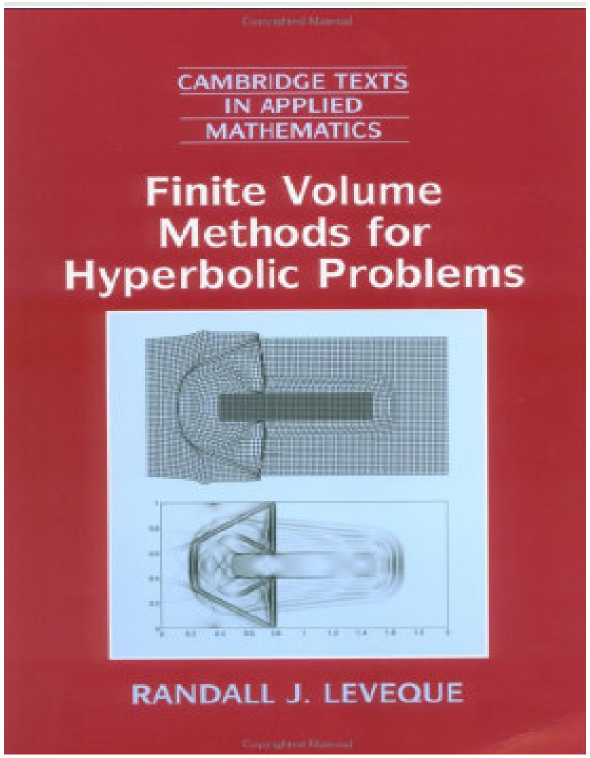
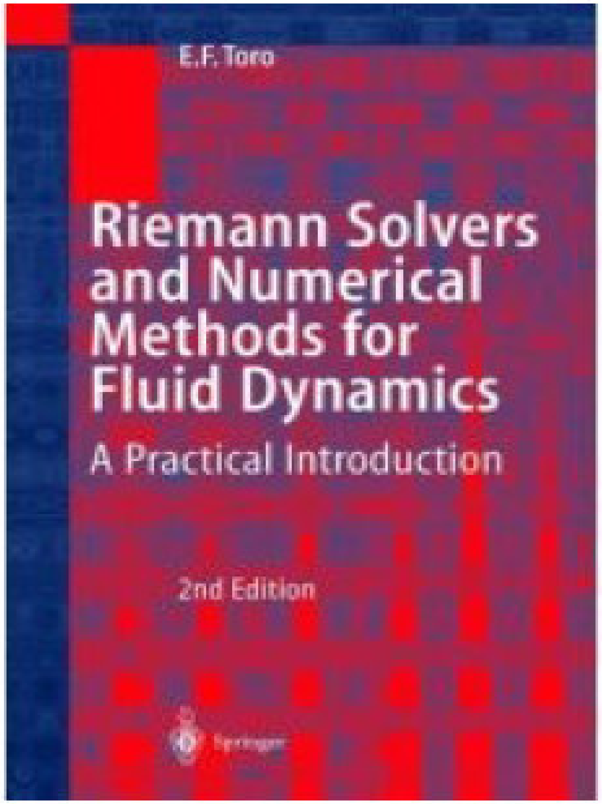
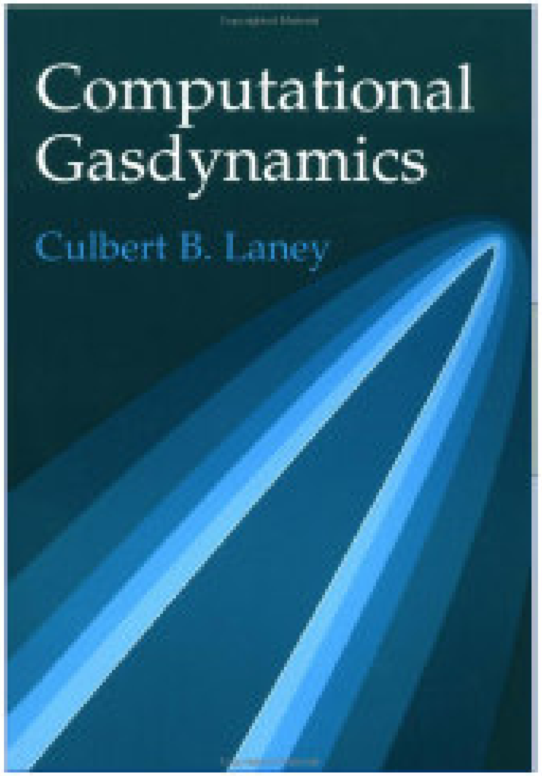
5-Nonlinear Systems: The Euler Equations

