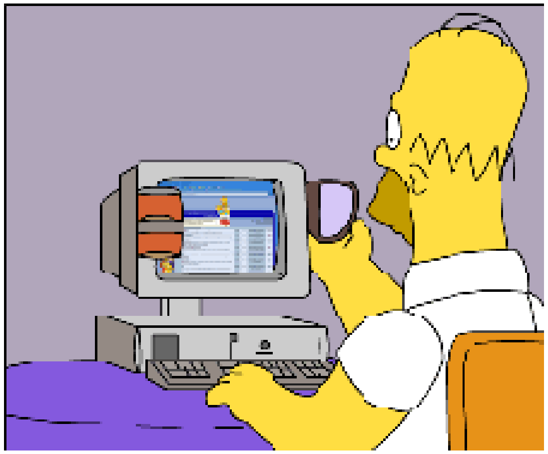
6-Higher order methods, multidimensional integration and the PLUTO Code

“Don’t worry brain, the computer will do our thinking now”

-- Homer Simpson



High Order Methods:
The PLUTO code

- Multi-physics, multi-algorithm modular code for the solution of compressible, highly supersonic flows, based on the RSA:

  - Newtonian Hydrodynamics
  - Relativistic Hydrodynamics
  - Ideal/resistive Magneto Hydrodynamics (MHD)
  - Relativistic MHD

- oriented towards the treatment of astrophysical flows in presence of discontinuities;

- For more information: *http://plutocode.to.astro.it*

The PLUTO code

- Pluto achieves second or better accuracy by following a strategy commonly adopted in shock-capturing schemes: the *Reconstruct*-Solve-*Average* (RSA).

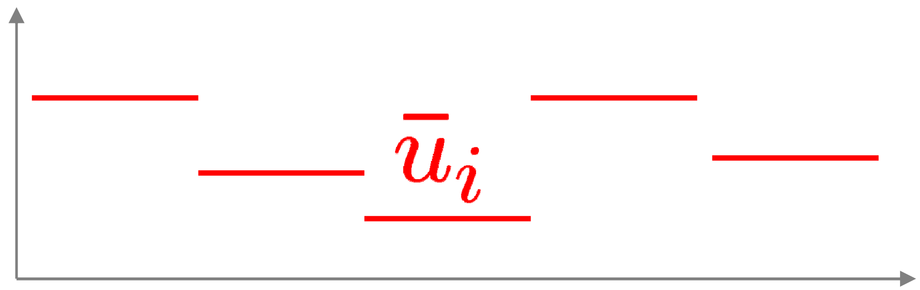
- *Reconstruct*: piecewise polynomial interpolation is provided inside each cell to increase the spatial order of integration;

- *Solve*: A Riemann problem is solved between left and right edge interpolated values;

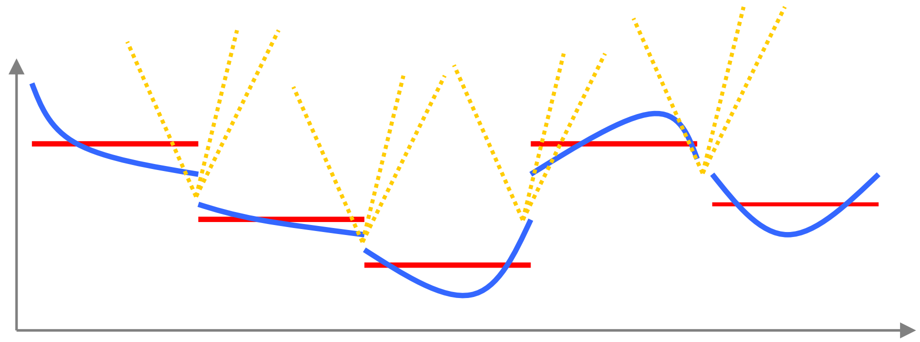
- *Average*: the final update is done in a conservative way by using either dimensionally split or unsplit schemes.

