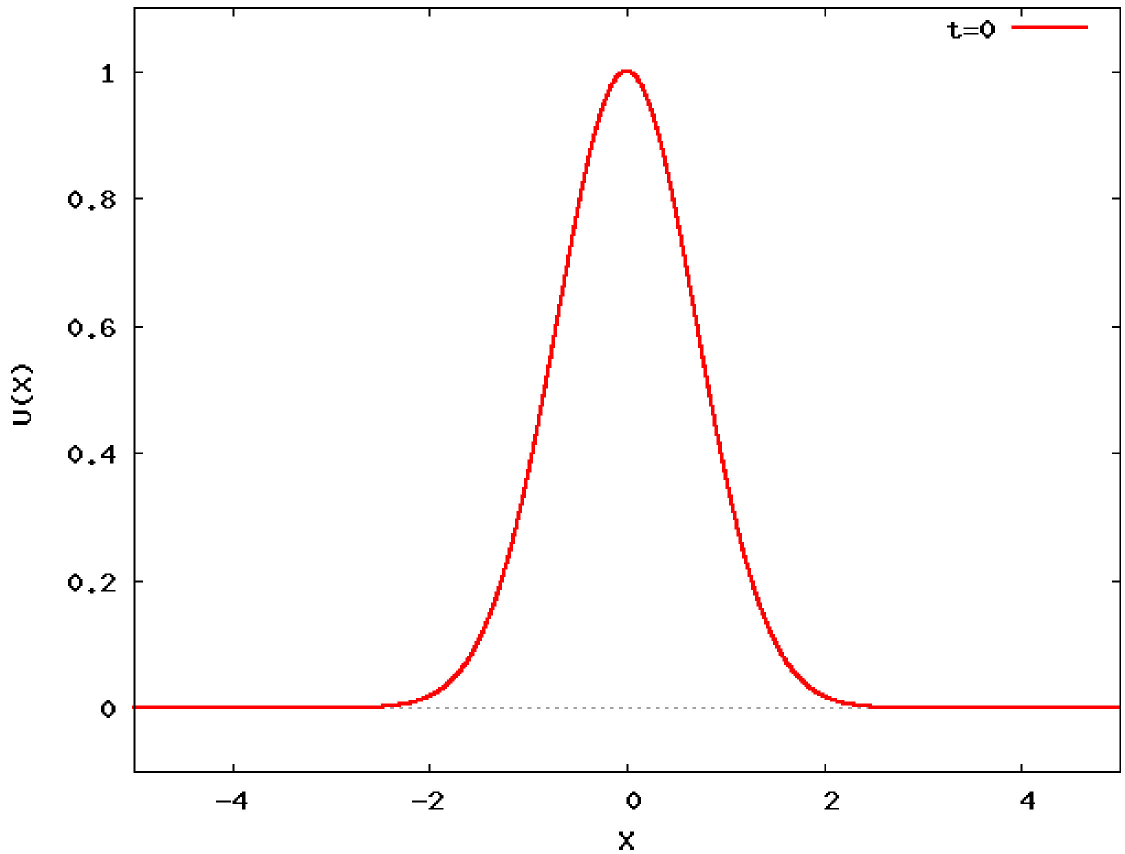
- In this simple case, *u* is a Riemann invariant.

- Considering that *dx/dt = u(x,t)* we deduce that characteristic curves are again straight lines: values of *u* associated with some fluid element do not change as that element moves.

- However, since *u(x,t)* can change in space, these lines are not necessarily parallel to each other as was the case for the linear advection equation.