Textbooks & References







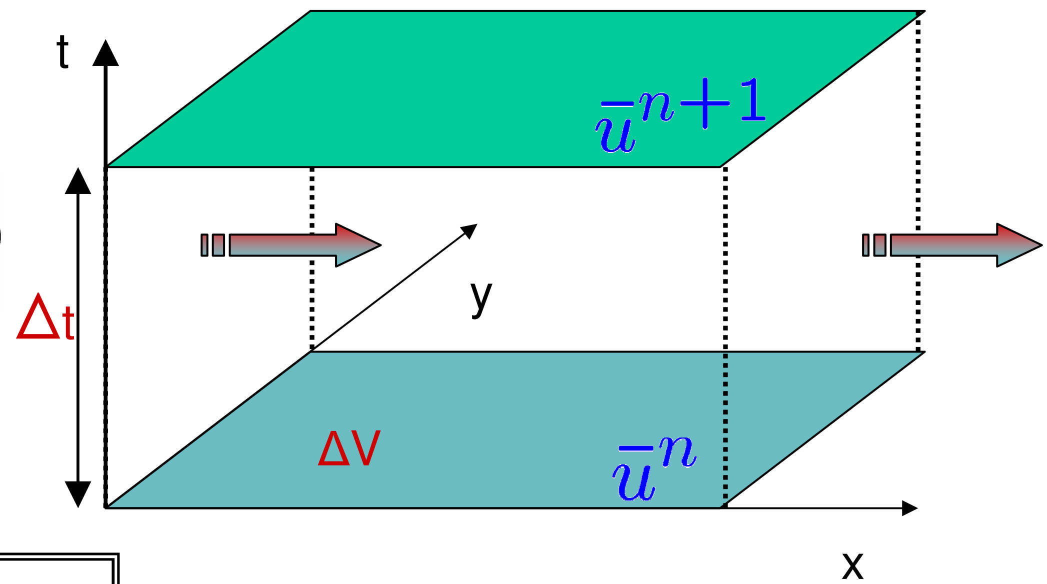
Nonlinear Systems

- Much of what is known about the numerical solution of hyperbolic systems of nonlinear equations comes from the results obtained in the linear case or simple nonlinear scalar equations.
- The key idea is to exploit the conservative form and assume the system can be locally "frozen" at each grid interface.
- However, this still requires the solution of the Riemann problem, which becomes increasingly difficult for complicated set of hyperbolic P.D.E.

Finite Volume Formulation

- Integral form of the equations:

$$\int dt \int dV \left\{ \frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}) \right\} = 0$$



$$\boxed{\frac{\bar{\mathbf{u}}^{n+1} - \bar{\mathbf{u}}^{n}}{\Delta t} + \frac{1}{\Delta V} \oint \tilde{\mathbf{F}} \cdot d\mathbf{S} = 0}$$

→ *"Finite volume"*

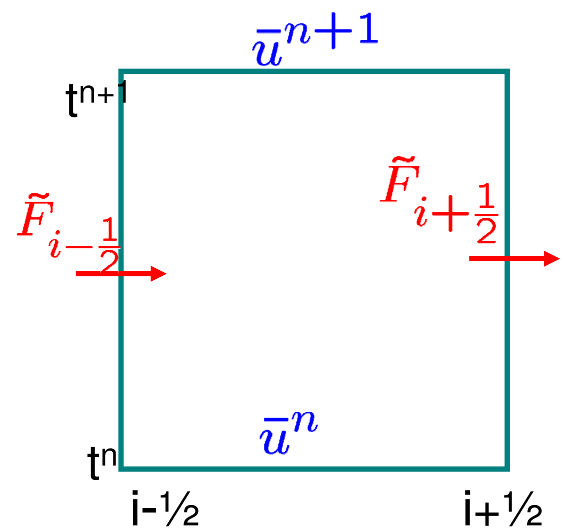
- Evolve volume averages instead of point values

$$\bar{\mathbf{u}}(t) = \frac{1}{\Delta V} \int \mathbf{u}(x,t) dV \; , \quad \tilde{\mathbf{F}}(x) = \frac{1}{\Delta t} \int \mathbf{F}(\mathbf{u}(x,t)) dt$$

One Dimension

- In 1-D only,

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{\Delta x}\left(\tilde{F}_{i+\frac{1}{2}} - \tilde{F}_{i-\frac{1}{2}}\right)$$



- Written in terms of averaged quantities

$$\bar{u}_i(t) = \frac{1}{\Delta x_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t)\,dx$$

$$\tilde{F}_{i+\frac{1}{2}} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} F(u(x_{i+\frac{1}{2}},t))\,dt$$

numerics here !

- Exact: no approximations introduced yet !

Flux Computation ⇔ Riemann Problem

- Computation of the flux requires the (exact or approximate) solution of the Riemann problem at zone edges…

