Magnetohydrodynamics

1 Basic equations

In galaxies, and indeed in many other astrophysical settings, the gas is partially or fully ionized and can carry electric currents that, in turn, produce magnetic fields. The associated Lorentz force exerted on the ionized gas (also called plasma) can in general no longer be neglected in the momentum equation for the gas. Magneto-hydrodynamics (MHD) is the study of the interaction of the magnetic field and the plasma treated as a fluid. In MHD we combine Maxwell's equations of electrodynamics with the fluid equations, including also the Lorentz forces due to electromagnetic fields. We first discuss Maxwell's equations that characterize the evolution of the magnetic field.

1.1 Maxwell's equations

In the Gaussian cgs units, Maxwell's equations can be written in the form

$$\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla}\times\boldsymbol{E}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \tag{1}$$

$$\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{\nabla}\times\boldsymbol{B} - \frac{4\pi}{c}\boldsymbol{J}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{E} = 4\pi\rho_{\rm e}, \tag{2}$$

where $\boldsymbol{B}$ is the magnetic flux density (usually referred to as simply the magnetic field), $\boldsymbol{E}$ is the electric field, $\boldsymbol{J}$ is the current density, c is the speed of light, and $\rho_{\rm e}$ is the charge density.

To ensure that $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ is satisfied at all times it is often convenient to define $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ and to replace Eq. (1) by the 'uncurled' equation for the magnetic vector potential, $\boldsymbol{A}$,

$$\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{E} - \boldsymbol{\nabla}\phi, \tag{3}$$

where ϕ is the scalar potential. Note that magnetic and electric fields are invariant under the gauge transformation

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\Lambda, \tag{4}$$

$$\phi' = \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}. \tag{5}$$

1.2 Resistive MHD and the induction equation

In order to close the system of equations, we need to relate the current density $\boldsymbol{J}$ back to the fields. For this we use the standard Ohm's law in a fixed frame of reference,

$$\boldsymbol{J} = \sigma\left(\boldsymbol{E} + \frac{\boldsymbol{V}\times\boldsymbol{B}}{c}\right), \tag{6}$$

where σ is the electric conductivity. This simple form arises for a conducting fluid moving in given electric and magnetic fields at non-relativistic velocities. The physical picture for the above expression is elaborated further below; the Lorentz force provides a relative acceleration between the positive and negative charges in the system, which is balanced by friction between them due to collisions. The resulting steady drift velocity between the negative and positive charges, corresponds to a current density proportional to the Lorentz force itself, the constant of proportionality being the conductivity.

Introducing the magnetic diffusivity $\eta = c^2/(4\pi\sigma)$ in cgs units, we can eliminate $\boldsymbol{J}$ from Eq. (2), so we have

$$\frac{\eta}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{E} = \frac{\eta}{c}\boldsymbol{\nabla}\times\boldsymbol{B} - \frac{\boldsymbol{V}\times\boldsymbol{B}}{c}. \tag{7}$$

This formulation shows that the time derivative term (arising from the Faraday displacement current) can be neglected if the relevant time scale over which the electric field varies, exceeds the Faraday time $\tau_{\text{Faraday}} = \eta/c^2$. Below we shall discuss that for ordinary Spitzer resistivity, η is proportional to $T^{-3/2}$ and varies between 10 and $10^7\,\text{cm}^2\,\text{s}^{-1}$ for temperatures between $T = 10^8$ and $T = 10^4\,\text{K}$. Thus, the displacement current can be neglected when the variation time scales are longer than $10^{-20}\,\text{s}$ (for $T \approx 10^8\,\text{K}$) and longer than $10^{-14}\,\text{s}$ (for $T \approx 10^4\,\text{K}$). For the applications discussed in this book, this condition is always met, even for neutron stars where the time scales of variation can be of the order of milliseconds, but the temperatures are very high as well. We can therefore safely neglect the displacement current and eliminate $\boldsymbol{E}$, so Eq. (2) can be replaced by Ampere's law $\boldsymbol{J} = (c/4\pi)\boldsymbol{\nabla}\times\boldsymbol{B}$.

Substituting Ohm's law into the Faraday's law of induction, and using Ampere's law to eliminate $\boldsymbol{J}$, one can write a single evolution equation for $\boldsymbol{B}$, which is called the induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times\left(\boldsymbol{V}\times\boldsymbol{B} - \eta\nabla\times\boldsymbol{B}\right). \tag{8}$$

The induction equation gives the evolution of the magnetic field given the velocity field. Taking the divergence of Eq. (8) we see that $\partial(\boldsymbol{\nabla}\cdot\boldsymbol{B})/\partial t = 0$, and the divergence free property of the magnetic field is preserved in time, as it must be.

2 The momentum equation

The magnetic field influences the fluid velocity in turn through the Lorentz force. On a single charged particle of charge q, the Lorentz force is $\boldsymbol{F}_L = q[\boldsymbol{E} + (\boldsymbol{V}\times\boldsymbol{B})/c]$. In a conducting fluid where there are say n_i ions per unit volume with charge q_i moving with a bulk velocity $\boldsymbol{V}_i$ and n_e electrons per unit volume with charge $-e$ and velocity $\boldsymbol{V}_e$, the Lorentz force density is,

$$\boldsymbol{f}_L = q_i n_i\left[\boldsymbol{E} + \frac{\boldsymbol{V}_i\times\boldsymbol{B}}{c}\right] - e n_e\left[\boldsymbol{E} + \frac{\boldsymbol{V}_e\times\boldsymbol{B}}{c}\right] = \rho_e\boldsymbol{E} + \frac{\boldsymbol{J}\times\boldsymbol{B}}{c}. \tag{9}$$

Here the charge density $\rho_e = (q_i n_i - e n_e)$ and the current density $\boldsymbol{J} = (q_i n_i \boldsymbol{V}_i - e n_e \boldsymbol{V}_e)$. Suppose we compare the electric and magnetic parts of the Lorentz force for a highly conducting fluid. We have for such a fluid $\boldsymbol{E} \approx -(\boldsymbol{V}\times\boldsymbol{B})/c$ from Ohms law. Using Gauss's law to relate ρ_e to $\boldsymbol{E}$ and Ampere's law to calculate $\boldsymbol{J}$ in terms of $\boldsymbol{B}$ (neglecting the displacement current as above), we then have

$$\frac{|\rho_e\boldsymbol{E}|}{|(\boldsymbol{J}\times\boldsymbol{B}/c)|} \sim \frac{V^2}{c^2} \ll 1, \tag{10}$$

where the last inequality holds for non-relativistic velocities and we have assumed similar scales of variation for the $\boldsymbol{E}$ and $\boldsymbol{B}$. Therefore, for highly conducting fluid moving with non-relativistic velocities, the part of the Lorentz force due to the electric field is negligible compared to the magnetic part. We therefore neglect it for most part of this book.

The momentum equation is then just the ordinary Navier-Stokes equation in fluid dynamics supplemented by the Lorentz force, $\boldsymbol{J}\times\boldsymbol{B}/c$, i.e.

$$\rho\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t} = -\boldsymbol{\nabla}p + \frac{\boldsymbol{J}\times\boldsymbol{B}}{c} + \boldsymbol{f} + \boldsymbol{F}_{\text{visc}}, \tag{11}$$

where $\boldsymbol{V}$ is the ordinary bulk velocity of the gas, ρ is the density, p is the pressure, $\boldsymbol{F}_{\rm visc}$ is the viscous force, and $\boldsymbol{f}$ subsumes all other body forces acting on the gas, including gravity and, in a rotating system also the Coriolis and centrifugal forces. (We use an upper case $\boldsymbol{V}$, because later on we shall use a lower case $\boldsymbol{v}$ for the fluctuating component of the velocity.) The Lorentz force can then also be written purely in terms of $\boldsymbol{B}$ by using Ampere's law to eliminate $\boldsymbol{J}$. We get,

$$\boldsymbol{f}_L = \frac{\boldsymbol{J}\times\boldsymbol{B}}{c} = \frac{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi} = -\underbrace{\boldsymbol{\nabla}\left(\frac{\boldsymbol{B}^2}{8\pi}\right)}_{\text{pressure}} + \underbrace{\frac{(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B}}{4\pi}}_{\text{tension}}, \tag{12}$$

where in the latter equality we have split the Lorentz force into a component, which is due to the gradient of a magnetic pressure, and one which a tension component due to variations of the field along itself.

The equations of MHD involve supplementing Eq. (11) by the continuity equation,

$$\frac{\partial\rho}{\partial t} = -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{V}), \tag{13}$$

an equation of state, $p = p(\rho, e)$, an energy equation for the internal energy e, and the induction equation giving the evolution of the magnetic field.

