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Accretion-Ejection Flows

Andria ROGAVA
KU Leuven, Belgium

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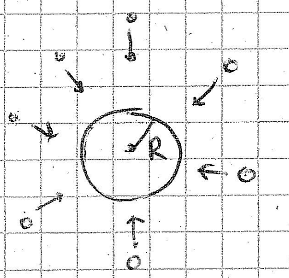
Lecture II "Spherical Accretion"

2.1. Collisionless, spherical accretion:

[Zeldovich, Novikov "Relativistic Astrophysics" Vol. 1 1971]



– Particles of mass m, falling on the central star with mass M and radius R.

– Particle distribution function:

$$f(\vec{r}, \vec{v}, t) \tag{1}$$

specifies the number of particles in the phase space volume element $d^3\vec{r}\, d^3\vec{v}$ centered about $\vec{r}, \vec{v}$ at time t.

– What defines the form of the f-function?

~ Collisionless Boltzmann equation

$$d_t f(\vec{r}, \vec{v}, t) = \partial_t f(\vec{r}, \vec{v}, t) + (\vec{v} \cdot \nabla) f + (\dot{\vec{v}}, \nabla_{\vec{v}}) f = 0 \tag{2}$$

– Macroscopic quantities:

(a) Number density:

$$n(\vec{r}, t) \equiv \int f(\vec{r}, \vec{v}, t)\, d^3\vec{v} \tag{3}$$

(b) Velocity dispersion ("temperature"):

$$T(\vec{r}, t) \sim \langle \vec{v}^2(\vec{r}, t) \rangle \equiv \frac{1}{n(r,t)} \int \vec{v}^2 f(\vec{r}, \vec{v}, t)\, d^3\vec{v} \tag{4}$$

– What defines $\dot{\vec{v}}$ – ? Gravitational potential $\Phi(r)$

$$\dot{\vec{v}} = -\nabla\Phi, \quad \Phi = -\frac{GM}{r} + \Phi_{sg} \tag{5}$$

where Φ_{sg} is the self-gravity $(\nabla^2\Phi_{sg} = 4\pi G m n)$. Neglected!

Note! (2) is simply the continuity equation for the flow of particles in 6D phase space (Reif, "Fundamentals of Statistical and Thermal Physics" 1965)

Note also! (2) is a statement of Liouville's theorem: the distribution function is conserved along the trajectory of each particle.

Jeans' Theorem: When flows are stationary, i.e., when f does not depend on time, it depends only on dynamical constants of motion. [Jeans, "Problems of Cosmogony and Stellar Dynamics" 1919]

$$\text{If } f(\vec{r}, \vec{v}, t) = \text{const}(t), \text{ then } f = f(E, J) \tag{6}$$

For spherically symmetric system there are two (2) dynamical constants: energy E and angular momentum J (both per unit mass).

$$E \equiv \frac{1}{2} v^2 + \Phi(r) = \frac{1}{2} v_r^2 + \frac{1}{2}\frac{J^2}{r^2} - \frac{GM}{r} \tag{7}$$

$$J \equiv r\, v_t \tag{8}$$

where v_r and v_t are radial and transversal velocity components, respectively.

- Stationary case:

$$f = f(E, J) \tag{9}$$

- Stationary and isotropic (velocity) case:

$$f = f(E) \tag{10}$$

Obviously $E \geq \Phi(r)$.

One can easily shew that in stationary, spherically symmetric case with isotropic velocities:

$$n(r) = 4\pi \int v^2 f \, dv = 4\pi \int_{E=\Phi}^{\infty} [2(E-\Phi)]^{1/2} f(E) \, dE \qquad (11)$$

$$\langle v^2(r) \rangle = \frac{4\pi}{n(r)} \int_{E=\Phi}^{\infty} [2(E-\Phi)]^{3/2} f(E) \, dE \qquad (12)$$

Problem 1.1. Derive (11) and (12)

Particles with angular momentum $J < J_{min}(E)$ are being captured by the accreting star. For nonrelativistic particles moving around a Newtonian star with radius R:

$$J_{min}(E) = \left[2\left(E + \frac{GM}{r}\right)\right]^{1/2} \cdot R \qquad (13)$$

Problem 1.2. Prove (13)