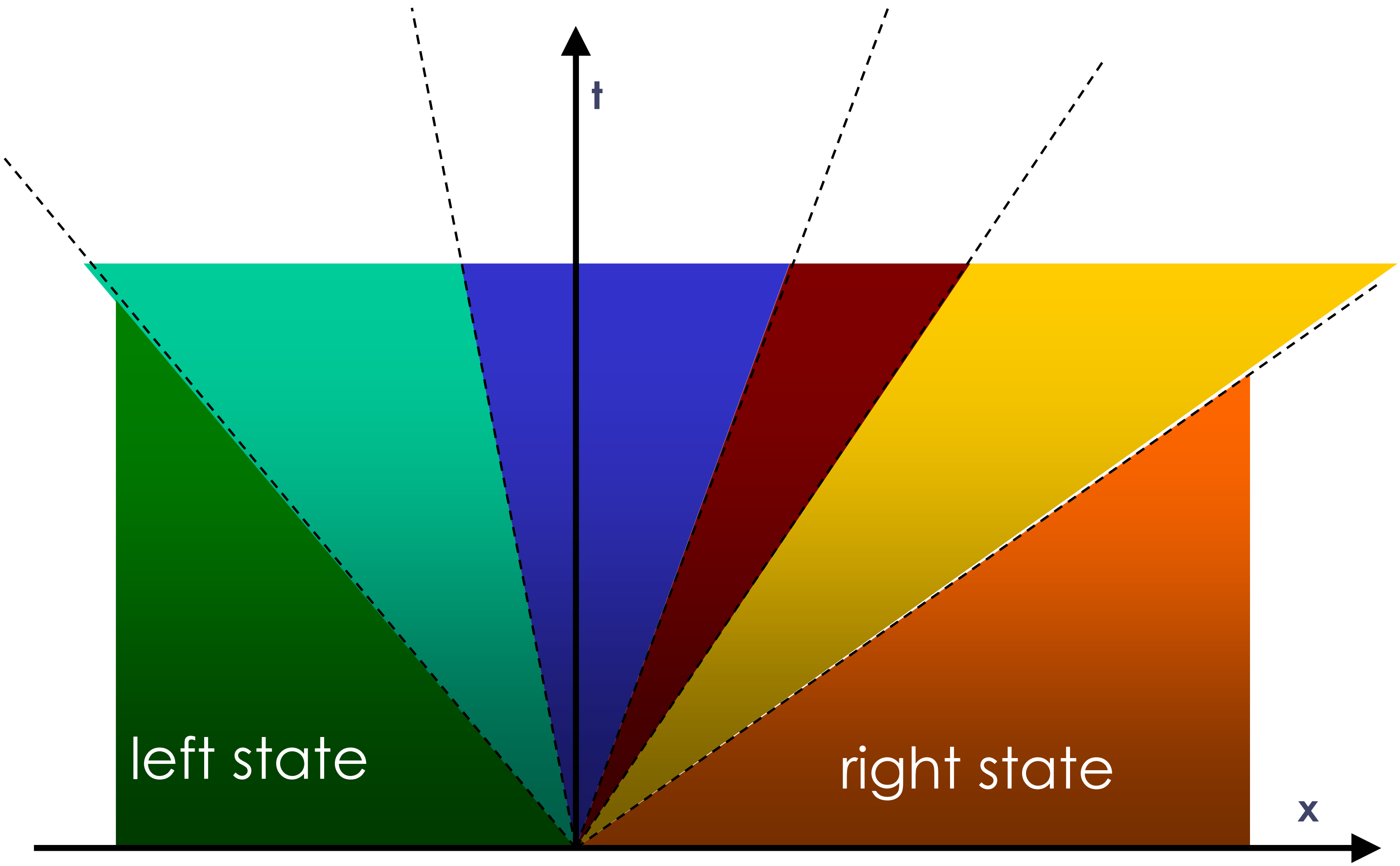
- Riemann Problem: given left and right states at a zone edge

$$U(x,0) = \begin{cases} U^L_{i+\frac{1}{2}} & \text{for} \quad x < x_{i+\frac{1}{2}} \\ U^R_{i+\frac{1}{2}} & \text{for} \quad x > x_{i+\frac{1}{2}} \end{cases}$$

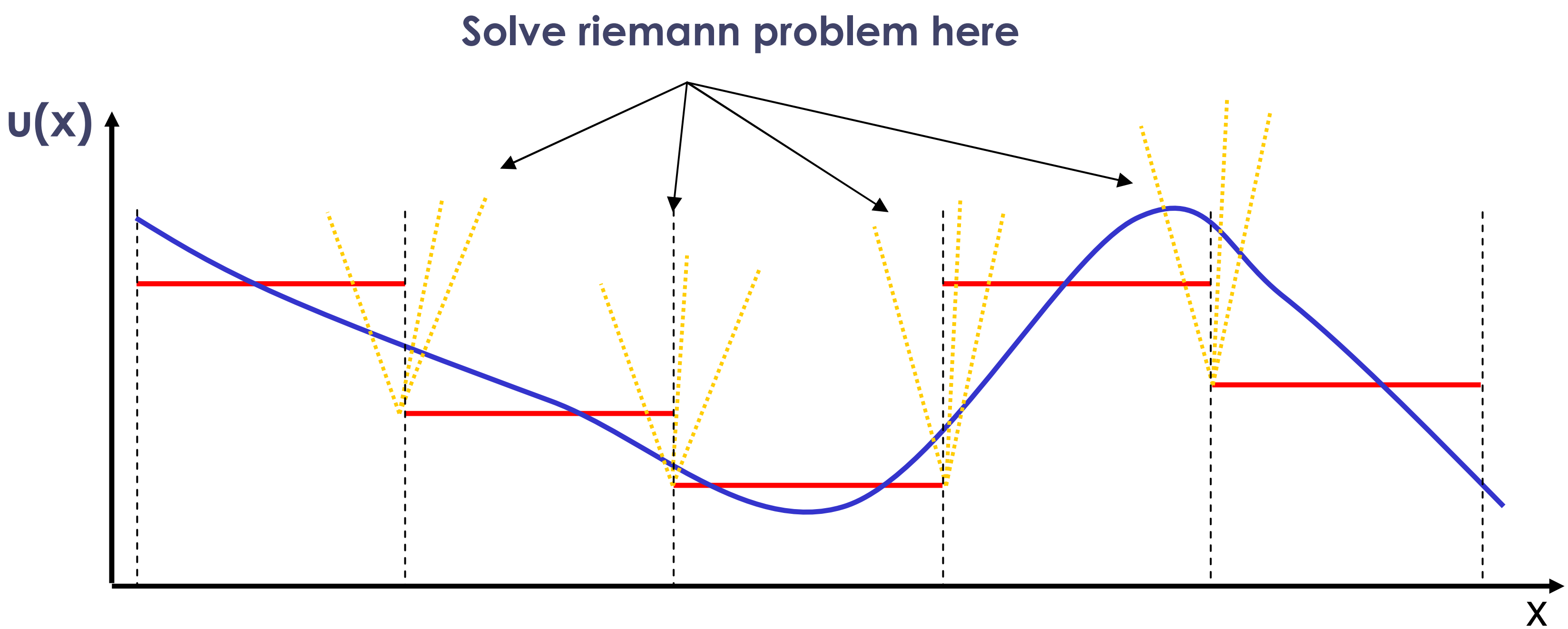
- → what is $U(x_{i+\frac{1}{2}}, t > 0) = ?$