An important quantity is the adiabatic sound speed, $c_{\rm s}$, defined as $c_{\rm s}^2 = (\partial p/\partial\rho)_s$, evaluated at constant entropy s. For a perfect gas with constant ratio γ of specific heats ($\gamma = 5/3$ for a monatomic gas) we have $c_{\rm s}^2 = \gamma p/\rho$. When the flow speed is much smaller than the sound speed, i.e. when the average Mach number $\mathrm{Ma} = \langle\boldsymbol{V}^2/c_{\rm s}^2\rangle^{1/2}$ is much smaller than unity, the assumption of incompressibility can be made. In addition, if the density was uniform initially, then it can be taken as approximately uniform at all times, i.e. $\rho \approx \rho_0 = \text{const}$. For incompressible motions, Eq. (13) can be *replaced* by $\boldsymbol{\nabla}\cdot\boldsymbol{V} = 0$, and the momentum equation then simplifies to

$$\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t} = -\frac{1}{\rho_0}\boldsymbol{\nabla}p + \frac{\boldsymbol{J}\times\boldsymbol{B}}{\rho_0} + \boldsymbol{f} + \nu\nabla^2\boldsymbol{V}, \tag{14}$$

where ν is the kinematic viscosity and $\boldsymbol{f}$ is now an external body force per unit mass. The ratio $P_{\rm m} = \nu/\eta$ is the magnetic Prandtl number; see Eq. (30).

The assumption of incompressibility is a great simplification that is useful for many analytic considerations, but for numerical solutions this restriction is often not necessary. As long as the Mach number is small, say below 0.3, the weakly compressible case is believed to be equivalent to the incompressible case (cf. Dobler, Brandenburg, Yousef, 2003).

3 Magnetic flux freezing

The $\boldsymbol{V}\times\boldsymbol{B}$ term in Eq. (8) is usually referred to as the induction term. To clarify its role we expand its curl as

$$\boldsymbol{\nabla}\times(\boldsymbol{V}\times\boldsymbol{B}) = -\underbrace{(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{B}}_{\text{advection}} + \underbrace{(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{V}}_{\text{stretching}} - \underbrace{\boldsymbol{B}(\boldsymbol{\nabla}\cdot\boldsymbol{V})}_{\text{compression}}, \tag{15}$$

where we have used the fact that $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$. As a simple example, we consider the effect of a linear shear flow, $\boldsymbol{V} = (0, Sx, 0)$ on the initial field $\boldsymbol{B} = (B_0, 0, 0)$. The solution is $\boldsymbol{B} = (1, St, 0)B_0$, i.e. the field component in the direction of the flow grows linearly in time.

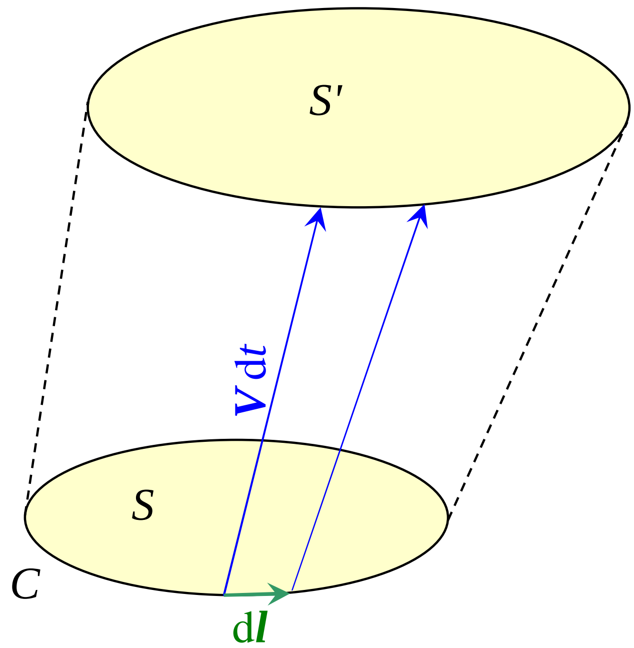

Figure 1: The surface S enclosed by the curve C is carried by fluid motion to the surface S' after a time dt. The flux through this surface Φ is frozen into the fluid for a perfectly conducting fluid.

The net induction term more generally implies that the magnetic flux through a surface moving with the fluid remains constant in the high-conductivity limit. Consider a surface S, bounded by a curve C, moving with the fluid, as shown in Fig. 1. Note that the surface S need not lie in a plane. Suppose we define the magnetic flux through this surface, $\Phi = \int_S \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S}$. Then after a time d$t$ the change in flux is given by

$$\Delta\Phi = \int_{S'} \boldsymbol{B}(t+\mathrm{d}t)\cdot \mathrm{d}\boldsymbol{S} - \int_S \boldsymbol{B}(t)\cdot \mathrm{d}\boldsymbol{S}. \tag{16}$$

Applying $\int \boldsymbol{\nabla}\cdot\boldsymbol{B}\,\mathrm{d}V = 0$ at time $t+\mathrm{d}t$, to the 'tube'-like volume swept up by the moving surface S, shown in Fig. 1, we also have

$$\int_{S'} \boldsymbol{B}(t+\mathrm{d}t)\cdot \mathrm{d}\boldsymbol{S} = \int_S \boldsymbol{B}(t+\mathrm{d}t)\cdot \mathrm{d}\boldsymbol{S} - \oint_C \boldsymbol{B}(t+\mathrm{d}t)\cdot(\mathrm{d}\boldsymbol{l}\times\boldsymbol{V}\mathrm{d}t), \tag{17}$$

where C is the curve bounding the surface S, and d$\boldsymbol{l}$ is the line element along C. (In the last term, to linear order in dt, it does not matter whether we take the integral over the curve C or C'.) Using the above condition in Eq. (16), we obtain

$$\Delta\Phi = \int_S [\boldsymbol{B}(t+\mathrm{d}t) - \boldsymbol{B}(t)]\cdot \mathrm{d}\boldsymbol{S} - \oint_C \boldsymbol{B}(t+\mathrm{d}t)\cdot(\mathrm{d}\boldsymbol{l}\times\boldsymbol{V})\mathrm{d}t. \tag{18}$$

Taking the limit of $\mathrm{d}t\to 0$, and noting that $\boldsymbol{B}\cdot(\mathrm{d}\boldsymbol{l}\times\boldsymbol{V}) = (\boldsymbol{V}\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{l}$, we have

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \int_S \frac{\partial\boldsymbol{B}}{\partial t}\cdot\mathrm{d}\boldsymbol{S} - \oint_C (\boldsymbol{V}\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{l} = -\int_S (\nabla\times(\eta\boldsymbol{\nabla}\times\boldsymbol{B}))\cdot\mathrm{d}\boldsymbol{S}. \tag{19}$$

In the second equality we have used $\oint_C (\boldsymbol{V}\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{l} = \int_S \boldsymbol{\nabla}\times(\boldsymbol{V}\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{S}$ together with the induction equation (8). One can see that, when $\eta\to 0$, $\mathrm{d}\Phi/\mathrm{d}t\to 0$ and so Φ is constant.

Now suppose we consider a small segment of a thin flux tube of length l and cross-section A, in a highly conducting fluid. Then, as the fluid moves about, conservation of flux implies

BA is constant, and conservation of mass implies ρAl is constant, where ρ is the local density. So $B \propto \rho l$. For a nearly incompressible fluid, or a flow with small changes in ρ, one will obtain $B \propto l$. Any shearing motion which increases l will also amplify B; an increase in l leading to a decrease in A (because of incompressibility) and hence an increase in B (due to flux freezing). This effect, also obtained in our discussion of stretching above, will play a crucial role in all scenarios involving dynamo generation of magnetic fields.

The concept of flux freezing can also be derived from the elegant Cauchy solution of the induction equation with zero diffusion. This solution is of use in several contexts and so we describe it briefly below. In the case $\eta = 0$, the $\boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{B})$ term in Eq. (8) can be expanded to give

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{V} - \boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{V}), \tag{20}$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \boldsymbol{V} \cdot \boldsymbol{\nabla}$ is the Lagrangian derivative. If we eliminate the $\boldsymbol{\nabla} \cdot \boldsymbol{V}$ term using the continuity equation for the fluid,

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}), \tag{21}$$

where ρ is the fluid density, then we can write

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \frac{\boldsymbol{B}}{\rho} \cdot \boldsymbol{\nabla}\boldsymbol{V}. \tag{22}$$