Reconstruct-Solve-Average

- Start from zone averages, break the problem into 3 pieces:



1 - Piecewise polynomial reconstruction



$$u_i(x) = P_i(x)\ , \text{ for } x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}}$$

2 - Solve Riemann problem between left and right states

$$\begin{cases} u_L = P_i\left(x_{i+\frac{1}{2}}\right) \\ u_R = P_{i+1}\left(x_{i+\frac{1}{2}}\right) \end{cases} \rightarrow f_{i+\frac{1}{2}}$$

3 – Form new averages (evolve):

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{\Delta x}\left[f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right]$$

Spatial Reconstruction

- Must be consistent with data representation

$$\frac{1}{\Delta x_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} P_i(x)dx = \bar{u}_i$$

- Satisfy monotonicity constraints:

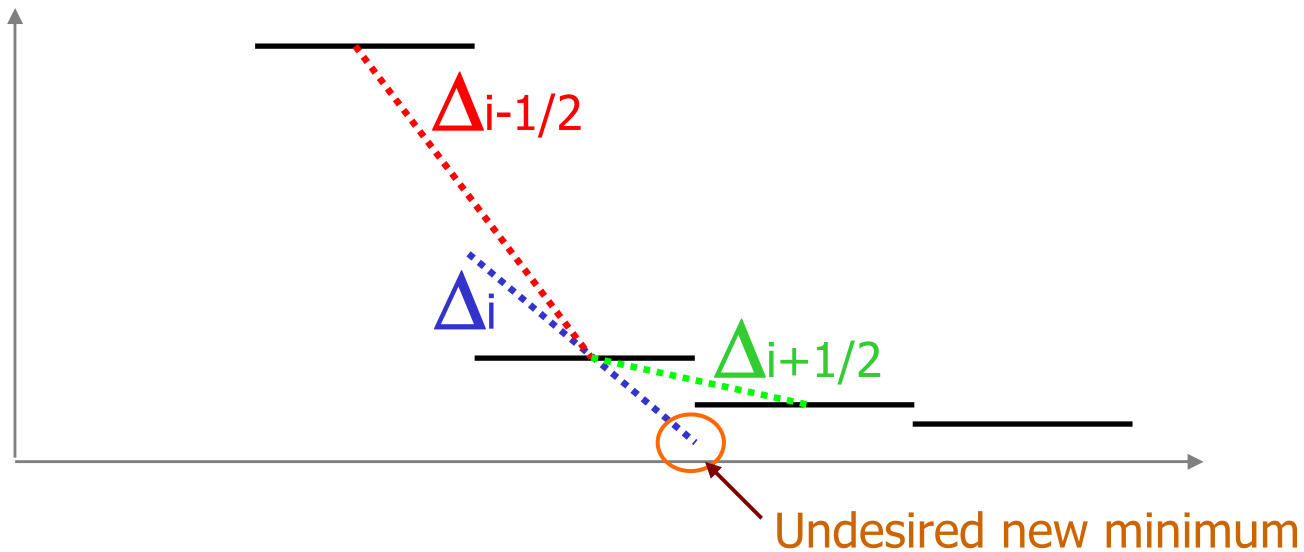
$$\min(P_i(x)) \geq \min\left(\bar{u}_{i-1}, \bar{u}_i, \bar{u}_{i+1}\right)$$
$$\max(P_i(x)) \leq \max\left(\bar{u}_{i-1}, \bar{u}_i, \bar{u}_{i+1}\right)$$

- *→ no new extrema allowed (Total Variation Diminishing (TVD) schemes)*
- *→ Oscillation free solution*

Spatial Reconstruction

- For 2nd-order interpolant, we use $V(x) = V_i + \frac{\delta V}{\Delta x}(x - x_i)$
- Use slope limiters to avoid introducing new extrema:

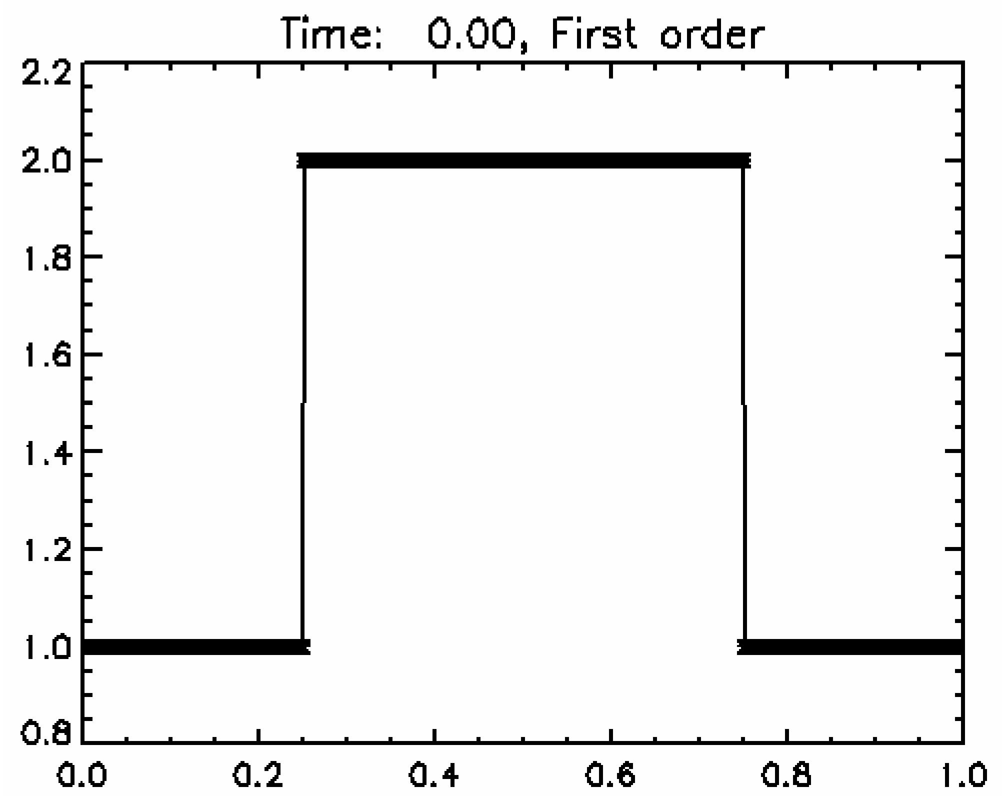
$$\delta V_i = \lim\left(\Delta_{\mathrm{i}-1/2}, \Delta_{\mathrm{i}+1/2}\right)$$



- Example: $$\mathrm{minmod}(x,y) = \begin{cases} x & \text{if} \quad |x| < |y|, xy > 0 \\ y & \text{if} \quad |y| < |x|, xy > 0 \\ 0 & \text{if} \quad xy < 0 \end{cases}$$

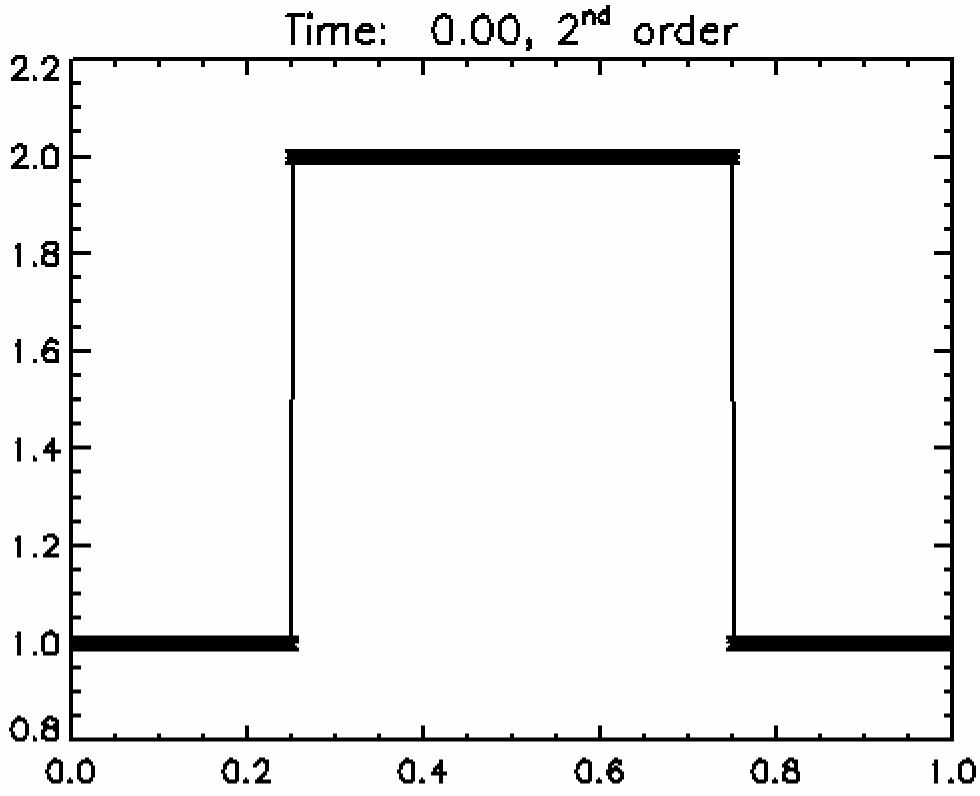
Spatial Reconstruction

- For example, consider the advection of a square pulse. For a first-order spatial reconstruction will evolve like:



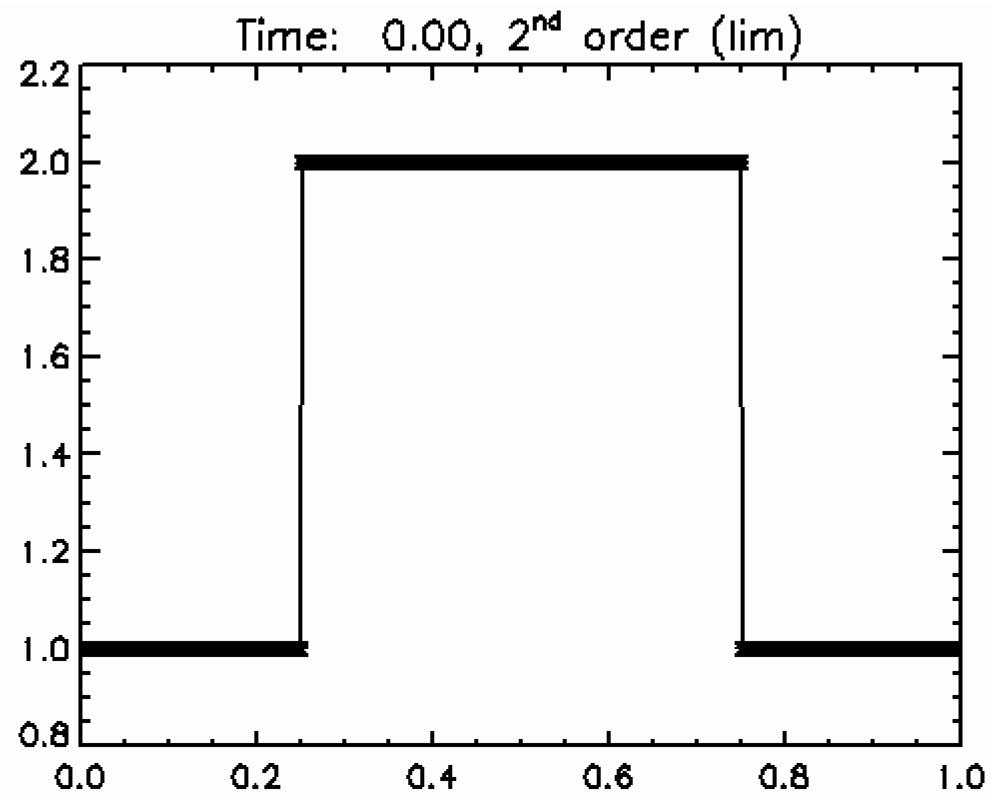
Spatial Reconstruction

- with a second-order unlimited reconstruction:



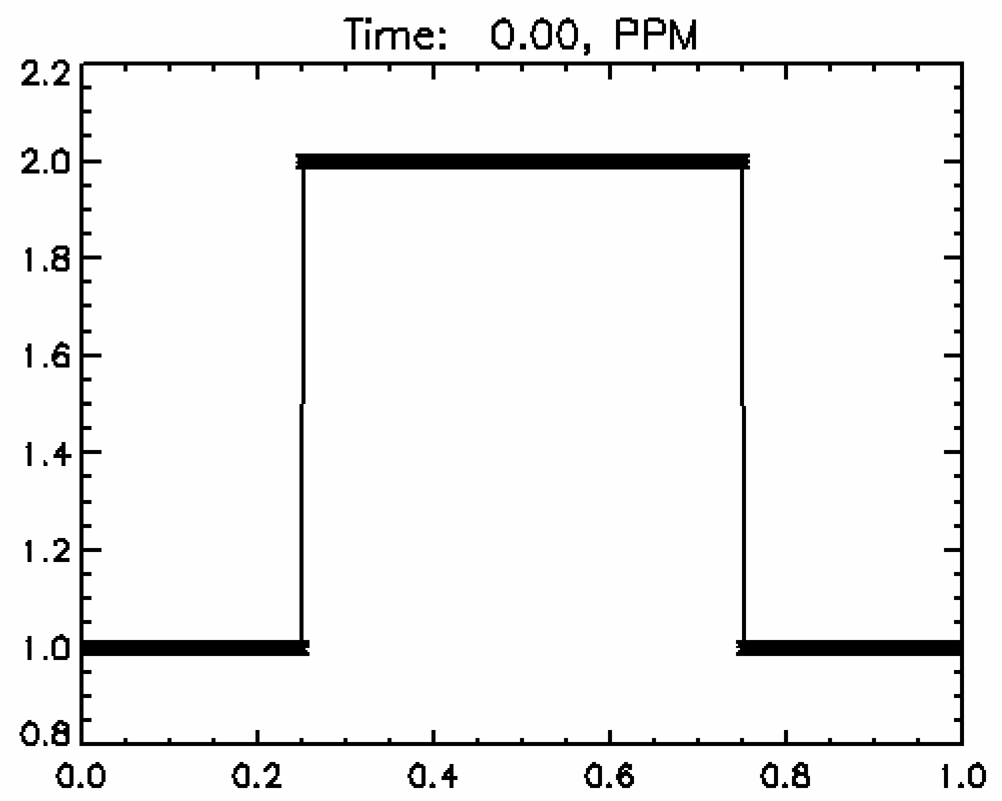
Spatial Reconstruction

- with 2nd order, limited reconstruction:



Spatial Reconstruction

- with limited parabolic reconstruction:

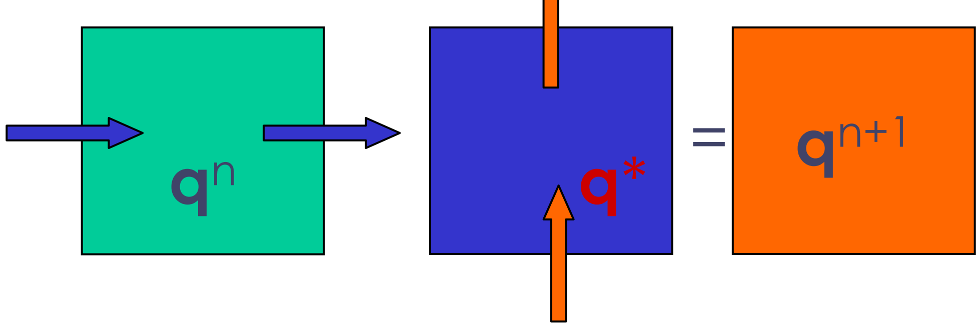


Multi Dimensional Integration

- Integration in more than one dimensions can be achieved using two distinct approaches:

1. Dimensionally Split schemes: solve the PDE as a sequence of 1-D sub-problems.

$$\mathbf{q}^* = \mathbf{q}^n - \Delta t \mathcal{L}_x(\mathbf{q}^n) \qquad \mathbf{q}^{n+1} = \mathbf{q}^* - \Delta t \mathcal{L}_y(\mathbf{q}^*)$$



2. Dimensionally Unsplit schemes: solve the full problem:

$$\mathbf{q}^{n+1} = \mathbf{q}^n - \Delta t \mathcal{L}_x(\mathbf{q}^n) - \Delta t \mathcal{L}_y(\mathbf{q}^n)$$

High order Integration in time

- A simple and effective way to achieve 2nd or 3rd order accuracy in time is to treat the PDE in semi-discrete form:

$$\int \left(\frac{\partial \mathbf{q}}{\partial t} + \nabla \cdot \mathbf{F} \right) dV = 0 \quad \Longrightarrow \quad \frac{d\bar{\mathbf{q}}}{dt} = -\oint \mathbf{F} \cdot d\mathbf{S}$$

- and treating the right hand side as an ODE, e.g., using predictor/corrector methods,

$$\begin{cases} \bar{\mathbf{q}}^* = \bar{\mathbf{q}}^n - \Delta t \left(\oint \mathbf{F}^n \cdot d\mathbf{S} \right) \\ \bar{\mathbf{q}}^{n+1} = \frac{1}{2} \left(\bar{\mathbf{q}}^n + \bar{\mathbf{q}}^* - \Delta t \oint \mathbf{F}^* \cdot d\mathbf{S} \right) \end{cases}$$

Practice Session: Shock-Tube integration with high order method

- file → pluto1d

Practice Session: 2D Jet Simulation

- file → pluto2d