- answer: the solution depends on the form of the conservation law

Flux Computation ⇔ Riemann Problem



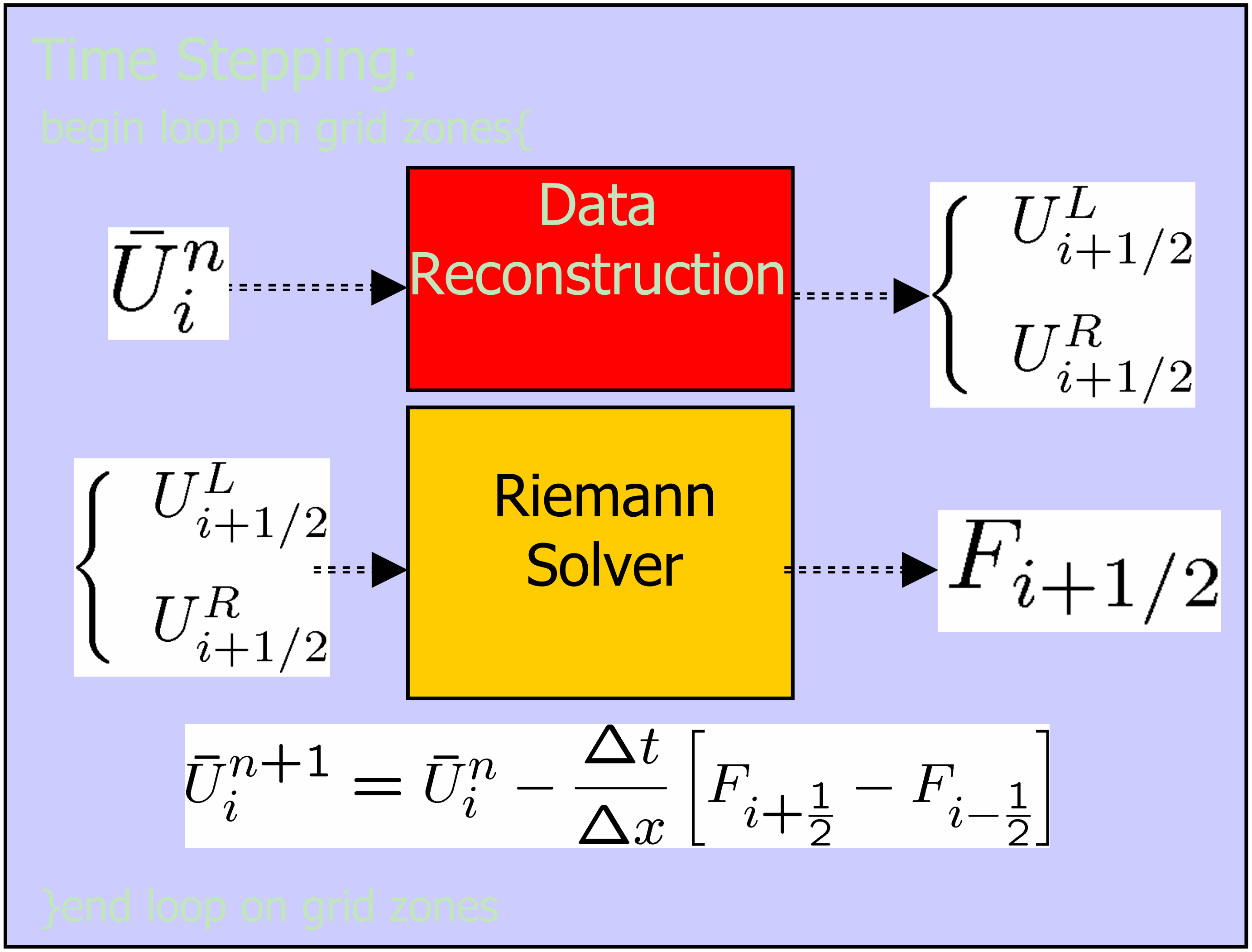
1st Order Godunov Formalism



- Start with zone averaged values: $\bar{u}_i(t) = \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t)dx$
- Solve riemann problem $(\bar{u}_i, \bar{u}_{i+1}) \Longrightarrow u^*_{i+\frac{1}{2}}$
- Compute fluxes $\tilde{F}_{i+\frac{1}{2}} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} F(u^*_{i+\frac{1}{2}})dt$

A “Pseudo-Code”...

for each dt {



}

Euler Equations

- The Euler equations of compressible gasdynamics are written as a system of conservation laws describing conservation of mass, momentum and energy:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \text{(mass)}$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} + \mathbf{I}p] = 0 \qquad \text{(momentum)}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\,\mathbf{v}] = 0 \qquad \text{(energy)}$$

- In total, this is a system of 5 equations: density, energy and the 3 components of velocity.

Euler Equations

- In the simple one-dimensional case, they reduce to

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} = 0$$

$$\frac{\partial\left(\rho v_x\right)}{\partial t} + \frac{\partial}{\partial x}\left[\rho v_x^2 + p\right] = 0$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[\left(E + p\right) v_x\right] = 0$$

- The total energy density *E* is the sum of internal + Kinetic terms:

$$E = \rho\epsilon + \rho\frac{\mathbf{v}^2}{2}$$

- In total, this is a system of 3 equations for density, the x-component of the momentum and energy.