Suppose we describe the evolution of a fluid element by giving its trajectory as $\boldsymbol{x}(\boldsymbol{x}_0, t)$, where $\boldsymbol{x}_0$ is its location at an initial time t_0. Consider further the evolution of two infinitesimally separated fluid elements, A and B, which, at an initial time $t = t_0$, are located at $\boldsymbol{x}_0$ and $\boldsymbol{x}_0 + \delta\boldsymbol{x}_0$, respectively. The subsequent location of these fluid elements will be, say, $\boldsymbol{x}_A = \boldsymbol{x}(\boldsymbol{x}_0, t)$ and $\boldsymbol{x}_B = \boldsymbol{x}(\boldsymbol{x}_0 + \delta\boldsymbol{x}_0, t)$ and their separation is $\boldsymbol{x}_B - \boldsymbol{x}_A = \delta\boldsymbol{x}(\boldsymbol{x}_0, t)$. Since the velocity of the fluid particles will be $\boldsymbol{V}(\boldsymbol{x}_A)$ and $\boldsymbol{V}(\boldsymbol{x}_A) + \delta\boldsymbol{x} \cdot \boldsymbol{\nabla}\boldsymbol{V}$, after a time δt, the separation of the two fluid particles will change by $\delta t\, \delta\boldsymbol{x} \cdot \boldsymbol{\nabla}\boldsymbol{V}$. The separation vector therefore evolves as

$$\frac{\mathrm{D}\delta\boldsymbol{x}}{\mathrm{D}t} = \delta\boldsymbol{x} \cdot \boldsymbol{\nabla}\boldsymbol{V}, \tag{23}$$

which is an evolution equation identical to that satisfied by $\boldsymbol{B}/\rho$. So, if initially, at time $t = t_0$, the fluid particles were on a given magnetic field line with $(\boldsymbol{B}/\rho)(\boldsymbol{x}_0, t_0) = c_0 \delta\boldsymbol{x}(t_0) = c_0 \delta\boldsymbol{x}_0$, where c_0 is a constant, then for all times we will have $\boldsymbol{B}/\rho = c_0 \delta\boldsymbol{x}$. In other words, 'if two infinitesimally close fluid particles are on the same line of force at any time, then they will always be on the same line of force, and the value of $\boldsymbol{B}/\rho$ will be proportional to the distance between the particles' (Landau and Lifshitz ..) Further, since $\delta x_i(\boldsymbol{x}_0, t) = \mathsf{G}_{ij} \delta x_{0j}$, where $\mathsf{G}_{ij} = \partial x_i / \partial x_{0j}$, we can also write

$$B_i(\boldsymbol{x}, t) = \rho c_0 \delta x_i = \frac{\mathsf{G}_{ij}(\boldsymbol{x}_0, t)}{\det \mathsf{G}} B_{0j}(\boldsymbol{x}_0), \tag{24}$$

where we have used the fact that $\rho(\boldsymbol{x}, t)/\rho(\boldsymbol{x}_0, t_0) = (\det \mathsf{G})^{-1}$. We will use this Cauchy solution later.

4 Dissipation in space plasmas

4.1 Resistivity and viscosity

Plasmas are usually far from ideal and will have finite resistivity (or conductivity) and viscosity. We first describe a simple physical picture for the conductivity in a plasma. The force due to an electric field $\boldsymbol{E}$ accelerates electrons relative to the ions; but they cannot move freely due to friction with the ionic fluid, caused by electron–ion collisions. They acquire a 'terminal' relative velocity $\mathbf{U}$ with respect to the ions, obtained by balancing the Lorentz force with friction. This velocity can also be estimated as follows. Assume that electrons move freely for about an electron–ion collision time τ_{ei}, after which their velocity becomes again randomized. Electrons of charge e and mass m_e in free motion during the time τ_{ei} acquire from the action of an electric field $\boldsymbol{E}$ an ordered speed $\mathbf{U} \sim \tau_{ei} e\boldsymbol{E}/m_e$. This corresponds to a current density $\boldsymbol{J} \sim en_e\mathbf{U} \sim (n_e e^2 \tau_{ei}/m)\boldsymbol{E}$ and hence leads to $\sigma \sim n_e e^2 \tau_{ei}/m_e$.

The electron–ion collision time scale (which determines σ) can also be estimated as follows. For a strong collision between an electron and an ion one needs an impact parameter b which satisfies the condition $Ze^2/b > m_e v^2$. This gives a cross section for strong scattering of $\sigma_t \sim \pi b^2$. Since the Coulomb force is a long range force, the larger number of random weak scatterings add up to give an extra 'Coulomb logarithm' correction to make $\sigma_t \sim \pi(Ze^2/mv^2)^2 \ln\Lambda$, where $\ln\Lambda$ is in the range between 5 and 20. The corresponding mean free time between collisions is

$$\tau_{ei} \sim \frac{1}{n_i \sigma_t v} \sim \frac{(k_{\rm B}T)^{3/2} m_e^{1/2}}{\pi Z e^4 n_e \ln\Lambda}, \tag{25}$$

where we have used the fact that $m_e v^2 \sim k_{\rm B}T$ and $Zn_i = n_e$. Hence we obtain the estimate

$$\sigma \sim \frac{(k_{\rm B}T)^{3/2}}{m_e^{1/2} \pi Z e^2 \ln\Lambda}, \tag{26}$$

where most importantly the dependence on the electron density has canceled out. A more exact calculation can be found, for example in Lifshitz and Pitaevskii (1993, Eq. 44.11), and gives an extra factor of $4(2/\pi)^{1/2}$ multiplying the above result. The above argument has ignored collisions between electrons themselves, and treated the plasma as a 'lorentzian plasma'. Electron–electron collisions further reduce the conductivity by a certain factor ranging from about 0.582 for $Z = 1$ to 1 for $Z \to \infty$ [see Table 5.1 and Eqs. (5)–(37) in Spitzer (1956), and leads to a diffusivity, in cgs units, of $\eta = c^2/(4\pi\sigma)$ given by

$$\eta = 10^4 \left(\frac{T}{10^6\,{\rm K}}\right)^{-3/2} \left(\frac{\ln\Lambda}{20}\right)\ {\rm cm^2\,s^{-1}}. \tag{27}$$

As noted above, the resistivity is independent of density, and is also inversely proportional to the temperature (larger temperatures implying larger mean free time between collisions, larger conductivity and hence smaller resistivity).

The corresponding expression for the kinematic viscosity ν is quite different. Simple kinetic theory arguments give $\nu \sim v_{\rm t} l_i$, where l_i is the mean free path of the particles which dominate the momentum transport and $v_{\rm t}$ is their random velocity. For a fully ionized gas the ions dominate the momentum transport, and their mean free path $l_i \sim (n_i\sigma_i)^{-1}$, with the cross-section σ_i, is determined again by the ion–ion 'Coulomb' interaction. From a reasoning very similar to the above for electron–ion collisions, we have $\sigma_i \sim \pi(Z^2e^2/k_{\rm B}T)^2 \ln\Lambda$, where we have used $m_i v_{\rm t}^2 \sim k_{\rm B}T$. Substituting for $v_{\rm t}$ and l_i, this then gives

$$\nu \sim \frac{(k_{\rm B}T)^{5/2}}{n_i m_i^{1/2} \pi Z^4 e^4 \ln\Lambda}. \tag{28}$$

More accurate evaluation using the Landau collision integral gives a factor 0.4 for a hydrogen plasma, instead of $1/\pi$ in the above expression (see the end of Section 43 in vol. 10 of Lifshitz and Pitaevskii 1993). This gives numerically

$$\nu = 6.5 \times 10^{22} \left(\frac{T}{10^6\,{\rm K}}\right)^{5/2} \left(\frac{n_i}{{\rm cm^{-3}}}\right)^{-1} \left(\frac{\ln\Lambda}{20}\right)^{-1}\ {\rm cm^2\,s^{-1}}, \tag{29}$$

Table 1: Summary of some important parameters in various astrophysical settings. The values given should be understood as rough indications only. In particular, the applicability of Eq. (30) is questionable in some cases and has therefore not been used for protostellar discs. We have assumed $\ln\Lambda = 20$ in computing $R_{\rm m}$ and $P_{\rm m}$. CZ means convection zone AGNs are active galactic nuclei. Numbers in parenthesis indicate significant uncertainty due to other effects.

	T [K]	ρ [g cm^{-3}]	$P_{\rm m}$	$u_{\rm rms}$ [cm s^{-1}]	L [cm]	$R_{\rm m}$
Solar CZ (upper part)	10^4	10^{-6}	10^{-7}	10^6	10^8	10^6
Solar CZ (lower part)	10^6	10^{-1}	10^{-4}	10^4	10^{10}	10^9
Protostellar discs	10^3	10^{-10}	10^{-8}	10^5	10^{12}	10
AGN discs	10^7	10^{-5}	10^4	10^5	10^9	10^{11}
Galaxy	10^4	10^{-24}	(10^{11})	10^6	10^{20}	(10^{18})
Galaxy clusters	10^8	10^{-26}	(10^{29})	10^8	10^{23}	(10^{29})

so the magnetic Prandtl number is

$$P_{\rm m} \equiv \frac{\nu}{\eta} = 4\times 10^{11}\left(\frac{T}{10^4\,{\rm K}}\right)^4\left(\frac{\rho}{10^{-24}\,{\rm g\,cm^{-3}}}\right)^{-1}\left(\frac{\ln\Lambda}{10}\right)^{-2}. \tag{30}$$

Thus, in the galaxy, using $T = 10^4\,{\rm K}$ and $\rho = 10^{-24}\,{\rm g\,cm^{-3}}$, $\ln\Lambda \sim 10$, this formula gives $P_{\rm m} = 4\times 10^{11}$. Applied to the sun and other stars ($T\sim 10^6\,{\rm K}$, $\rho\sim 0.1\,{\rm g\,cm^{-3}}$) the magnetic Prandtl number is much less than unity. The reason $P_{\rm m}$ is so large in galaxies is mostly because of the very long mean free path caused by the low density. For galaxy clusters, the temperature of the gas is even larger and the density smaller, making the medium much more viscous and having even larger $P_{\rm m}$.