Using (7) and (8) we find:

$$d^3v = 2\pi v_t \, dv_t \, dv_r = \frac{4\pi J}{r^2 |v_r|} \qquad (14)$$

Now we can define $N^-(r, E, J)$ – number of particles with the velocity directed inwards per unit r, J, E:

$$N^-(r,E,J) \, dr \, dE \, dJ = \frac{1}{2} f(E,J) \, d^3r \, d^3v = 8\pi^2 \frac{J}{|v_r|} f \, dr \, dE \, dJ \qquad (15)$$

Total particle capture rate is:

$$\dot{N}_{tot} = \int\limits_{\Phi(r)}^{\infty} dE \int\limits_{J=0}^{J_{min}} dJ\, |v_r|\, N^-(r,E,J)\Big|_{r=R} = 8\pi^2 \int\limits_{E=\Phi(r)}^{\infty} dE \int\limits_{0}^{J_{min}} f J\, dJ \quad (16)$$

And the corresponding accretion rate is: $\dot{M}_{tot} = m\dot{N}_{tot}$.

Concrete example: Consider infinite number of particles, surrounding the star with isotropic distribution. On infinity they have density n_∞ and velocities $v_\infty \ll c$. Obviously, each of them has energy $E_\infty = v_\infty/2$.

- The distribution function:

$$f = f(E) = n_\infty \frac{\delta(E - E_\infty)}{4\pi (2E_\infty)^{1/2}} \quad (17)$$

It allows to calculate the particle capture rate:

$$\dot{N}(E>0) = 8\pi^2 \int\limits_{0}^{\infty} dE\, d f(E) \int\limits_{0}^{J_{min}} J\, dJ = 4\pi^2 \int\limits_{0}^{\infty} dE\, f(E)\, J^2_{min}(E) \quad (18)$$

Using (17) we can calculate $n(r)$ and $\langle v^2(r) \rangle$:

$$n_{E>0}(r) = n_\infty \left(1 + \frac{2GM}{v_\infty^2 r}\right)^{1/2} \quad (19)$$

$$\langle v^2(r) \rangle_{E>0} = v_\infty^2 \left(1 + \frac{2GM}{v_\infty^2 r}\right) \quad (20)$$

and the latter defines effective temperature:

$$T_{E>0}(r) = T_\infty \left(1 + \frac{2GM}{v_\infty^2 r}\right), \quad \frac{3}{2} kT \equiv \frac{1}{2} m v^2 \quad (21)$$

- Accretion radius r_a is defined as a distance at which particle kinetic energy is equal to its potential energy:

$$r_a \equiv \frac{2GM}{v_\infty^2} \tag{22}$$

$\boxed{\text{Problem 1.3. Derive (19) and (20)}}$

- What happens?

(a) At $r \gg r_a$ values are close to asymptotic ones;

(b) At $r \ll r_a$ gravity focuses particles, increasing their density and temperature.

Note! Bounded ($E<0$) particles, if collisionless are absent, have no impact.

- What about collisions? Mean free path λ!

(a) Collision-less case:

$$\lambda_c \to \infty \tag{23}$$

(b) Weakly collisional case:

$$\infty > \lambda > r$$

One has to use Collisional Boltzmann equation:

$$\frac{Df}{Dt} = \left(\frac{\partial f}{\partial t}\right)_c \tag{24}$$

Relatively easy when: $t_d \equiv (r/v) \ll t_r \equiv (\lambda/v)$.

- Characteristic dynamic time scale - t_d - corresponds to particle orbital period

$$t_d \sim \frac{r}{v} \tag{25}$$

is much less than characteristic timescale of collisional relaxation:

$$t_r \sim \frac{\lambda}{v} \tag{26}$$

I.e., $t_d \ll t_r$ and at times comparable with t_d the f-function satisfies the Vlasov equation and for spherically symmetric case it has the $f = f(E, J)$ form.

However f undergoes slow, secular variation on timescales comparable with relaxation time.

(c) When $\lambda \ll r$ - hydrodynamic accretion.

Astrophysical regimes?

- Typical cases of accretion on compact objects are hydrodynamic:

$$\lambda \ll r \tag{27}$$

- Dynamics of stars in globular clusters:

$$\infty \gg \lambda \gg r\ , \text{ or } t_d \ll t_r \tag{28}$$

- Normal galaxies:

$$t_d \ll t_H \ll t_r \tag{29}$$

where $t_H \sim 1/H \sim 10^{10}$ years

2.2. Hydrodynamic, spherical accretion

- When accretion is hydrodynamic?

$$\lambda_{eff} \ll r \tag{1}$$

- What makes effective free path small?