Euler Equations

- Since we have 3 P.D.E. in the 4 unknowns *ρ, vx, p* and *E*, one must provide an additional relation to close the system.
- This is achieved by thermodynamical considerations, providing an equation of state (EoS) relating pressure and internal energy.
- Astrophysical flows are well described by using the ideal gas approximation, where

$$\rho\epsilon = \frac{p}{\Gamma - 1}$$

- Where $\Gamma = C_p/C_v$ is the ratio of specific heats, equal to *5/3* for a monoatomic gas.

Euler Equations

- Alternatively, the equations of gasdynamics can also be written in quasi-linear or *primitive* form, as

$$\frac{\partial \mathbf{V}}{\partial t} + A \cdot \frac{\partial \mathbf{V}}{\partial x} = 0 \,, \quad A = \left(\begin{array}{ccc} v_x & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho c_s^2 & u \end{array}\right)$$

  where $V = [\rho, vx, p]$ is a vector of *primitive* variable (as opposed to the *conservative* variables $q = [\rho, \rho u, E]$). Here $c_s = (\gamma p/\rho)^{1/2}$ is the adiabatic speed of sound.

- It is called "quasi-linear" since, differently from the linear case where we hd $A=const$, here $A = A(V)$.

Euler Equations

- The quasi-linear form can be used to find the eigenvector decomposition of the matrix *A*:

$$\mathbf{r}^1 = \begin{pmatrix} 1 \\ -c_s/\rho \\ c_s^2 \end{pmatrix}, \quad \mathbf{r}^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{r}^3 = \begin{pmatrix} 1 \\ c_s/\rho \\ c_s^2 \end{pmatrix}$$

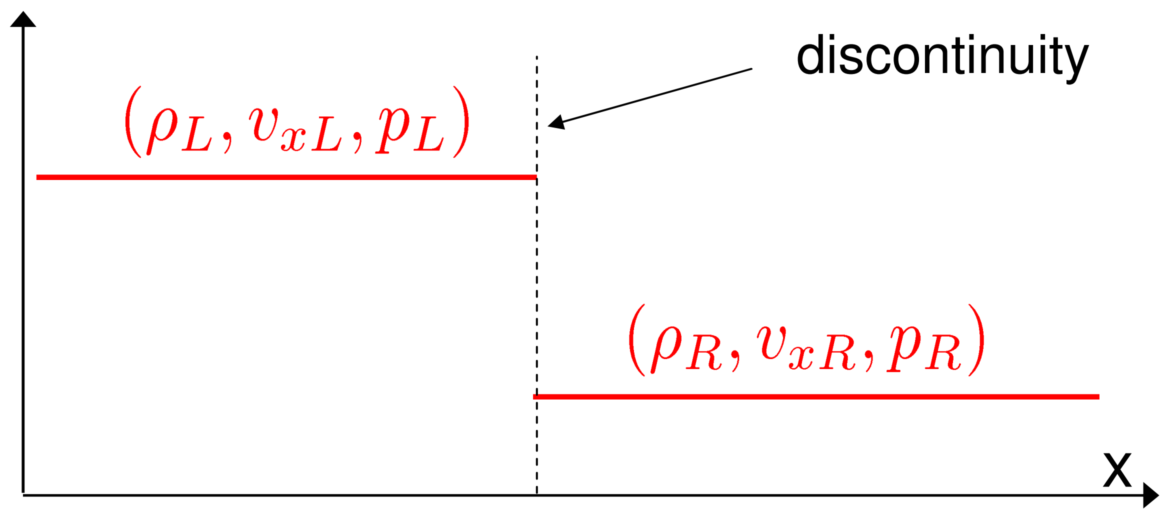
- Associated with the eigenvalues:

$$\lambda^1 = v_x - c_s\,, \quad \lambda^2 = v_x\,, \quad \lambda^3 = v_x + c_s$$

- These are the characteristic speeds of the system, i.e., the speeds at which information propagates. They tell us a lot about the structure of the solution.

Euler Equations: Riemann Problem

- We now wish to study the break of a discontinuity separating two constant states,



complemented with the Euler equations of fluid dynamics.

Euler Equations: Riemann Problem

- If the system was linear, this jump could be broken down into a series of jumps across each of the characteristics,

$$\mathbf{q}_R - \mathbf{q}_L = \alpha^1 \mathbf{r}^1 + \alpha^2 \mathbf{r}^2 + \alpha^3 \mathbf{r}^3$$

- Where the jumps associated with each wave is just the jump in the characteristic variable corresponding to that wave:

$$\alpha^k = w_L^k - w_R^k = \mathbf{l}^k \cdot (\mathbf{q}_R - \mathbf{q}_L)$$

- We know the initial jump, and we computed the left eigenvectors $\boldsymbol{l}^k$, so we know how to write this expansion.
- Note that the variables that jump across each wave is given by the right eigenvectors $\boldsymbol{r}^k$ in the above expression

Euler Equations: Riemann Problem

- By looking at the expressions for the right eigenvectors,

$$\mathbf{r}^1 = \begin{pmatrix} 1 \\ -c_s/\rho \\ c_s^2 \end{pmatrix}, \quad \mathbf{r}^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{r}^3 = \begin{pmatrix} 1 \\ c_s/\rho \\ c_s^2 \end{pmatrix}$$