In protostellar discs, on the other hand, the gas is mostly neutral with low temperatures. In this case, the electrical conductivity is given by $\sigma = n_e e^2\tau_{en}/m_e$, where τ_{en} is the rate of collisions between electrons and neutral particles.

In Table 1 we summarize typical values of temperature and density in different astrophysical settings and calculate the corresponding values of $P_{\rm m}$. Here we also give rough estimates of typical rms velocities, $u_{\rm rms}$, and eddy scales, L, which allow us to calculate the magnetic Reynolds number as

$$R_{\rm m} = u_{\rm rms}/(\eta k_{\rm f}), \tag{31}$$

where $k_{\rm f} = 2\pi/L$. This number characterizes the relative importance of magnetic induction relative to magnetic diffusion. A similar number is the fluid Reynolds number, ${\rm Re} = R_{\rm m}/P_{\rm m}$, which characterizes the relative importance of inertial forces to viscous forces. (We emphasize that in the above table, Reynolds numbers are defined based on the inverse wavenumber; our values may therefore be 2π times smaller that those by other authors. The present definition is a natural one in simulations where one forces power at a particular wavenumber around $k_{\rm f}$.)

4.2 The effect of ambipolar drift

In a partially ionized medium the magnetic field evolution is governed by the induction equation (8), but with $\boldsymbol{V}$ replaced by the velocity of the ionic component of the fluid, $\boldsymbol{v}_i$. The ions experience the Lorentz force due to the magnetic field. This will cause them to drift with respect to the neutral component of the fluid. If the ion-neutral collisions are sufficiently frequent, one can assume that the Lorentz force on the ions is balanced by their friction with the neutrals. Under this approximation, the Euler equation for the ions reduces to

$$\rho_i \nu_{in} (\boldsymbol{v}_i - \boldsymbol{v}_n) = \boldsymbol{J} \times \boldsymbol{B} \quad \text{(strong coupling approximation)}, \tag{32}$$

where ρ_i is the mass density of ions, ν_{in} the ion-neutral collision frequency and $\boldsymbol{v}_n$ the velocity of the neutral particles. For gas with nearly primordial composition and temperature $\sim 10^4\,\mathrm{K}$, one gets the estimate of $\rho_i \nu_{in} = n_i \rho_n \langle \sigma v \rangle_{\mathrm{eff}}$, with $\langle \sigma v \rangle_{\mathrm{eff}} \sim 4 \times 10^{-9}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$, in cgs units. Here, n_i is the number density of ions and ρ_n the mass density of neutrals.

In a weakly ionized gas, the bulk velocity is dominated by the neutrals, and Eq. (32) substituted into the induction equation Eq. (8) then leads to a modified induction equation,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times \left[(\boldsymbol{V} + a \boldsymbol{J} \times \boldsymbol{B}) \times \boldsymbol{B} - \eta \boldsymbol{J} \right], \tag{33}$$

where $a = (\rho_i \nu_{in})^{-1}$. The modification is therefore an addition of an extra drift velocity, proportional to the Lorentz force. One usually refers to this drift velocity as ambipolar drift (and sometimes as ambipolar diffusion) in the astrophysical community (cf. Mestel (1999) for a more detailed discussion).

Ambipolar drift is important in the magnetic field evolution in protostars, and also in the neutral component of the galactic gas. In the classical (non-turbulent) picture of star formation, ambipolar diffusion regulates a slow infall of the gas, which was originally magnetically supported (Mestel 1999)

5 Energetics

Important insight can be gained by considering the magnetic energy equation. By taking the dot product of Eq. (8) with $\boldsymbol{B}/(8\pi)$, using the vector identity $\boldsymbol{B} \cdot \boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{B}) + \boldsymbol{E} \cdot \boldsymbol{\nabla} \times \boldsymbol{B}$ and integrating over the volume V, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \frac{\boldsymbol{B}^2}{8\pi}\,\mathrm{d}V = -\int_V \boldsymbol{V} \cdot \frac{(\boldsymbol{J} \times \boldsymbol{B})}{c}\,\mathrm{d}V - \int_V \frac{\boldsymbol{J}^2}{\sigma}\,\mathrm{d}V - \oint_{\partial V} \frac{c}{4\pi} \boldsymbol{E} \times \boldsymbol{B}\,\mathrm{d}\boldsymbol{S}. \tag{34}$$

This equation shows that the magnetic energy can be increased by doing work against the Lorentz force, provided this term exceeds resistive losses (second term) or losses through the surface (Poynting flux, last term). Likewise, by taking the dot product of Eq. (11) with $\rho \boldsymbol{V}$ and integrating, one arrives at the kinetic energy equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_V \tfrac{1}{2} \rho \boldsymbol{V}^2\,\mathrm{d}V = &+ \int_V p \boldsymbol{\nabla} \cdot \boldsymbol{V}\,\mathrm{d}V + \int_V \boldsymbol{V} \cdot \frac{(\boldsymbol{J} \times \boldsymbol{B})}{c}\,\mathrm{d}V \\ &+ \int_V \rho \boldsymbol{V} \cdot \boldsymbol{g}\,\mathrm{d}V - \int_V 2\nu\rho [S]^2\,\mathrm{d}V, \end{aligned} \tag{35}$$

where $\mathsf{S}_{ij} = \frac{1}{2}(V_{i,j} + V_{j,i}) - \frac{1}{3}\delta_{ij} V_{k,k}$ is the traceless rate of strain tensor, and commas denote derivatives. In deriving Eq. (35) we have assumed stress-free boundary conditions, so there are no surface terms and no kinetic energy is lost through the boundaries. Equations Eq. (34) and Eq. (35) show that the generation of magnetic energy goes at the expense of kinetic energy, without loss of net energy.

6 Two-fluid approximation

The simplest generalization of the one-fluid model is to consider the electrons and ions as separate fluids which are interacting with each other through collisions. This two-fluid model is also essential for deriving the general form of Ohm's law and for describing battery effects, that generate fields *ab initio* from zero initial field. We therefore briefly consider it below.

For simplicity assume that the ions have one charge, and in fact they are just protons. That is the plasma is purely ionized hydrogen. It is straightforward to generalize these considerations to several species of ions. For our purpose it suffices first to follow the simple treatment of Spitzer (1956) where we take an isotropic pressure, leave out non-ideal terms, and also adopt a simple form for the collision term between electrons and protons. The equations of motion for the electron and proton fluids may then be written as

$$\frac{D_e \boldsymbol{v}_e}{Dt} = -\frac{\boldsymbol{\nabla} p_e}{n_e m_e} - \frac{e}{m_e}\left(\boldsymbol{E} + \boldsymbol{v}_e \times \boldsymbol{B}\right) - \boldsymbol{\nabla}\phi_g - \frac{(\boldsymbol{v}_e - \boldsymbol{v}_p)}{\tau_{ei}}, \tag{36}$$

$$\frac{D_i \boldsymbol{v}_i}{Dt} = -\frac{\boldsymbol{\nabla} p_i}{n_i m_i} + \frac{e}{m_i}\left(\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{B}\right) - \boldsymbol{\nabla}\phi_g + \frac{m_e n_e}{m_i n_i}\frac{(\boldsymbol{v}_e - \boldsymbol{v}_i)}{\tau_{ei}}. \tag{37}$$

Here $D_j\boldsymbol{v}_j/Dt = \partial\boldsymbol{v}_j/\partial t + \boldsymbol{v}_j \cdot \boldsymbol{\nabla}\boldsymbol{v}_j$ and we have included the forces due to the pressure gradient, gravity, electromagnetic fields and electron–proton collisions. Further, m_j, n_j, u_j, p_j are respectively the mass, number density, velocity, and the partial pressure of electrons ($j = e$) and protons ($j = i$), ϕ_g is the gravitational potential, and τ_{ei} is the e–i collision time scale. One can also write down a similar equation for the neutral component n. Adding the e, i and n equations we can recover the standard MHD Euler equation.

