MHD Assignment Sheet

1. **Flux Freezing**

   Consider a constant initial magnetic field $\mathbf{B} = B_0\mathbf{i}$ along the x-direction in a plasma of negligible resistivity. Suppose a velocity field $\mathbf{v}(x) = v_0 \exp(-x^2)\mathbf{j}$ is switched at time $t = 0$. How does this magnetic field evolve with time. Make a sketch of the magnetic field lines some time after switching on the velocity field. What could happen at very long times?

2. Make a simple estimate of resistivity in a fully ionized plasma. Find the magnetic Reynolds numbers for the ISM of our galaxy, a cluster plasma, a conductor like copper of 1 m dimension.

3. Express the MHD equations in conservation form.

4. **More on helicity**

   Magnetic helicity is the volume integral

   $$H = \int_V \mathbf{A} \cdot \mathbf{B}\, dV$$

   over a closed or periodic volume V.

   - Show that H is invariant under the gauge transformation $\mathbf{A'} = \mathbf{A} + \nabla\Lambda$.
   - Consider an *untwisted flux tube* described by say, $\mathbf{B}(R, z) = B(R, z)\mathbf{e}_\phi$, where all the field lines lie on circles. Here $B(R, z) = constant$ within $(R - R_0)^2 + z^2 \leq a^2$ and vanishes for $(R - R_0)^2 + z^2 > a^2$. Show that the magnetic helicity of this "flux tube" is equal to zero.
   - Derive an evolution equation for the helicity and show that it is conserved in the ideal case, when the resistivity $\eta = 0$.
   - Show from helicity conservation that any single, closed, thin, untwisted and unknotted flux tube will also have zero helicity.
   - Consider a volume which contains two such flux tubes T_1 and T_2. let Φ_1 and Φ_2 be the magnetic fluxes associated with T_1 and T_2. Show that $H = \pm 2\Phi_1\Phi_2$ if the tubes are simply interlinked and $H = 0$ if the two tubes are not linked at all.

5. Consider a horizontal magnetic flux tube of radius a with a uniform magnetic field B pointing in the x- direction. This tube is embedded in an isothermal atmosphere in a constant gravitational field g pointing in the negative z-direction. The tube starts from rest at $z = z_0$ and then rises due to magnetic buoyancy. It also experiences a drag force per unit length given by $C_D v^2 a/2$, where v is its instantaneous velocity and C_D is a constant. Assume that the pressure scale height of the atmosphere $H \gg a$, so that the gas pressure and density ρ can be taken as constant for a significant period of time T. Estimate the asymptotic speed acquired by the tube.

6. **Friedrichs diagrams**

   The dispersion relation for magneto acoustic waves in a uniform medium are given by

   $$\omega^2 = k^2 v_A^2 \cos^2\theta$$

   $$\omega^4 - (c_s^2 + v_A^2)k^2\omega^2 + c_s^2 v_A^2 k^4 \cos^2\theta = 0$$

where c_s is the sound speed and v_A the Alfvén velocity and θ is the angle between $\mathbf{k}$ and $\mathbf{B}$. From the form of $\omega(k, \theta)$ calculate the components of the group velocity, parallel and perpendicular to the magnetic field. Sketch the phase and group velocity over the full range of θ. Treat the cases $c_s > v_A$ and $c_s < v_A$ separately.

7. **Nonlinear Alfvén waves**

   Show that the equations of ideal MHD in the case of an incompressible fluid of uniform density ρ can be written in the symmetrical form

$$\frac{\partial \mathbf{z}_\pm}{\partial t} + \mathbf{z}_\mp \cdot \nabla \mathbf{z}_\pm = -\nabla(p/\rho)$$

$$\nabla \cdot \mathbf{z}_\pm = 0$$

   where

$$\mathbf{z}_\pm = \mathbf{v} \pm \mathbf{v}_A$$

   are the *Elsasser variables*, $\mathbf{v}_A = \mathbf{B}/(4\pi\rho)^{1/2}$ is the vector Alfvén velocity. Consider a static zeroth order state in which the magnetic field and the pressure are uniform. Write down the exact equations (without linearizing) governing departures from this base state. Hence show that disturbances of arbitrary amplitude can propagate along the magnetic field lines in either direction without change of form. Think about why this non-linear wave does not steepen with time (unlike in the case of a sound wave).

8. **Generalized Ohm's law**

   Write down the equations of motion for the electron and proton fluids separately, including pressure forces, the Lorentz force, and a collision term coupling the two fluids. You can take a simple form for this collision term; assuming that the electrons feel a drag force per unit mass of $-(\mathbf{v}_e - \mathbf{v}_p)/\tau_{ei}$, where $\mathbf{v}_e$ and $\mathbf{v}_p$ are the electron fluid and proton fluid velocities, and τ_{ei} is the collision time for the electron fluid. Using the approximation $m_e/m_p \ll 1$, and neglecting nonlinear terms in the momentum equations, derive the generalized Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B}/c = -\frac{\nabla p_e}{e n_e} + \frac{\mathbf{J}}{\sigma} + \frac{1}{e n_e}\mathbf{J} \times \mathbf{B} + \frac{m_e}{e^2}\frac{\partial}{\partial t}\left(\frac{\mathbf{J}}{n_e}\right),$$

   where $\sigma = \frac{n_e e^2 \tau_{ei}}{m_e}$ is the conductivity of the plasma. Estimate the relative importance of the different terms.

9. Mechanical analogy for the MRI: Two particles of mass m are orbiting around a much more masive body of mass M. The two particles are connected by a spring that obeys Hooke's law. The masses are originally in Keplerian orbits around M. Suppose one gives a small perturbation to this motion, describe the subsequent dynamics.