1b-(An extremely short introduction to)
The Diffusion Equation

The Diffusion Equation: Motivation

- Diffusion is the spontaneous net movement of particles from an area of high concentration to an area of low concentration. Diffuse operates towards shallowing the gradient between two different concentration (Wikipedia).
- Diffusion is an everyday experience; a hot cup of tea distributes its thermal energy or an uncorked perfume bottle distributes its scent throughout a room.
- In astrophysics, examples of processes regulated by diffusion are:
  - Viscosity
  - Thermal Conduction
  - Magnetic Resistivity
  - Photon diffusion (very high optical depth)

The Diffusion Equation

- The prototype equation in 1-D is $$\boxed{\frac{\partial q}{\partial t} = \nu \frac{\partial^2 q}{\partial x^2}}$$

- ν depends on the physical process.

- The diffusion equation is parabolic. A 2nd order PDE of the form

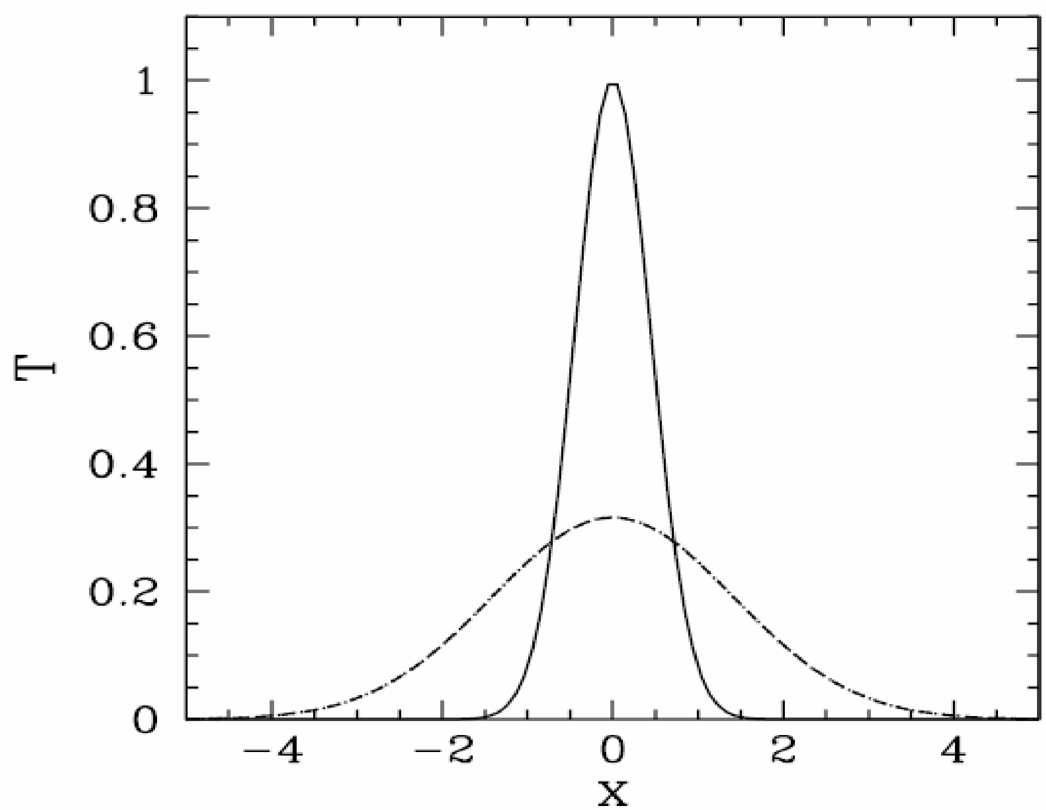
$$Aq_{tt} + 2Bq_{tx} + Cq_{xx} + Du_t + Eu_x + F = 0$$

  is parabolic if $M = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$ satisfies *det(M) = 0* .

- Parabolic equations are typical for dissipative processes.