More interesting in the present context is the difference between the electron and proton fluid equations. Using the approximation $m_e/m_i \ll 1$, this gives the generalized Ohms law; see the book by Spitzer (1956), and Eqs. (2)–(12) therein,

$$\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{B} = -\frac{\boldsymbol{\nabla} p_e}{e n_e} + \frac{\boldsymbol{J}}{\sigma} + \frac{1}{e n_e}\boldsymbol{J} \times \boldsymbol{B} + \frac{m_e}{e^2}\frac{\partial}{\partial t}\left(\frac{\boldsymbol{J}}{n_e}\right), \tag{38}$$

where $\boldsymbol{J} = (e n_i \boldsymbol{v}_i - e n_e \boldsymbol{v}_e)$ is the current density and

$$\sigma = \frac{n_e e^2 \tau_{ei}}{m_e} \tag{39}$$

is the electrical conductivity. [If $n_e \neq n_i$, additional terms arise on the RHS of Eq. (38) with $\boldsymbol{J}$ in Eq. (38) replaced by $-e\boldsymbol{v}_i(n_e - n_i)$. These terms can usually be neglected since $(n_e - n_i)/n_e \ll 1$. Also negligible are the effects of nonlinear terms $\propto u_j^2$.]

The first term on the RHS of Eq. (38), representing the effects of the electron pressure gradient, is the 'Biermann battery' term. It provides the source term for the thermally generated electromagnetic fields (Biermann 1950). If $\boldsymbol{\nabla} p_{\rm e}/e n_{\rm e}$ can be written as the gradient of some scalar function, then only an electrostatic field is induced by the pressure gradient. On the other hand, if this term has a curl then a magnetic field can grow. The next two terms on the RHS of Eq. (38) are the usual Ohmic term $\boldsymbol{J}/\sigma$ and the Hall electric field $\boldsymbol{J} \times \boldsymbol{B}/(n_{\rm e} e)$, which arises due to a non-vanishing Lorentz force. Its ratio to the Ohmic term is of order $\omega_{\rm e}\tau_{\rm ei}$, where $\omega_{\rm e} = eB/m_{\rm e}$ is the electron gyro-frequency. The last term on the RHS is the inertial term, which can be neglected if the macroscopic time scales are large compared to the plasma oscillation periods.

From the generalized Ohm's law one can formally solve for the current components parallel and perpendicular to $\boldsymbol{B}$ (cf. the book by Mestel 1999). Defining an 'equivalent electric field'

$$\boldsymbol{E}' = \frac{\boldsymbol{J}}{\sigma} + \frac{\boldsymbol{J} \times \boldsymbol{B}}{e n_e}, \tag{40}$$

one can rewrite the generalized Ohms law as

$$\boldsymbol{J} = \sigma \boldsymbol{E}'_{\parallel} + \sigma_1 \boldsymbol{E}'_{\perp} + \sigma_2 \frac{\boldsymbol{B} \times \boldsymbol{E}'}{B}, \tag{41}$$

where

$$\sigma_1 = \frac{\sigma}{1+(\omega_e\tau_{ei})^2}, \quad \sigma_2 = \frac{(\omega_e\tau_{ei})\sigma}{1+(\omega_e\tau_{ei})^2}. \tag{42}$$

The conductivity becomes increasingly anisotropic as $\omega_{\mathrm{e}}\tau_{\mathrm{ei}}$ increases. Assuming numerical values appropriate to the galactic interstellar medium, say, we have

$$\omega_e\tau_{ei} \approx 4\times 10^5 \left(\frac{B}{1\,\mu\mathrm{G}}\right)\left(\frac{T}{10^4 K}\right)^{3/2}\left(\frac{n_e}{1\,\mathrm{cm}^{-3}}\right)^{-1}\left(\frac{\ln\Lambda}{20}\right)^{-1}. \tag{43}$$

The Hall effect and the anisotropy in conductivity are therefore important in the galactic interstellar medium and in the cluster gas with high temperatures $T \sim 10^8\,\mathrm{K}$ and low densities $n_e \sim 10^{-2}\,\mathrm{cm}^{-3}$. Of course, in absolute terms, neither the resistivity nor the Hall field are important in these systems, compared to the inductive electric field or turbulent diffusion. For the solar convection zone with $n_e \sim 10^{18} - 10^{23}\,\mathrm{cm}^{-3}$, $\omega_e\tau_{ei} \ll 1$, even for fairly strong magnetic fields. On the other hand, in neutron stars, the presence of strong magnetic fields $B \sim 10^{13}\,\mathrm{G}$, could make the Hall term important, especially in their outer regions, where there are also strong density gradients.

A strong magnetic field also suppresses other transport phenomena like the viscosity and thermal conduction perpendicular to the field. These effects are again likely to be important in rarefied and hot plasmas such as in galaxy clusters.

7 Magnetic helicity

Magnetic helicity plays an important role in dynamo theory. We therefore give here a brief account of its properties. Magnetic helicity is the volume integral

$$H = \int_V \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}V \tag{44}$$

over a closed or periodic volume V. By a closed volume we mean one in which the magnetic field lines are fully contained, so the field has no component normal to the boundary, i.e. $\boldsymbol{B}\cdot\hat{\boldsymbol{n}} = 0$. The volume V could also be an unbounded volume with the fields falling off sufficiently rapidly at spatial infinity. In these particular cases, H is invariant under the gauge transformation Eq. (4), because

$$H' = \int_V \boldsymbol{A}'\cdot\boldsymbol{B}'\,\mathrm{d}V = H + \int_V \boldsymbol{\nabla}\Lambda\cdot\boldsymbol{B}\,\mathrm{d}V = H + \oint_{\partial V}\Lambda\boldsymbol{B}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S = H, \tag{45}$$

where $\hat{\boldsymbol{n}}$ is the unit outward normal to the closed surface ∂V. Here we have made use of $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$.

Magnetic helicity has a simple topological interpretation in terms of the linkage and twist of isolated (non-overlapping) flux tubes. For example consider the magnetic helicity for an interlocked, but untwisted, pair of thin flux tubes as shown in Fig. 2, with Φ_1 and Φ_2 being the fluxes in the tubes around C_1 and C_2 respectively. For this configuration of flux tubes, $\boldsymbol{B}\,\mathrm{d}^3x$ can be replaced by $\Phi_1\mathrm{d}\boldsymbol{l}$ on C_1 and $\Phi_2\mathrm{d}\boldsymbol{l}$ on C_2. The net helicity is then given by the sum

$$H = \Phi_1\oint_{C_1}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l} + \Phi_2\oint_{C_2}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}, = 2\Phi_1\Phi_2 \tag{46}$$

where we have used Stokes theorem to transform

$$\oint_{C_1}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l} = \int_{S(C_1)}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} \equiv \Phi_2, \qquad \oint_{C_2}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l} = \int_{S(C_2)}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} \equiv \Phi_1. \tag{47}$$

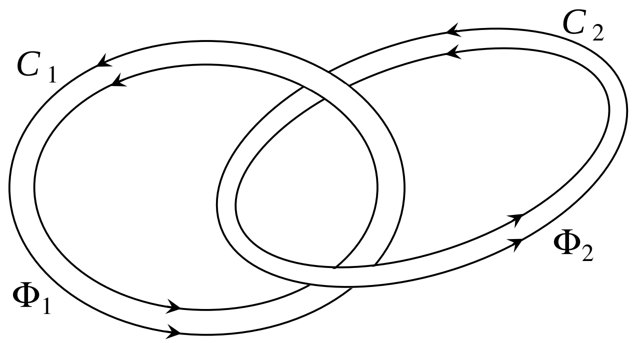

Figure 2: Two flux tubes with fluxes Φ_1 and Φ_2 are linked in such a way that they have a helicity $H = +2\Phi_1\Phi_2$. Interchanging the direction of the field in one of the two rings changes the sign of H.

For a general pair of non-overlapping thin flux tubes, the helicity is given by $H = \pm 2\Phi_1\Phi_2$; the sign of H depending on the relative orientation of the two tubes (Moffatt 1978).

The evolution equation for H can be derived from Faraday's law and its uncurled version for $\boldsymbol{A}$, Eq. (3), so we have

$$\begin{aligned}\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{A}\cdot\boldsymbol{B}) &= (-\boldsymbol{E}+\boldsymbol{\nabla}\phi)\cdot\boldsymbol{B}+\boldsymbol{A}\cdot(-\boldsymbol{\nabla}\times\boldsymbol{E}) \\ &= -2\boldsymbol{E}\cdot\boldsymbol{B}+\boldsymbol{\nabla}\cdot(\phi\boldsymbol{B}+\boldsymbol{A}\times\boldsymbol{E}). \end{aligned} \tag{48}$$

Integrating this over the volume V, the magnetic helicity satisfies the evolution equation

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -2c\int_V \boldsymbol{E}\cdot\boldsymbol{B}\mathrm{d}V + c\oint_{\partial V}(\boldsymbol{A}\times\boldsymbol{E}+\phi\boldsymbol{B})\cdot\hat{\boldsymbol{n}}\mathrm{d}S = -2\eta\frac{4\pi C}{c}, \tag{49}$$

where $C = \int_V \boldsymbol{J}\cdot\boldsymbol{B}\ \mathrm{d}V$ is the current helicity. Here we have used Ohm's law, $\boldsymbol{E} = -(\boldsymbol{V}\times\boldsymbol{B})/c+\boldsymbol{J}/\sigma$, in the volume integral and we have assumed that the surface integral vanishes for closed domains.