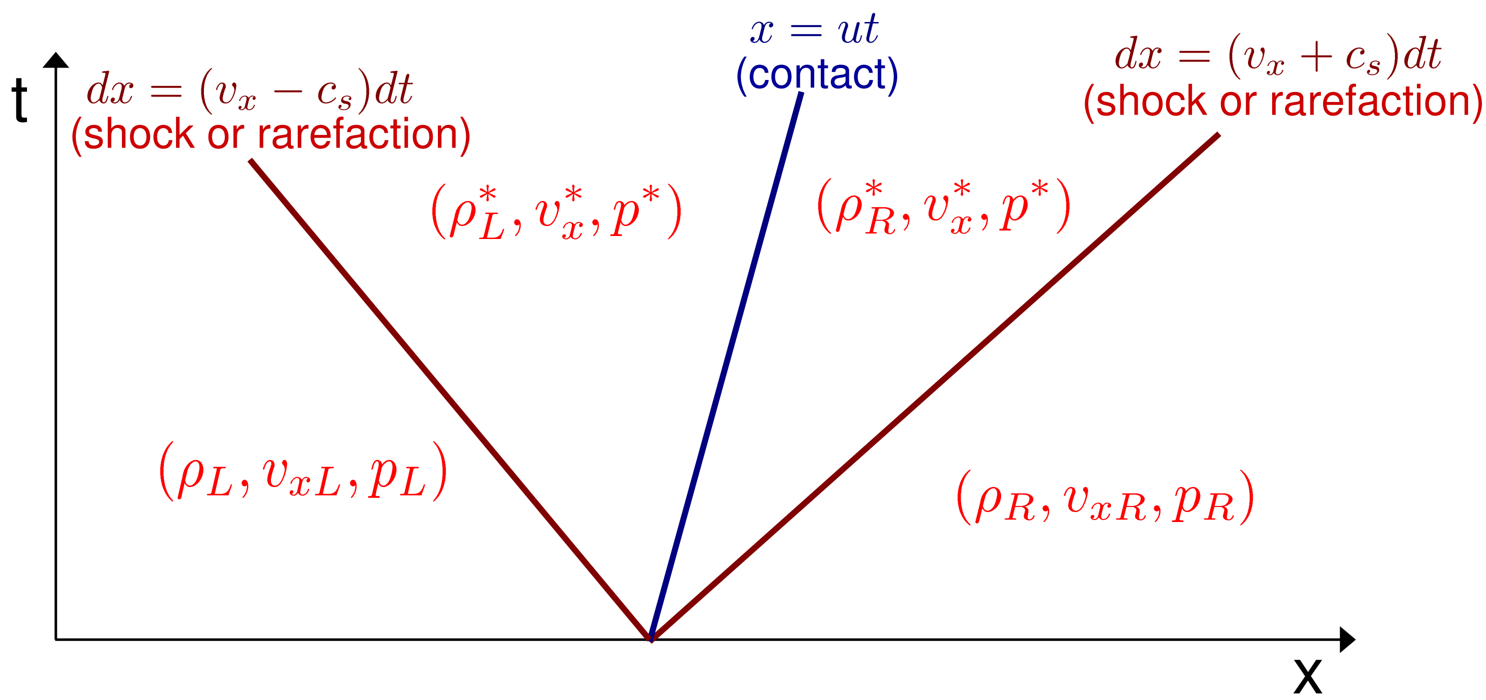
  we see that across waves 1 and 3, all variables jump. These are *nonlinear* waves, either *shock* or *rarefactions* waves.

- Across wave 2, only the density jumps. Velocity and pressure are constant. This defines the *contact discontinuity*.

- The characteristic curve associated with this linear wave is dx/dt = u, and it is a straight line. Since v_x is constant across this wave, the flow is neither converging or diverging.

Euler Equations: Riemann Problem

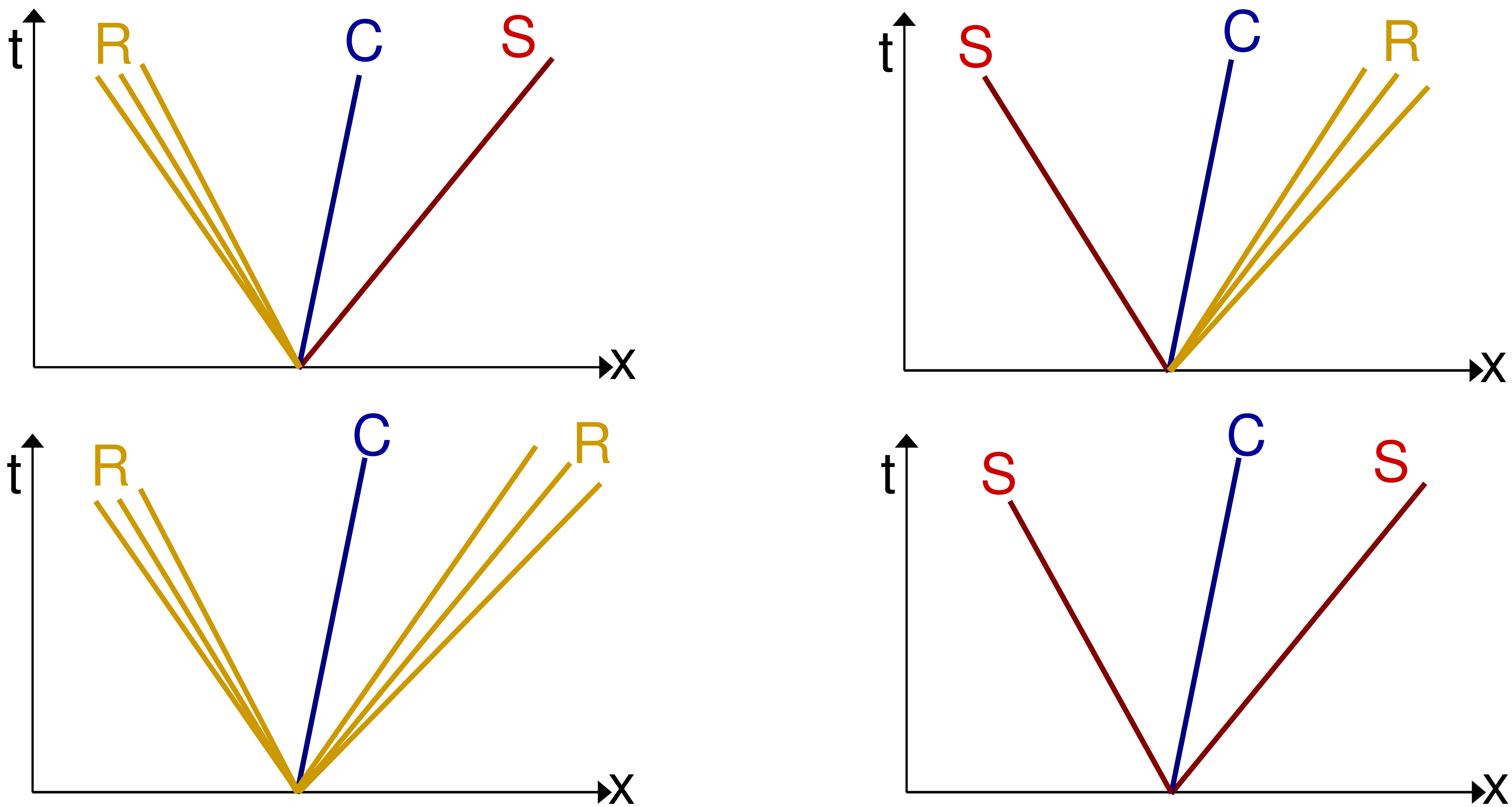
- Thus the solution to the Riemann problem should look like