Collisions, macroscopic but weak magnetic fields, two-stream plasma instabilities ...

- We consider stationary ($\partial_t = 0$), spherically symmetric accretion on a star of mass M.

- Continuity equation:

$$\cancel{\partial_t \rho} + \nabla(\rho \vec{v}) = 0$$

$$\partial_\theta = \partial_\varphi = 0 \quad ; \quad v_r \equiv u$$

$$\frac{1}{r^2}\partial_r (r^2 \rho u) = 0 \tag{2}$$

Integrating we get:

$$\boxed{\dot{M} \equiv 4\pi r^2 \rho u = \text{const}} \tag{3}$$

- Adiabatic flow assumption:

Polytropic equation of state:

$$P = K\rho^\Gamma \quad , \quad K, \Gamma - \text{const} \tag{4}$$

Sound speed:

$$\boxed{c_s^2 \equiv \frac{dP}{d\rho} = \Gamma \frac{P}{\rho} = K\Gamma\rho^{\Gamma-1}} \tag{5}$$

- Boundary conditions at infinity:

At $r \to \infty$ the gas is at rest ($u_\infty = 0$) and has P_∞, ρ_∞ and c_∞.

- Euler equation:

$$\cancel{\partial_t \vec{v}}^{\,0} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla P + \vec{g} \qquad (6)$$

reducing to:

$$u \frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{GM}{r^2} \qquad (7)$$

Note!

$$u \frac{\partial u}{\partial r} = \frac{1}{2}\frac{\partial}{\partial r}(u^2)$$

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{1}{\rho}\frac{\partial P}{\partial \rho}\,\frac{\partial \rho}{\partial r} = K\Gamma \rho^{\Gamma-2}\frac{\partial \rho}{\partial r} = \frac{K\Gamma}{\Gamma-1}\frac{\partial}{\partial r}\left(\rho^{\Gamma-1}\right) =$$

$$= \frac{1}{\Gamma-1}\frac{\partial}{\partial r}\left(c_s^2\right)$$

$$\frac{GM}{r^2} = -\frac{\partial}{\partial r}\left(\frac{GM}{r}\right)$$

Combining we get:

$$\boxed{\frac{1}{2}u^2 + \frac{1}{\Gamma-1}c_s^2 - \frac{GM}{r} = \text{const} = \frac{1}{\Gamma-1}c_\infty^2} \qquad (8)$$

(3), (5) and (8) constitute the set of three algebraic equations for three variables: u, ρ, and c_s.

Bondi found that different values of $\dot{M}$ lead to physically different classes of solutions for the same boundary conditions at infinity!

Let us find the unique solution for which the velocity monotonously increases from zero at infinity up to "free fall" velocity at small radii

$$\lim_{r \to 0} u(r) = \left(\frac{2GM}{r}\right)^{1/2} \qquad (9)$$

Bondi (1952) found that such a peculiar solution is realized for a particular value of $\dot{M}$.

Let us rewrite (2) as: $(\partial_r f \equiv f')$

$$\frac{2}{r} + \frac{u'}{u} + \frac{\rho'}{\rho} = 0 \qquad (10)$$

and (7) as:

$$u \cdot u' + C_s^2 \frac{\rho'}{\rho} + \frac{GM}{r^2} = 0 \qquad (11)$$

One can show that from (10) and (11) it follows:

$$\boxed{\begin{aligned} \rho' &= \frac{\rho}{C_s^2 - u^2} \cdot \left(\frac{2u^2}{r} - \frac{GM}{r^2}\right) \qquad (12) \\ u' &= \frac{u}{C_s^2 - u^2} \left(\frac{GM}{r^2} - \frac{2C_s^2}{r}\right) \qquad (13) \end{aligned}}$$

Problem 2.1 Derive (12) and (13)

From (12) and (13) we see that for the smooth, monotonous increase of $u(r)$ with decreasing r the solution should pass through the <u>critical point</u>:

$$\boxed{C_s^2 = u^2 = \frac{1}{2}\frac{GM}{r} \quad \text{at } r = r_{cr} \, !} \qquad (14)$$

Problem 2.2 Calculate how the location of the sonic point r_{cr} and values of $C_s(r_{cr})$ and $U(r_{cr})$ depend on the asymptotic value of the speed of sound C_s.