Nonlinear Advection Equation

- Now consider the initial Gaussian profile at t=0:

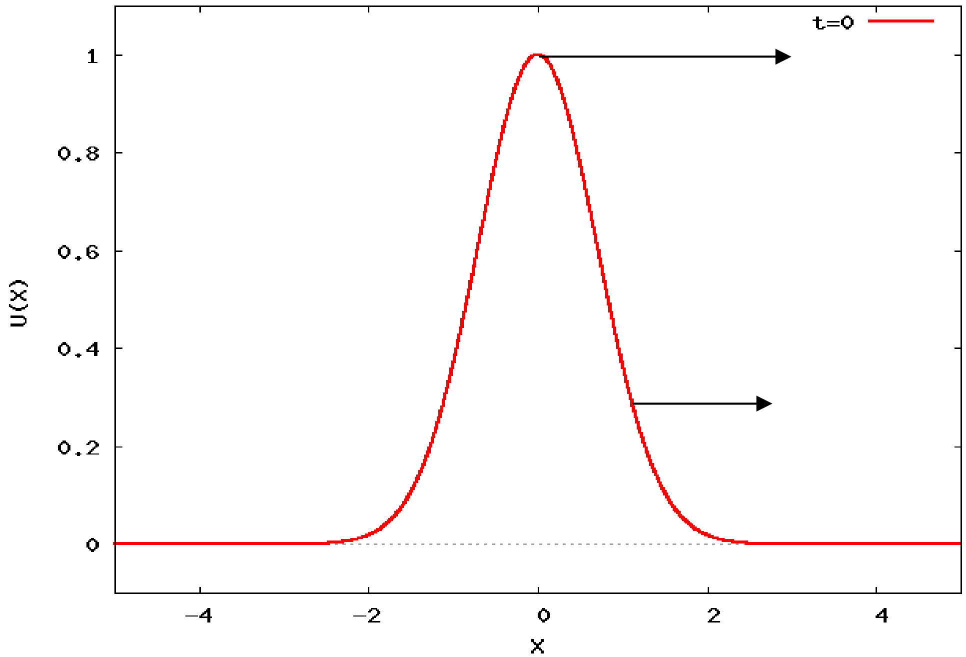


- What's going to happen at t > 0 ?

Nonlinear Advection Equation

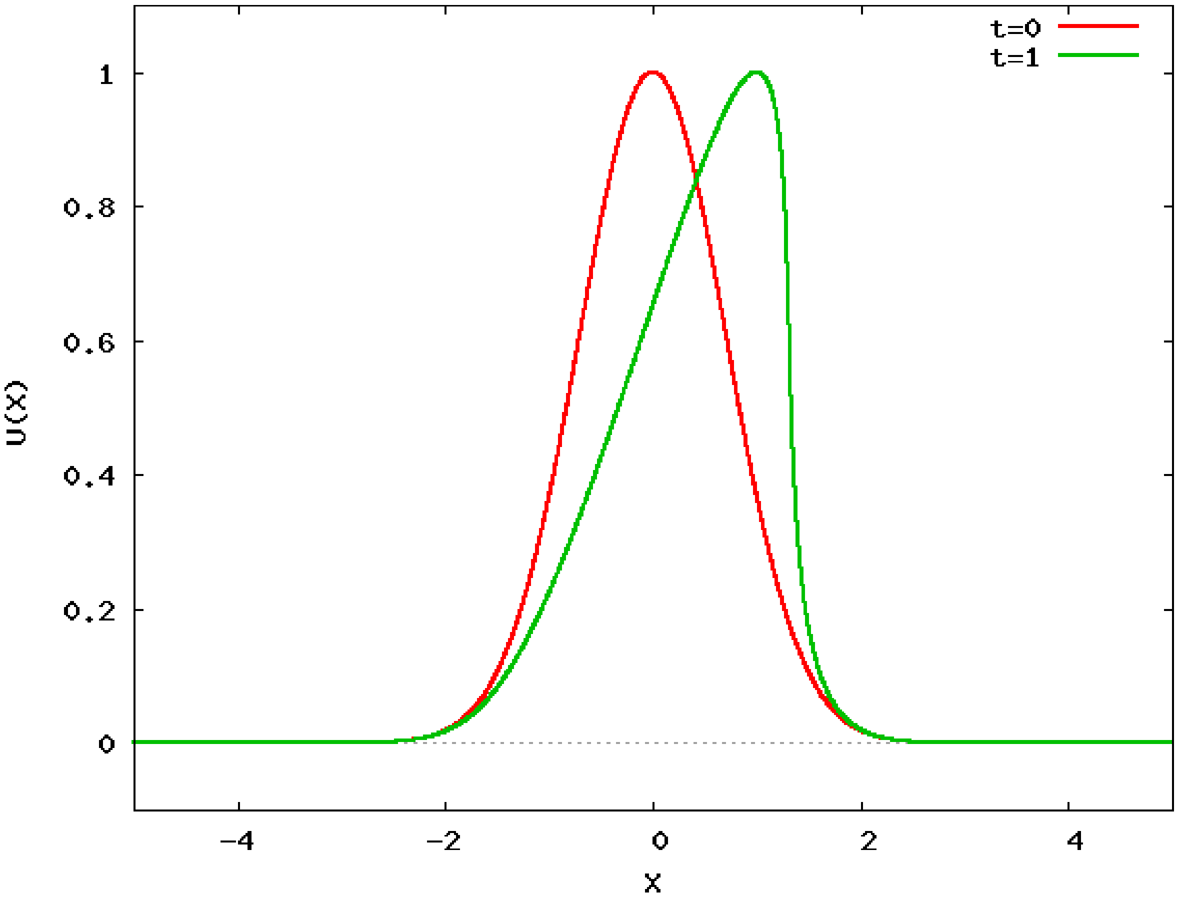
- From $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