- The outer waves can be either shocks or rarefactions.
- The middle wave is always a contact discontinuity.
- In total one has 4 unknowns: $\rho_L^*, \rho_R^*, v_x^*, p^*$, since only density jumps across the contact discontinuity.

Euler Equations: Riemann Problem

- Depending on the initial discontinuity, a total of 4 patterns can emerge from the solution:



Euler Equations: Shock Tube Problem

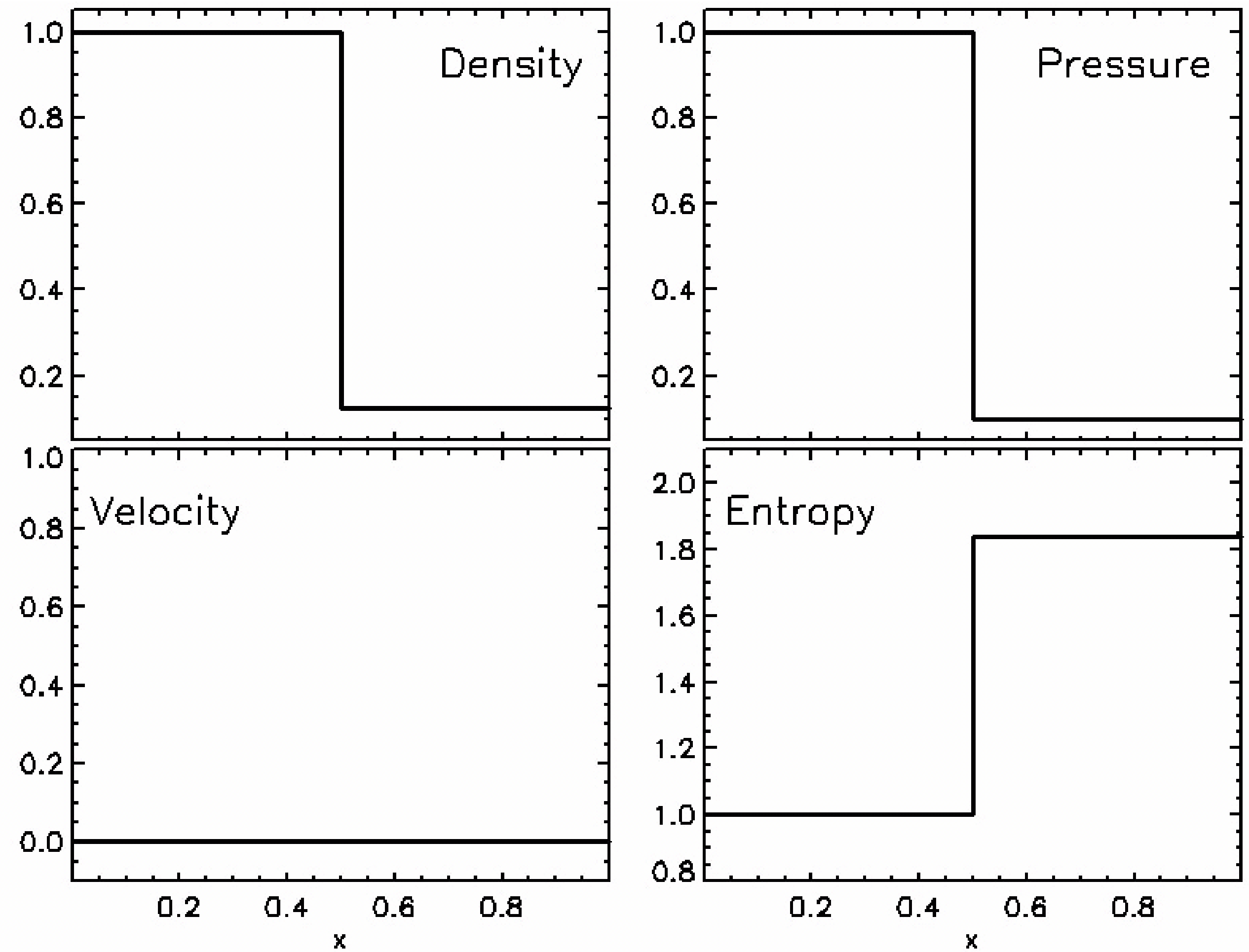
- The decay of the discontinuity defines what is usually called the “shock tube problem”,