In the non-resistive case, $\eta = 0$, the magnetic helicity is conserved, i.e. $\mathrm{d}H/\mathrm{d}t = 0$. However, this does not guarantee conservation of H in the limit $\eta \to 0$, because the current helicity, $\int \boldsymbol{J}\cdot\boldsymbol{B}\,\mathrm{d}V$, may in principle still become large. For example, the Ohmic dissipation rate of magnetic energy $Q_{\mathrm{Joule}} \equiv (4\pi/c^2)\int \eta\mathbf{J}^2\mathrm{d}V$ can be finite and balance magnetic energy input by motions, even when $\eta \to 0$. This is because small enough scales develop in the field (current sheets) where the current density increases with decreasing η as $\propto \eta^{-1/2}$ as $\eta \to 0$, whilst the rms magnetic field strength, B_{rms}, remains essentially independent of η. Even in this case, however, the rate of magnetic helicity dissipation *decreases* with η, with an upper bound to the dissipation rate $\propto \eta^{+1/2} \to 0$, as $\eta \to 0$. Thus, under many astrophysical conditions where R_{m} is large (η small), the magnetic helicity H, is almost independent of time, even when the magnetic energy is dissipated at finite rates. This robust conservation of magnetic helicity is an important constraint on the nonlinear evolution of dynamos and will play a crucial role below in determining how large scale turbulent dynamos saturate. Indeed, it is also at the heart of Taylor relaxation in laboratory plasmas, where an initially unstable plasma relaxes to a stable 'force-free' state, dissipating energy, while nearly conserving magnetic helicity (Taylor 1974).

We also note the very important fact that the fluid velocity completely drops out from the helicity evolution equation Eq. (49), since $(\boldsymbol{V}\times\boldsymbol{B})\cdot\boldsymbol{B} = 0$. Therefore, any change in the nature of the fluid velocity, for example due to turbulence (turbulent diffusion), the Hall effect, or ambipolar drift (see below), does not affect magnetic helicity conservation. We will discuss in more detail the concept of turbulent diffusion in a later section, and its role in dissipating the mean magnetic field. However, such turbulent magnetic diffusion does *not* dissipate the net magnetic helicity. This property is crucial for understanding why, in spite of the destructive properties of turbulence, large scale spatio-temporal coherence can emerge if there is helicity in the system.

Although the Hall electric field does not alter the volume dissipation and/or generation of helicity, the battery term however can in principle contribute to helicity dissipation and/or generation. But this contribution is generally expected to be small. To see this, rewrite this contribution to helicity generation, say $(\mathrm{d}H/\mathrm{d}t)_{\mathrm{Batt}}$, using $p_e = n_e k_{\mathrm{B}} T_e$, as

$$\frac{1}{c}\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)_{\mathrm{Batt}} = 2\int \frac{\boldsymbol{\nabla} p_e}{e n_e}\cdot \boldsymbol{B}\,\mathrm{d}V = -2\int \frac{\ln n_e}{e}\,\boldsymbol{B}\cdot\boldsymbol{\nabla}(k_{\mathrm{B}}T_e)\,\mathrm{d}V, \tag{50}$$

where k_{B} is the Boltzmann constant, and the integration is assumed to extend over a closed or periodic domain, so there are no surface terms.[1] We see from Eq. (50) that generation/dissipation of helicity can occur only if there are temperature gradients parallel to the magnetic field. Such parallel gradients are in general very small due to fast electron flow along field lines. We will see below that the battery effect can provide a small but finite seed field; this can also be accompanied by the generation of a small but finite magnetic helicity.

We should point out that it is also possible to define magnetic helicity as linkages of flux analogous to the Gauss linking formula for linkages of curves. Given the magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$ the magnetic helicity is given

$$H_{\mathrm{G}} = \frac{1}{4\pi}\int\int \boldsymbol{B}(\boldsymbol{x})\cdot\left[\boldsymbol{B}(\boldsymbol{y})\times\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right]\mathrm{d}^3x\;\mathrm{d}^3y, \tag{51}$$

where both integrations extend over the full volume. Suppose we define an auxiliary field

$$\boldsymbol{A}_C(\boldsymbol{x}) = \frac{1}{4\pi}\int \boldsymbol{B}(\boldsymbol{y})\times\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}\,\mathrm{d}^3y, \tag{52}$$

then this field satisfies $\boldsymbol{\nabla}\times\boldsymbol{A}_C = \boldsymbol{B}$, and $\boldsymbol{\nabla}\cdot\boldsymbol{A}_C = 0$, and one can write $H_{\mathrm{G}} = \int \boldsymbol{A}_C\cdot\boldsymbol{b}\,\mathrm{d}^3x$. This is the origin of the textbook definition of magnetic helicity in what is known as the Coulomb gauge for the vector potential. Provided the field is closed over the integration volume, this definition can be applied in any other gauge and $H \equiv H_G$.

8 Magnetohydrodynamic waves

In fluids without a magnetic field small perturbations propagate isotropically as sound waves. On introducing a magetic field into the system the number of possible wave modes increases and wave propagation also becomes anisotropic, depending on the direction of the magnetic field. Consider the simple case where the unperturbed fluid is at rest and has uniform density (ρ_0), pressure (p_0) and magnetic field ($\boldsymbol{B}_0$). Examine the evolution of small perturbations: $\boldsymbol{v}$ in velocity, ρ_1 in density, p_1 in pressure and $\boldsymbol{b}$ in the magnetic field about this base state. The continuity, momentum and the induction equations can be linearized to give,

$$\begin{aligned}
\frac{\partial\rho_1}{\partial t} &= -\rho_0\boldsymbol{\nabla}\cdot\boldsymbol{v}, \qquad p_1 = c_{\mathrm{s}}^2\rho_1 \\
\frac{\partial\boldsymbol{v}}{\partial t} &= -\boldsymbol{\nabla}\left[\frac{p_1}{\rho_0}+\frac{\boldsymbol{B}_0\cdot\boldsymbol{b}}{4\pi\rho_0}\right]+\frac{\boldsymbol{B}_0\cdot\boldsymbol{\nabla}\boldsymbol{b}}{4\pi\rho_0}, \\
\frac{\partial\boldsymbol{b}}{\partial t} &= \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}_0) = \boldsymbol{B}_0\cdot\boldsymbol{\nabla}\boldsymbol{v}-\boldsymbol{B}_0\boldsymbol{\nabla}\cdot\boldsymbol{v}.
\end{aligned} \tag{53}$$

[1] Note that n_{e} in the above equation can be divided by an arbitrary constant density, say n_0 to make the argument of the log term dimensionless since, on integrating by parts, $\int \ln(n_0)\boldsymbol{B}\cdot\boldsymbol{\nabla}(k_{\mathrm{B}}T_{\mathrm{e}})\,\mathrm{d}V = 0$.

We have adopted here adiabatic pressure perturbations and c_s is the adiabatic sound speed as before. We look for solutions where all perturbed quantities are expanded in Fourier modes in the form, $f(\boldsymbol{x},t) = \widehat{f} \exp \mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)$ and use Eq. (53) to eliminate all other variable except $\widehat{\boldsymbol{v}}$ to get,

$$\omega^2 \widehat{\boldsymbol{v}} = (c_s^2 + V_A^2)(\boldsymbol{k}\cdot\widehat{\boldsymbol{v}})\boldsymbol{k} + (\boldsymbol{k}\cdot\boldsymbol{V}_A)\left[(\boldsymbol{k}\cdot\boldsymbol{V}_A)\widehat{\boldsymbol{v}} - (\boldsymbol{V}_A\cdot\widehat{\boldsymbol{v}})\boldsymbol{k} - (\boldsymbol{k}\cdot\widehat{\boldsymbol{v}})\boldsymbol{V}_A\right]. \tag{54}$$

Here we have defined the Alfvén velocity $\boldsymbol{V}_A = \boldsymbol{B}_0/\sqrt{4\pi\rho_0}$ and $V_A = |\boldsymbol{V}_A|$. These are 3 linear homogeneous equations for the three components of $\widehat{\boldsymbol{v}}$ and so will give 3 possible independent modes of oscillations. To elucidate their properties we proceed as follows:

First consider the case where $\widehat{\boldsymbol{v}} \perp \boldsymbol{k}$ and $\widehat{\boldsymbol{v}} \perp \boldsymbol{B}_0$, which implies $\boldsymbol{k}\cdot\widehat{\boldsymbol{v}} = 0$ and $\boldsymbol{V}_A\cdot\widehat{\boldsymbol{v}} = 0$. Then from Eq. (54) such modes have the dispersion relation,

$$\omega^2 = (\boldsymbol{k}\cdot\boldsymbol{V}_A)^2 = (kV_A\cos\theta)^2, \tag{55}$$

where $\theta = \hat{\boldsymbol{B}}_0\cdot\hat{\boldsymbol{k}}$ is the angle between the propagation direction $\hat{\boldsymbol{k}}$ and the zeroth order magnetic field $\boldsymbol{B}_0$. For this mode, the phase velocity $\omega/k = \pm V_A\cos\theta$ and the group velocity $\boldsymbol{\nabla}_{\boldsymbol{k}}\omega = \pm\boldsymbol{V}_A$. Thus they propagate along the magnetic field with the Alfvén velocity. These are called Alfvén waves. The Alfvén wave is incompressible since $\widehat{\boldsymbol{v}}\cdot\boldsymbol{k} = 0$, $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$ and thus $\rho_1 = 0 = p_1$. The Fourier transform of the perturbed induction equation in Eq. (53) gives $\widehat{\boldsymbol{v}} \parallel \widehat{\boldsymbol{b}}$ and so $\widehat{\boldsymbol{b}}\cdot\boldsymbol{B}_0 = 0$, from which it follows that the magnetic pressure perturbation also vanishes to linear order. The restoring force for these oscillation comes from the tension component of the Lorentz force.