  one can predict that, higher values of *u* will propagate faster than lower values: this leads to a wave steepening, since upstream values will advances faster than downstream values.



Nonlinear Advection Equation

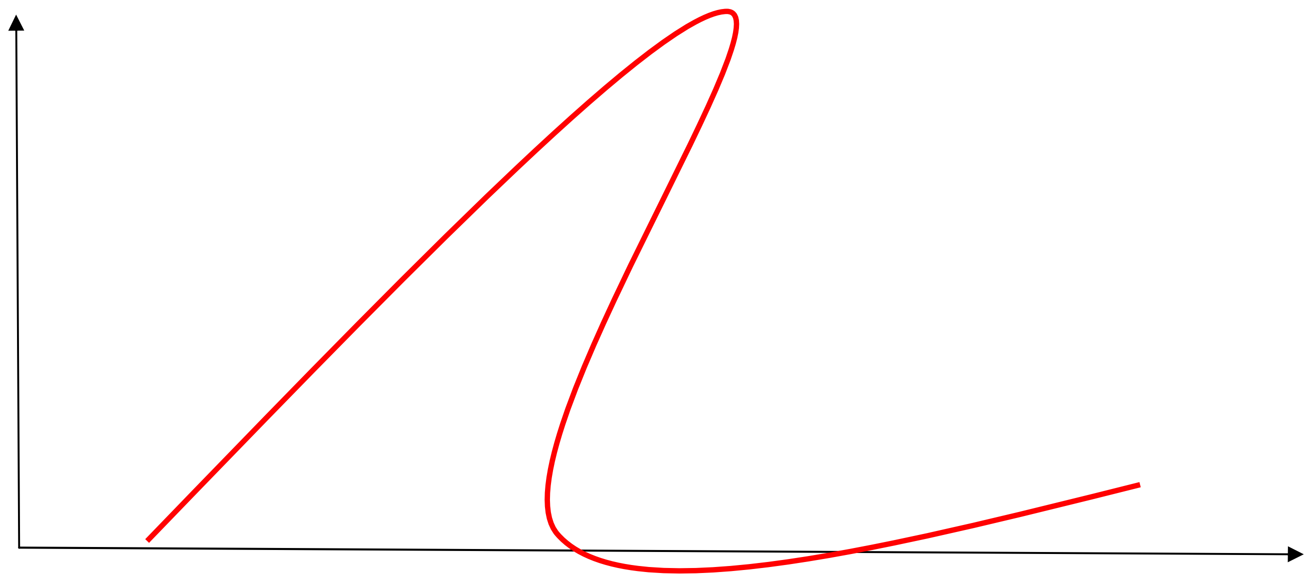
- Indeed, at t=1 the wave profile will look like:



- the wave steepens…

Nonlinear Advection Equation

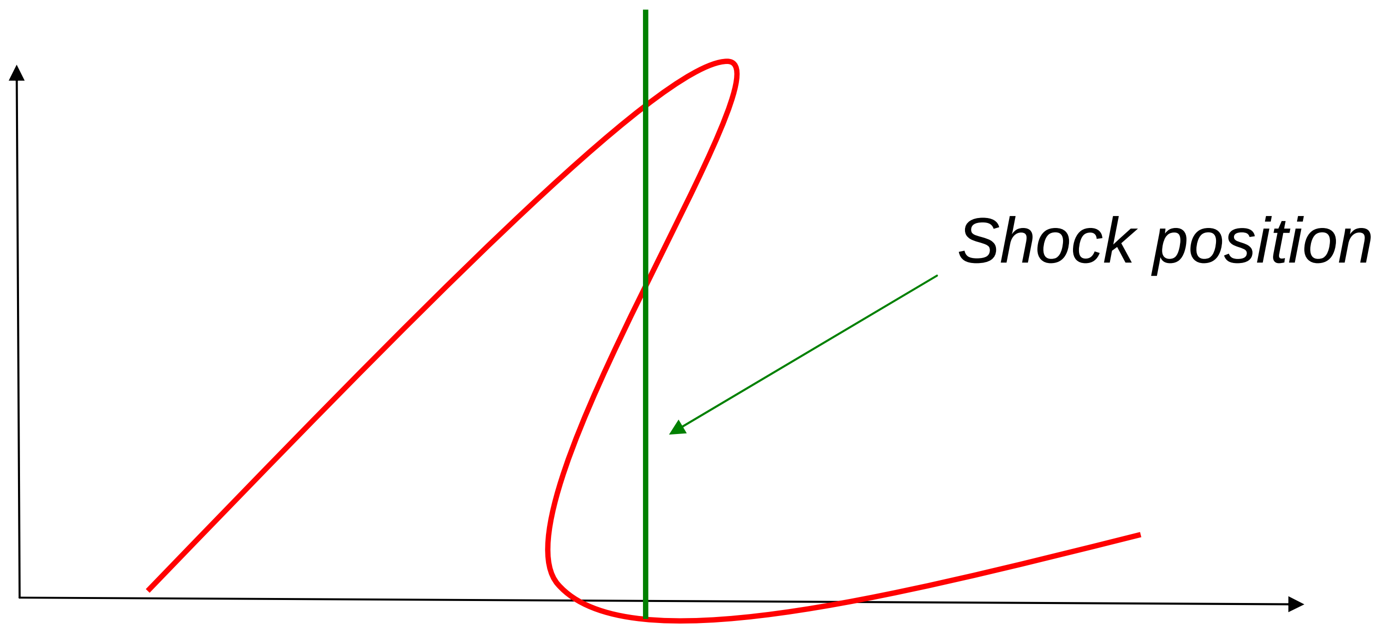
- If wait more, we should get something like this:



- A multivalue functions ??!!??! → Clearly Unphysical !!