-Left Values:

$$(\rho_L, v_{xL}, p_L) = (1, 0, 1)$$

-Right Values:

$$(\rho_R, v_{xR}, p_R) = \left(\frac{1}{8}, 0, \frac{1}{10}\right)$$



Euler Equations: Shock Tube Problem

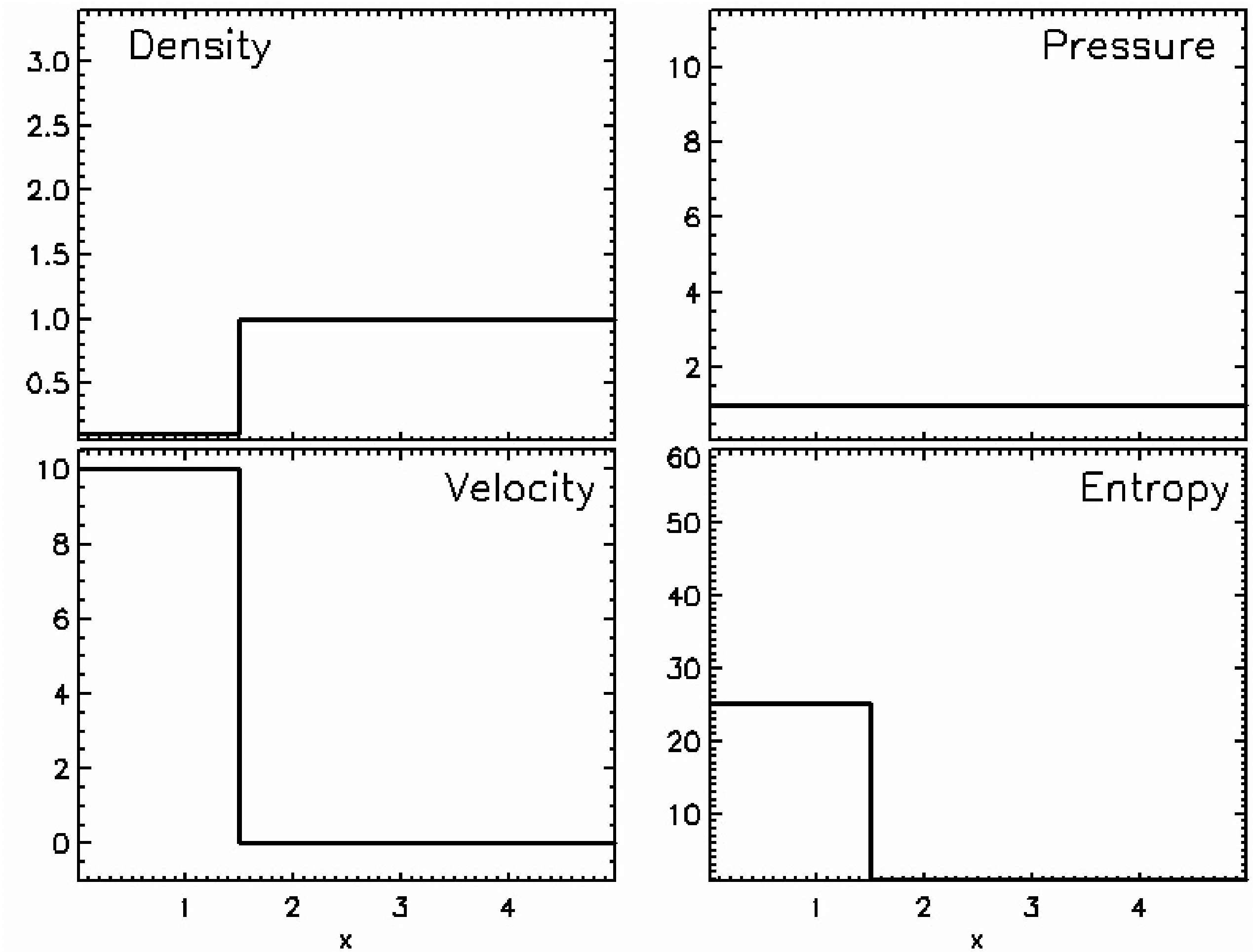
- The one dimensional jet problem reduces to a shock-tube with a S-C-S structure:

-Left Values:

$$(\rho_L, v_{xL}, p_L) = (0.1, 10, 1)$$

-Right Values:

$$(\rho_R, v_{xR}, p_R) = (1, 0, 1)$$



Euler Equations: Riemann Problem

- The full analytical solution to the Riemann problem for the Euler equation can be found, but this is a rather complicated task (see the book by Toro).

- In general, approximate methods of solution are preferred.

- The advantage of using approximate solvers is the reduced computational costs and the ease of implementation.

- The degree of approximation reflects on the ability to “capture” and spread discontinuities over few or more computational zones.

Euler Equations: Riemann Problem

- A practical and simple Riemann solver is the Lax-Friedrichs solver, by which the solution inside the Riemann fan is approximated by:

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{\mathbf{F}_i + \mathbf{F}_{i+1}}{2} - \frac{|\lambda|_{\max}}{2}(\mathbf{q}_{i+1} - \mathbf{q}_i)$$

- Where

$$\mathbf{F} = \left(\rho v_x, \rho v_x^2 + p, (E+p)v_x\right) \;, \quad \mathbf{q} = (\rho, \rho v_x, E)$$

- Example program → *euler.f*