To find the other two modes consider the component of $\widehat{\boldsymbol{v}}$ parallel to $\boldsymbol{k}$ and the one parallel to $\boldsymbol{V}_A$. Taking dot products of Eq. (54) with $\boldsymbol{k}$ and with $\boldsymbol{V}_A$ gives two equations in the form

$$\begin{pmatrix} \omega^2 - k^2(c_s^2 + V_A^2) & k^2(\boldsymbol{V}_A\cdot\boldsymbol{k}) \\ -(\boldsymbol{V}_A\cdot\boldsymbol{k})c_s^2 & \omega^2 \end{pmatrix} \begin{pmatrix} \boldsymbol{k}\cdot\widehat{\boldsymbol{v}} \\ \boldsymbol{V}_A\cdot\widehat{\boldsymbol{v}} \end{pmatrix} = 0 \tag{56}$$

For a non-trivial solution one must demand that the determinant of the (2×2) matrix in Eq. (56) vanish, which gives the dispersion relations

$$\left(\frac{\omega}{k}\right)^2 = \tfrac{1}{2}(V_A^2 + c_s^2) \pm \left[\tfrac{1}{4}(V_A^2 + c_s^2)^2 - V_A^2 c_s^2\cos^2\theta\right]^{1/2}. \tag{57}$$

The $\pm$ signs in the above dispersion relation corresponds to the fast and slow magnetosonic waves.

9 Magneto-Rotational Instability

Consider an accretion (or galactic) disk where in the zeroth order state, the fluid goes around a central mass on circular orbits, with gravity providing the centripetal force. The pressure forces are required to maintain vertical equilibrium, but are sub-dominant for the radial balance of forces. We wish to consider a simple example where the Magneto-Rotational Instability (MRI) operates. Let the disk be threaded by a uniform field in the vertical direction $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$. Consider perturbations to this base state, which are incompressible, 'local' (in radius) and axisymmetric, with all perturbed variables only dependent on z. That is we consider perturbations in cylindrical-polar co-ordinates, of the form,

$$\boldsymbol{V} = r\Omega(r)\hat{\boldsymbol{\phi}} + \boldsymbol{v}(z,t); \qquad \boldsymbol{B} = B_0\hat{\boldsymbol{z}} + \boldsymbol{b}(z,t); \qquad \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0. \tag{58}$$

We will also neglect the dissipative effects for the present; their effects will be discussed later below. To linear order the momentum equation simplifies to,

$$\frac{\partial \boldsymbol{v}}{\partial t} - 2\Omega v_\phi\hat{\boldsymbol{r}} + 2v_r\hat{\boldsymbol{\phi}}[\Omega + \tfrac{1}{2}r\Omega'] = -\frac{\boldsymbol{\nabla}p_1}{\rho_0} + \frac{B_0}{4\pi\rho_0}\frac{\partial \boldsymbol{b}}{\partial z} \tag{59}$$

Here $\Omega' = d\Omega/dr$, p_1 is the perturbed pressure and we have taken the perturbed density to be zero (because of incompressibility). Using $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$ and since all perturbed variables have only z-dependence, the above equation gives $\nabla^2 p_1 = 0$, $p_1 =$ constant and so $\boldsymbol{\nabla}p_1 = 0$. The perturbed ideal induction equation is

$$\frac{\partial \boldsymbol{b}}{\partial t} = B_0\frac{\partial \boldsymbol{v}}{\partial z} + b_r(r\Omega')\hat{\boldsymbol{\phi}} \tag{60}$$

We can also take consistently $b_z = 0 = v_z$ in the above equations, and so the velocity and magnetic perturbations are purely in the horizontal directions. In component form the perturbed momentum and induction equations become,

$$\begin{aligned}
\frac{\partial v_r}{\partial t} - 2\Omega v_\phi &= \frac{B_0}{4\pi\rho_0}\frac{\partial b_r}{\partial z}, \\
\frac{\partial v_\phi}{\partial t} + 2v_r[\Omega + \tfrac{1}{2}r\Omega'] &= \frac{B_0}{4\pi\rho_0}\frac{\partial b_\phi}{\partial z}, \\
\frac{\partial b_r}{\partial t} &= B_0\frac{\partial v_r}{\partial z}, \\
\frac{\partial b_\phi}{\partial t} &= B_0\frac{\partial v_\phi}{\partial z} + b_r(r\Omega').
\end{aligned} \tag{61}$$

We again look for solutions where all perturbed quantities are expanded in Fourier modes in the form, $f(\boldsymbol{x},t) = \widehat{f}\exp\mathrm{i}(kz - \omega t)$ and use Eq. (61) to eliminate the magnetic field perturbations. We are then left with the following two equations written in matrix form,

$$\begin{pmatrix} (\omega^2 - k^2V_A^2) & -2\mathrm{i}\Omega\omega \\ [\mathrm{i}\omega(2\Omega + r\Omega') + k^2V_A^2(r\Omega')/(\mathrm{i}\omega)] & (\omega^2 - k^2V_A^2) \end{pmatrix}\begin{pmatrix} \widehat{v}_r \\ \widehat{v}_\phi \end{pmatrix} = 0. \tag{62}$$

Nontrivial solutions obtain only when the determinant of the (2×2) matrix in Eq. (62) vanishes. This gives the condition

$$[\omega^2 - k^2V_A^2]^2 - 2[\omega^2 - k^2V_A^2]\kappa^2 - 4k^2V_A^2\Omega^2 = 0 \tag{63}$$

where we have defined the epicyclic frequency, $\kappa = \sqrt{(4\Omega^2 + 2r\Omega\Omega')}$. The dispersion relation for the perturbations is then given by

$$\omega_\pm^2 = k^2V_A^2 + \tfrac{1}{2}\kappa^2 \pm \left[\tfrac{1}{4}\kappa^4 + 4k^2V_A^2\Omega^2\right]^{1/2}. \tag{64}$$

One can therefore potentially get an instability with $\omega_-^2 < 0$. It is clear that both roots in Eq. (64) are real and their sum is positive and so they both can not be negative. The condition for one of them to be negative is therefore $\omega_+^2\omega_-^2 < 0$. This translates into the condition

$$k^2V_A^2 = \frac{k^2B_0^2}{4\pi\rho_0} < -2r\Omega\frac{d\Omega}{dr}, \qquad (65)$$

that is the angular velocity must *decrease* with radius for instability. This instability is called the Magnetorotational instability.

A number of points are worth noting: If $\Omega = 0$, the dispersion relation in Eq. (64) reduces to $\omega = \pm kV_A$; these modes represent simply the Alfvén wave and the incompressible limit of the slow wave modes. If on the other hand, $B_0 = 0$, then one has simply "epicyclic" motion with frequency κ as is the case for the perturbed motion of stars in a galaxy. Also if one has rotation but no shear, i.e. $d\Omega/dr = 0$ then again the perturbations are stable. The MRI also obtains for an arbitrarily small value of B_0 in principle, but in practice, as we see below, is limited by the maximum value of k. We can also calculate the maximum value of the growth rate Γ_{max} by finding the value of ω where $d\omega_-/dk = 0$. This implies $k^2V_A^2 = \Omega^2 - \kappa^4/(16\Omega^2)$, and using this in the expression for ω_- gives $\Gamma_{max} = -(r/2)(d\Omega/dr) = 3\Omega/4$. Here the last equality is for Kepler rotation law which obtains for rotation around a point mass. One sees therefore that the instability grows very rapidly on the rotation rate of the disk. For every rotation this corresponds to a growth factor $\exp(3\pi/2) \sim 111$. For Kepler rotation, the wavenumber at which the maximum growth occurs is $k_{max} = \sqrt{15/16}(\Omega/V_A)$. Thus as $B_0 \to 0$, $V_A \to 0$ and $k_{max} \to \infty$; so for such weak fields diffusion and viscosity cannot be ignored.