Nonlinear Advection Equation

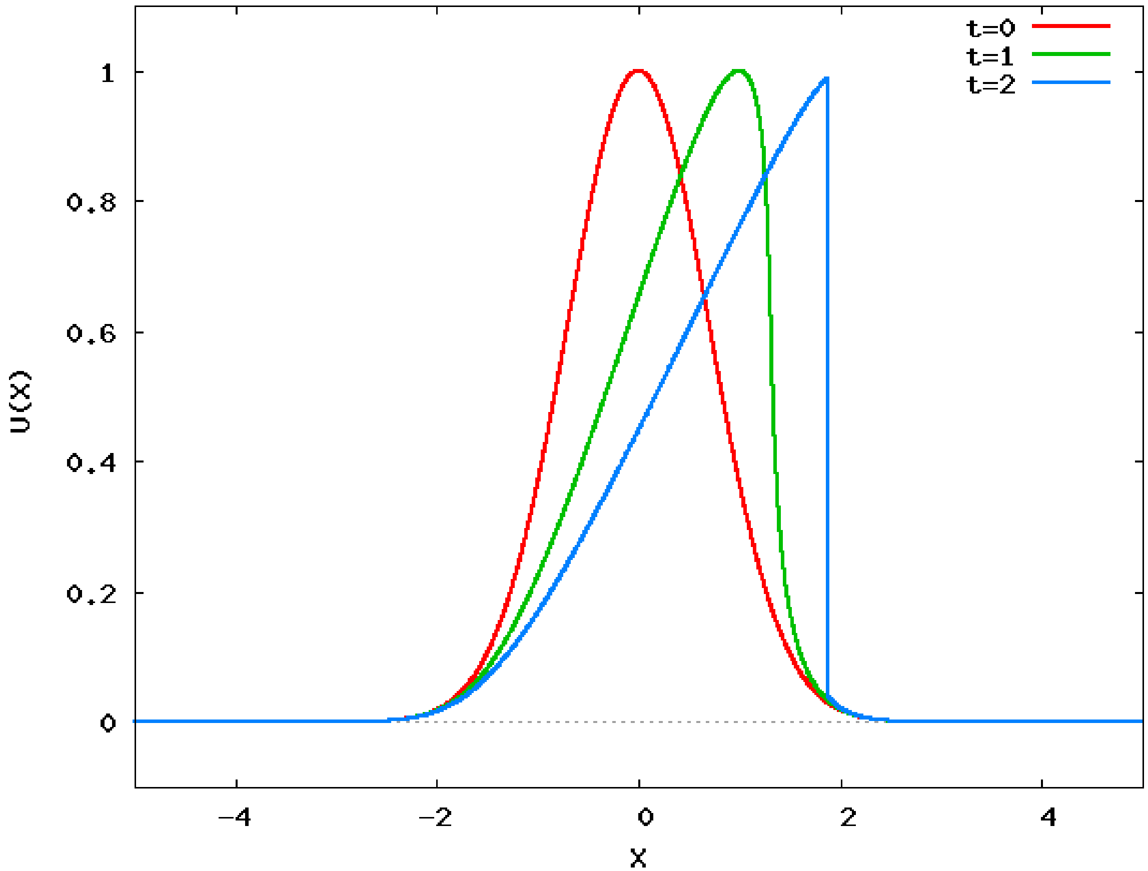
- The physical solution is to place a discontinuity there: a *shock wave*.



- Since the solution is no longer smooth, the *differential form* is not valid anymore and we need to consider the *integral form*.

Nonlinear Advection Equation

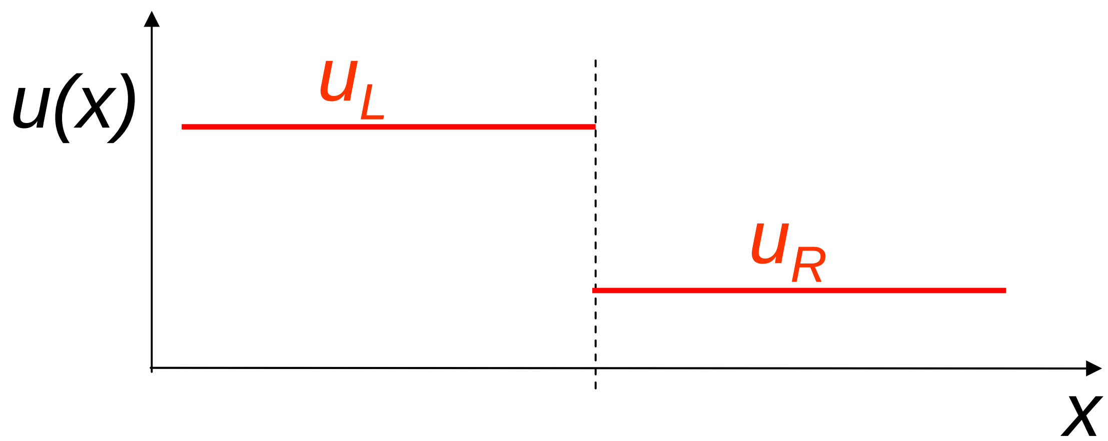
- This is how the solution should look like:



- Such solutions to the PDE are called weak solutions.