The answers on Prob. 2.2 question are:

$$r_{cr} = \left(\frac{5-3\Gamma}{4}\right)\frac{GM}{C_\infty^2} \tag{15}$$

$$C_s(r_{cr}) = U(r_{cr}) = \left(\frac{2}{5-3\Gamma}\right) C_\infty^2 \tag{16}$$

Now we can calculate the accretion rate $\dot{M}$:

Since $C_s^2 = K\Gamma\rho^{\Gamma-1}$, we can see that $C_\infty^2 = K\Gamma\rho_\infty^{\Gamma-1}$ so that $K\Gamma = C_\infty^2/\rho_\infty^{\Gamma-1}$, yielding:

$$\rho = \rho_\infty\left(\frac{C_s}{C_\infty}\right)^{\frac{2}{\Gamma-1}} \tag{17}$$

Plugging this into the definition of $\dot{M}$ we find

$$\dot{M} = 4\pi r_{cr}^2 U(r_{cr}) \cdot \rho_\infty \left(\frac{C_s(r_{cr})}{C_\infty}\right)^{\frac{2}{\Gamma-1}} = 4\pi\lambda_s\left(\frac{GM}{C_\infty^2}\right)^2 \rho_\infty C_\infty \tag{18}$$

Problem 2.3 Show that:

$$\lambda_s \equiv \left(\frac{1}{2}\right)^{\frac{(\Gamma+1)}{2(\Gamma-1)}} \cdot \left(\frac{5-3\Gamma}{4}\right)^{-\frac{(5-3\Gamma)}{2(\Gamma-1)}} \tag{19}$$

The accretion rate can be written also as:

$$\dot{M} = 4\pi\lambda_s (GM)^2 \rho_\infty c_\infty c^{-2} \left(\frac{c}{c_\infty}\right)^2 \qquad (20)$$

when $c_\infty \approx u_\infty$ we see that $\dot{M}$ is larger in the case of hydrodynamic accretion than in the case of collisionless accretion. It is larger $\sim (c/c_\infty)$ times! For typical ISM $c_\infty \sim 10\ \mathrm{km\,s^{-1}}$ and this factor is $\sim 10^9$

- For the ideal gas:

$$P = \frac{\rho k T}{\mu m}, \quad c_s^2 = \frac{\Gamma k T}{\mu m}, \quad T = T_\infty \left(\frac{\rho}{\rho_\infty}\right)^{\Gamma - 1} \qquad (21)$$

One can find out for the $\dot{M}$:

$$\dot{M} = 8{,}77 \cdot 10^{-16} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{\rho_\infty}{10^{-24}\ \mathrm{g\,cm^{-3}}}\right) \left(\frac{c_\infty}{10\ \mathrm{km\,s^{-1}}}\right)^{-3} \frac{M_\odot}{\mathrm{year}} \qquad (22)$$

Problem 2.4. Show that the transsonic accretion rate, specified by (18), is the maximum possible accretion rate for the given value of Γ.

Problem 2.5. Show that when $\Gamma = 1$ $\lambda_s = e^{3/2}/4$, while when $\Gamma = 5/3$ $\lambda_s = 1/4$.

Let us see how the transsonic accretion flow behaves at different radial distances

- When $r \gg r_{cr}$: gravity of the central star does not affect much the thermodynamic quantities:

$$\rho \approx \rho_\infty \ , \ T \approx T_\infty \ , \ c_s \approx c_\infty \quad \text{at} \quad r \gg r_{cr} \tag{23}$$

From (23), (18) and (3) we get:

$$\dot{M} = \cancel{4\pi}\, \lambda_s \left(\frac{GM}{c_\infty^2}\right)^2 \cancel{\rho_\infty}\, c_\infty \simeq \cancel{4\pi}\, r^2 \cancel{\rho_\infty}\, u(r)$$

leading to:
$$\frac{u(r)}{c_\infty} \simeq \lambda_s \left(\frac{GM}{c_\infty^2}\right)^2 r^{-2} \ , \ \text{at} \ r \gg r_{cr} \tag{24}$$

When $r \ll r_{cr}$ gravity dominates over deceleration due to gas pressure and the flow velocity approaches the free-fall speed:

$$u \simeq \left(\frac{2GM}{r}\right)^{1/2} , \ r \ll r_{cr} \quad \left(1 \leq \Gamma < \frac{5}{3}\right) \tag{25}$$

and for the density and the temperature we get:

$$\frac{\rho}{\rho_\infty} \simeq \frac{\lambda_s}{\sqrt{2}} \left(\frac{GM}{c_\infty^2}\right)^{3/2} r^{-3/2} , \tag{26}$$

$$\frac{T}{T_\infty} \simeq \left[\frac{\lambda_s}{\sqrt{2}} \left(\frac{GM}{c_\infty^2}\right)^{3/2} \right]^{\Gamma - 1} r^{-3(\Gamma-1)/2} \tag{27}$$

- What happens when $\Gamma = 5/3$ and $r_{cr} = 0$!?