The non-ideal case is simplest to treat for the case $\nu = \eta$, for which one simply replaces $-\mathrm{i}\omega$ by $-\mathrm{i}\omega + \eta k^2$ in the dispersion relation. This implies $\omega = \omega_{ideal} - \mathrm{i}\eta k^2$and so viscosity and resistivity simply damp all perturbations by a further factor of $\exp(-\eta k^2)$. Thus very small scales cannot substain MRI.

There is also a minimum $k = k_{min}$ allowed in a disk of finite thickness h given by $k_{min} \sim 1/h$. Using this in conjunction with the condition for instability given in Eq. (65) implies $v_A < \Omega/k_{min} \sim c_s$, since $h \sim c_s/\Omega$.

How weak a field can then lead to MRI? For MRI to operate one requires that atleast the growth rate $\Gamma > \eta k^2$. In the limit of small fields, or $kV_A \ll \Omega^2$, one can approximate the dispersion relation for ω_-^2 to get $\omega_-^2 \sim -k^2V_A^2$. Thus $\Gamma \sim kV_A$, and if one requires this to be larger than the damping rate ηk^2, one needs $V_A > \eta k$. The smallest value of the field will obtain for $k = k_{min} \sim h^{-1}$. One requires therefore fields strong enough such that $V_A > \eta/h$ for MRI to obtain. Thus the field satisfies the limits $\eta/h < V_A < c_s$ for MRI to operate. Note that even for a small field satisfying the above limit, with $V_A/c_s << 1$, one can still have rapid growth at the rotation rate for $k = k_{max} \sim (\Omega/V_A)$.

10 Batteries and seed magnetic fields

Note that $\boldsymbol{B} = 0$ is a perfectly valid solution of the induction equation (8), so no magnetic field would be generated if one were to start with zero magnetic field. The universe probably did not start with an initial magnetic field. One therefore needs some way of violating the induction equation to produce a cosmic battery effect, and to drive currents from a state with initially no current. There are a number of such battery mechanisms which have been suggested. Almost all of them lead to only weak fields, much weaker than the observed fields. Therefore, dynamo action due to a velocity field acting to exponentiate small seed fields efficiently, is needed to explain observed field strengths. We briefly comment on one cosmic battery, the Biermann battery.

The basic problem any battery has to address is how to produce finite currents from zero currents? Most astrophysical mechanisms use the fact that positively and negatively charged particles in a charge-neutral universe, do not have identical properties. For example if one considered a gas of ionized hydrogen, then the electrons have a much smaller mass compared to protons. This means that for a given pressure gradient of the gas the electrons tend to be accelerated much more than the ions. This leads in general to an electric field, which couples back positive and negative charges. This is exactly the thermally generated field we found in deriving the generalized Ohm's law.

Taking the curl of Eq. (38), using Maxwell's equations (Faraday's and Ampere's law), and writing $p_e = n_e k_{\rm B} T$, where $k_{\rm B}$ is the Boltzmann constant, we obtain

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{B}) - \boldsymbol{\nabla} \times \eta (\boldsymbol{\nabla} \times \boldsymbol{B}) - \frac{c k_{\rm B}}{e} \frac{\boldsymbol{\nabla} n_e}{n_e} \times \boldsymbol{\nabla} T. \tag{66}$$

Here we have taken the velocity of the ionic component to be also nearly the bulk velocity in a completely ionized fluid, so we put $\boldsymbol{v}_i = \boldsymbol{V}$. We have neglected the Hall effect and inertial effects as they are generally very small for the fields one generates.

We see that over and above the usual flux freezing and diffusion terms we have a *source term* for the magnetic field evolution, even if the initial field were zero. This source term is nonzero if and only if the density and temperature gradients, $\boldsymbol{\nabla} n_e$ and $\boldsymbol{\nabla} T$, are not parallel to each other. The resulting battery effect, known as the Biermann battery, was first proposed as a mechanism for the thermal generation of stellar magnetic fields (Biermann, 1950; Mestel and Roxburgh (1962).

10.1 Seed field generation during re-ionization

In the cosmological context, the Biermann battery can also lead to the thermal generation of seed fields in cosmic ionization fronts (Subramanian, Narasimha and Chitre, 1994). These ionization fronts are produced when the first ultraviolet photon sources, like quasars, turn on to ionize the intergalactic medium (IGM). The temperature gradient in a cosmic ionization front is normal to the front. However, a component to the density gradient can arise in a different direction, if the ionization front is sweeping across arbitrarily laid down density fluctuations. Such density fluctuations, associated with protogalaxies/clusters, in general have no correlation to the source of the ionizing photons. Therefore, their density gradients are not parallel to the temperature gradient associated with the ionization front. The resulting thermally generated electric field has a curl, and magnetic fields on galactic scales can grow. After compression during galaxy formation, they turn out to have a strength $B \sim 3 \times 10^{-20}$ G. A similar effect was considered earlier in the context of generating fields in the interstellar medium in Lazarian (1992). (This mechanism also has analogues in some laboratory experiments, when laser generated plasmas interact with their surroundings (Stamper, 1971, 1975). Indeed, our estimate for the generated field is very similar to the estimate in Stamper (1971). This field by itself falls far short of the observed microgauss strength fields in galaxies, but it can provide a seed field, coherent on galactic scales, for a dynamo. Indeed the whole of the IGM is seeded with magnetic fields of small strength but coherent on megaparsec scales.

This scenario has in fact been confirmed in detailed numerical simulations of IGM reionization (Gnedin, Ferrara and Zweibel, 2000), where it was found that the breakout of ionization fronts from protogalaxies and their propagation through the high-density neutral filaments that are part of the cosmic web, and that both generate magnetic fields. The field strengths increase further due to gas compression occurring as cosmic structures form.

The magnetic field at a redshift $z \sim 5$ closely traces the gas density, and is highly ordered on megaparsec scales. Gnedin et al.(2000) found a mean mass-weighted field strength of $B \sim 10^{-19}$ G in their simulation box.

10.2 Seed fields from structure formation shocks

The Biermann battery has also been shown to generate both vorticity and magnetic fields in oblique cosmological shocks which arise during cosmological structure formation (Kulsrud et. al., 1997, Davis and Widrow, 2000). In fact, Kulsrud et al. (1997) point out that the well-known analogy between the induction equation and the vorticity equation (without Lorentz force) extends even to the case where a battery term is present. Suppose we assume that the gas is pure hydrogen, has a constant (in space) ionization fraction χ, and has the same temperature for electrons, protons and hydrogen, it follows that $p_e = \chi p/(1+\chi)$ and $n_e = \chi\rho/m_p$. Defining $\boldsymbol{\omega}_B = e\boldsymbol{B}/m_p$, the induction equation with the thermal battery term can then be written as

$$\frac{\partial \boldsymbol{\omega}_B}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{\omega}_B - \eta \boldsymbol{\nabla} \times \boldsymbol{\omega}_B) + \frac{\boldsymbol{\nabla} p \times \boldsymbol{\nabla} \rho}{\rho^2} \frac{1}{1+\chi}. \tag{67}$$

The last term, without the extra factor of $-(1+\chi)^{-1}$, corresponds to the baroclinic term in the equation for the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{V}$,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{\omega} - \nu \boldsymbol{\nabla} \times \boldsymbol{\omega}) - \frac{\boldsymbol{\nabla} p \times \boldsymbol{\nabla} \rho}{\rho^2}. \tag{68}$$

So, provided viscosity and magnetic diffusivity were negligible, both $\boldsymbol{\omega}_B(1+\chi)$ and $-\boldsymbol{\omega}$ satisfy the same equation. Furthermore, if they were both zero initially then, for subsequent times, we have $e\boldsymbol{B}/m_p = -\boldsymbol{\omega}/(1+\chi)$. Numerically, a value of $\omega \sim 10^{-15}\,\mathrm{s}^{-1}$ corresponds to a magnetic field of about $\sim 10^{-19}$ G.

11 References

These notes are extracted from a book in preparation by A. Shukurov, K. Subramanian and D. Sokoloff on the "The magnetic universe". Much of the material and the references are contained in a review by A. Brandenburg and K. Subramanian, "Astrophysicl magnetic fields and nonlinear dynamo theory", Physics Reports, volume 417, pages 1-209 (2005). Please use the notes for your personal use and do not circulate without prior permission.

A good reference for MRI is the review by S. A. Balbus and J. F. Hawley, "Instabilities and turbulence in accretion disks", Reviews of Modern Physics, Vol 70, , pg 1 (1998). There are also some good notes on the web by Gordon Ogilvie. The Book by Leon Mestel, on "Stellar Magnetism" Clarendon Press, Oxford, 1999, has good outline of basic MHD an also its application to stellar problems.