Nonlinear Advection Equation

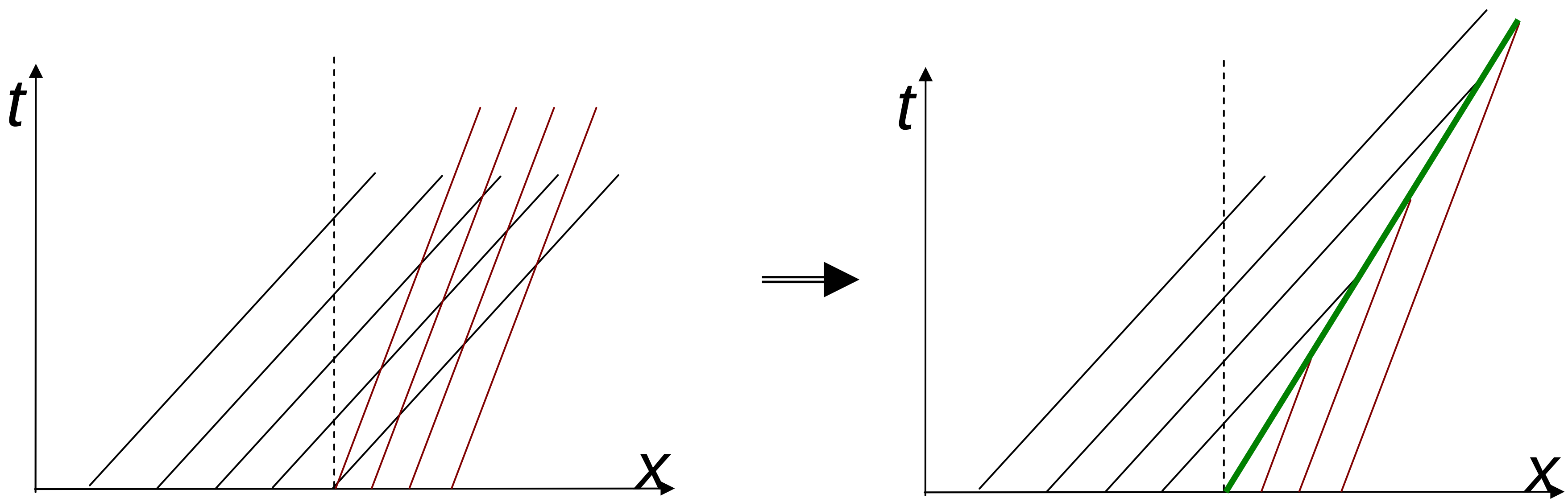
- Let's try to understand what happens by looking at the characteristics.
- Consider two states initially separated by a jump at an interface:



- Here, the characteristic velocities on the left are greater than those on the right.

Nonlinear Advection Equation

- The characteristic will intersect, creating a *shock*:



- The shock speed is such that $\lambda(u_L) > S > \lambda(u_R)$. This is called the *entropy condition*.

Nonlinear Advection Equation

- The shock speed S can be found using the Rankine-Hugoniot jump conditions, obtained from the integral form of the equation:

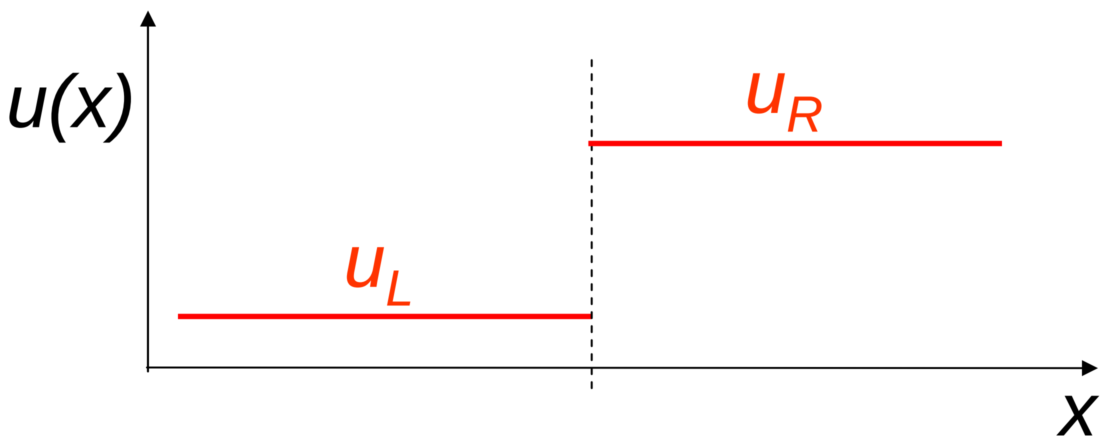
$$f(u_R) - f(u_L) = S(u_R - u_L)$$

- For Burger's equation $f(u) = u^2/2$ so that one finds the shock speed as

$$S = \frac{u_L + u_R}{2}$$

Nonlinear Advection Equation

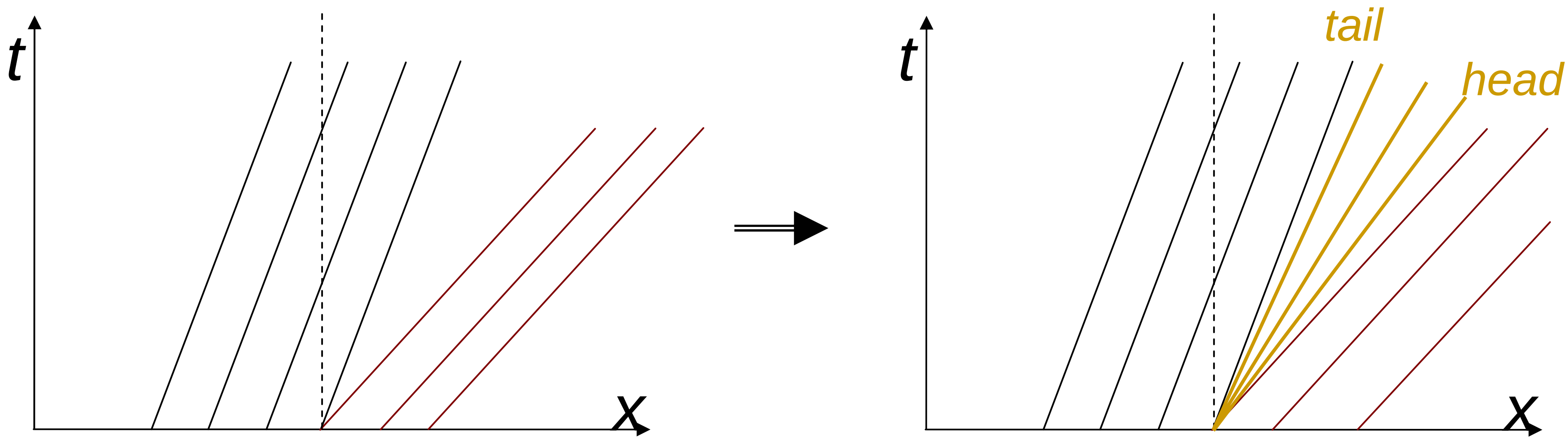
- Let's consider the opposite situation:



- Here, the characteristic velocities on the left are smaller than those on the right.

Nonlinear Advection Equation

- Now the characteristics will diverge:



- Putting a shock wave between the two states would be incorrect, since it would violate the entropy condition. Instead, the proper solution is a *rarefaction wave*.

Nonlinear Advection Equation

- A rarefaction wave is a nonlinear wave that smoothly connects the left and the right state. It is an expansion wave.
- The solution between the states can only be self-similar and takes on the range of values between u_L and u_R
- The head of the rarefaction moves at the speed $\lambda(u_R)$, whereas the tail moves at the speed $\lambda(u_L)$.
- The general condition for a rarefaction wave is $\lambda(u_L)<\lambda(u_R)$
- Both rarefactions and shocks are present in the solutions to the Euler equation. Both waves are nonlinear.