The Diffusion Equation

- A typical example is the diffusion of a Gaussian profile:



- with analytical solution given by $q(x,t) = \sqrt{\frac{t_0}{t}} \exp\left(-\frac{x^2}{4\nu t}\right)$

Finite Difference Methods

- An interpretation in terms of characteristics is useless because there is only one characteristic. Any local perturbation influences the entire domain with decreasing magnitude at larger distances from the source of the perturbation.

- We will try to use again FTCS (forward in time, centered in space) discretization

$$\frac{\partial q}{\partial t} \approx \frac{q_i^{n+1} - q_i^n}{\Delta t}$$

$$\frac{\partial^2 q}{\partial x^2} \approx \frac{q_{i+1}^n - 2q_i^n + q_{i-1}^n}{\Delta x^2}$$

Finite Difference Methods

- Putting these approximation in the original PDE and solving for the solution at the next time step gives

$$\boxed{q_i^{n+1} = q_i^n - \nu \frac{\Delta t}{\Delta x^2} \left(q_{i+1}^n - 2q_i^n + q_{i-1}^n \right)}$$

- This scheme is explicit and it is 1st order accurate in time and 2nd order in space.

- The stability analysis (http://farside.ph.utexas.edu/teaching/329/329.html) shows that the explicit FTCS is stable under the Courant condition

$$\Delta t \leq \frac{\Delta x^2}{2\nu}$$

- If the grid is refined by a factor of 2, the time step will decrease accordingly by a factor of 4 !

Finite Difference Methods

- To overcome the time step limitation, diffusion problems are better solved using *implicit* schemes.
- In an implicit scheme, the spatial derivative is computed at time level *n+1*, instead of time level *n*:

$$\frac{\partial^2 q}{\partial x^2} \approx \frac{q_{i+1}^{n+1} - 2q_i^{n+1} + q_{i-1}^{n+1}}{\Delta x^2}$$

- This yields a the first-order implicit scheme

$$\boxed{q_i^{n+1} = q_i^n - \nu \frac{\Delta t}{\Delta x^2}\left(q_{i+1}^{n+1} - 2q_i^{n+1} + q_{i-1}^{n+1}\right)}$$

also known as the backward Euler or implicit Euler method.

- It is called “implicit” since the unknowns appear on both side of the equation.

Finite Difference Methods

- A stability analysis on the implicit scheme reveals, in fact, that there is no time step limitation: the backward Euler method is thus *unconditionally stable* for any time step *Δt* !

- However, the solution is more involved, since each equation becomes coupled with it neighbor:

$$q_i^{n+1} = q_i^n - \nu \frac{\Delta t}{\Delta x^2} \left(q_{i+1}^{n+1} - 2q_i^{n+1} + q_{i-1}^{n+1} \right) \quad \text{for} \quad i = 1, 2, \cdots, N_x$$

- This is a linear system of equations for all grid nodes *i* and can be cast as

$$\alpha q_{i+1}^{n+1} + (1 - 2\alpha)\, q_i^{n+1} + \alpha q_{i-1}^{n+1} = q_i^n \ , \quad \text{with} \quad \alpha = \frac{\nu \Delta t}{\Delta x^2}$$

Finite Difference Methods

- In matrix form this is equivalent to

$$A \cdot \{q\}^{n+1} = \{q\}^n$$

where A is a tri-diagonal matrix of the form

$$A = \begin{pmatrix} (1-2\alpha) & \alpha & & \\ \alpha & (1-2\alpha) & \alpha & \\ & \alpha & (1-2\alpha) & \alpha \\ & & & \ddots \end{pmatrix}$$

Summary

- Diffusion equations can be solved with either *explicit* or *implicit* differencing.
- The *explicit* FTCS is stable under a severe restriction: the square of the grid-spacing. This is a very unfavorable scaling, since a doubling of the spatial resolution requires a simultaneous reduction in the time-step by a factor of four in order to maintain numerical stability.

- Alternatively, *implicit* schemes do not suffer from this restriction and the time step can be arbitrarily large.
- However, this requires the inversion of tri-diagonal systems of equations of very high dimensionality. The situation gets even more involved in more than one dimension. This is the price one has to pay to obtain more efficient and stable types of discretization.