In this case for $r \ll GM/c_\infty^2$ we get:

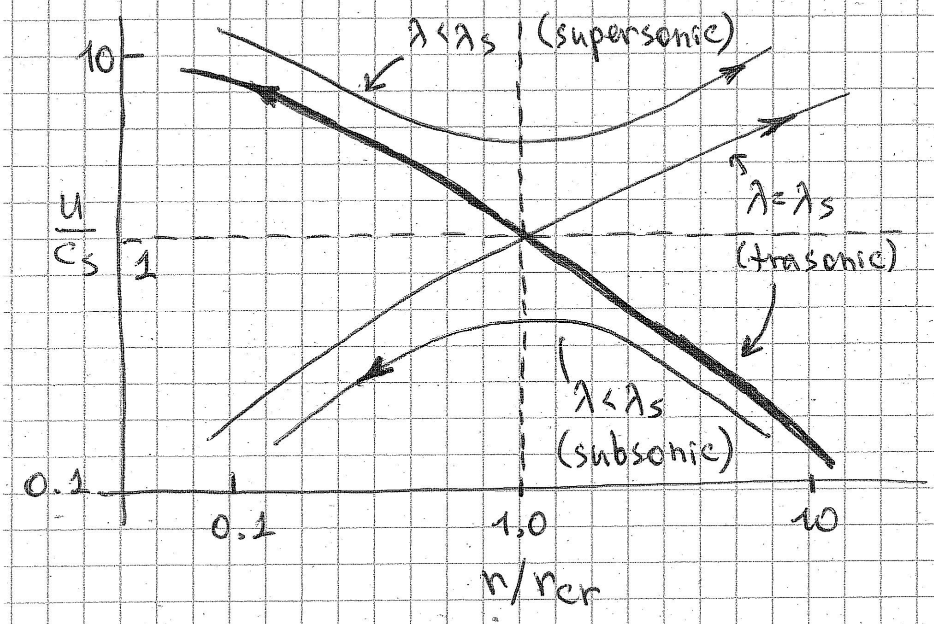
$$c_s \simeq u \approx \left(\frac{GM}{2r}\right)^{1/2} \tag{28}$$

and corresponding ρ and T are given by:

$$\frac{\rho}{\rho_\infty} = \frac{1}{2\sqrt{2}} \left(\frac{GM}{c_\infty^2}\right)^{3/2} r^{-3/2} \ ; \ \frac{T}{T_\infty} \simeq \frac{1}{2} \frac{GM}{c_\infty^2} r^{-1} \tag{29}$$

Problem 2.6 Consider transonic accretion with $\Gamma = 1$ (isothermal flow).
(a) How (8) is modified?
(b) Will (23-27) still be valid?

The boundary conditions at infinity ($u=0$, $c=c_\infty$, $\rho=\rho_\infty$) do not specify uniquely the accretion solution. Bondi found out that there exists a second, subsonic, class of solutions!



At $r \gg r_{cr}$ (23) and (24) are still valid. For $r \ll r_{cr}$ the second term in (8) dominates the first one and the equation leads to the condition of hydrostatic equilibrium:

$$\frac{1}{\Gamma-1}\left[\left(\frac{c_s}{c_\infty}\right)^2 - 1\right] \approx \frac{GM}{c_\infty r} \tag{30}$$

Thus the flow is "choked off by back pressure on the subsonic regime. From (30) we find:

$$\frac{\rho}{\rho_\infty} = \left(\frac{c_s}{c_\infty}\right)^{\frac{2}{\Gamma-1}} \approx \left[1 + \frac{(\Gamma-1)GM}{c_\infty^2 r}\right]^{\frac{1}{\Gamma-1}} \tag{31}$$

In the "no accretion" limit $\lambda = 0 = u$ we get "extended atmosphere" solution.