Nonlinear Advection Equation

- These results can be used to write the general solution to the Riemann problem for the Burger's equation:
  - If $u_L > u_R$ the solution is a *shock wave*. In this case

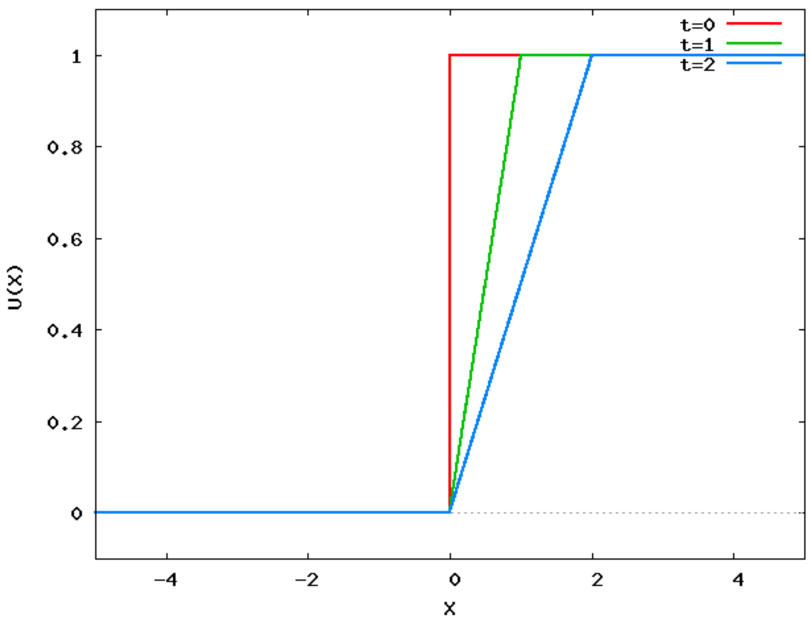
$$u(x,t) = \begin{cases} u_L & \text{if} \quad x - St < 0 \\ u_R & \text{if} \quad x - St > 0 \end{cases} , \qquad S = \frac{u_L + u_R}{2}$$

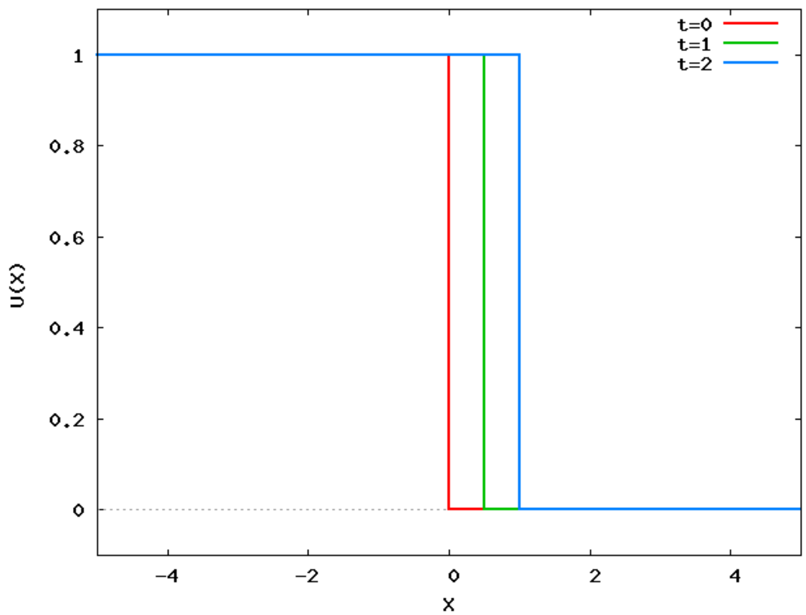
  - If $u_L < u_R$ the solution is a *rarefaction wave*. In this case

$$u(x,t) = \begin{cases} u_L & \text{if} \quad x/t \le u_L \\ x/t & \text{if} \quad u_L < x/t < u_R \\ u_R & \text{if} \quad x/t > u_R \end{cases}$$

Nonlinear Advection Equation

- Solutions look like





for a rarefaction and a shock, respectively.

Nonlinear Advection Equation

- An implementation is given in burger.c.