Lecture 3: Disc Accretion – nonrelativistic theory

– Heuristics:

(a) The matter accreting onto a compact object will have significant angular momentum and may form a disc;

(b) The gas elements in the disc lose angular momentum due to friction between adjacent layers;

(c) Part of the released gravitational energy increases the kinetic energy – gas particles spiral inwards;

(d) The other part is converted into thermal energy which is radiated from the disc surfaces.

– Final outcome: (turbulent ?) viscosity converts gravitational potential energy into radiation.

Note! Rotating gas masses and the formation of disks have been studied before the concept of accretion discs has appeared.

– In connection with early solar nebula:

1) C.F. von Weizsäcker: Z. Naturforsch, 3a, 524 (1948) "The rotation of cosmic gas masses"

2) R. Lüst, Z. Naturforsch, 7a, 87 (1952)

– Foundations of the standard disc accretion model:

1) J.E. Pringle, M. Rees: Astron. Astrophys, 21, 1, (1972)

2) N.I. Shakura: Astron Zh., 49, 921 (1972)

- Basic build-up of the model:

- The disc is assumed to be axisymmetric

$$\partial_\varphi = 0 \qquad (1)$$

(r, φ, z) cylindrical coordinates withe the z-axis chosen as the axis of rotation: $z=0$ is the central, "equatorial" plane of the disc.

- The disc is supposed to be quasi-Keplerian:

$$v_z \ll v_r \ll v_\varphi \equiv \left(\frac{GM}{r^3}\right)^{1/2} \qquad (2)$$

- The disc is assumed to be geometrically thin:

$$h(r) \ll r \,, \quad \text{at } \forall\, r\text{'s} \qquad (3)$$

(a) Equation of state:

- The total pressure is the sum of the gas and radiation pressures:

$$P = P_g + P_{rad} = \frac{k}{\mu m_H}\rho T + \frac{a}{3}T^4 \qquad (4)$$

while the internal specific energy is:

$$\varepsilon = C_v T + \frac{aT^4}{\rho} \qquad (5)$$

Problem 3.1. Show that if $\gamma \equiv C_p/C_v$ and $P_g \equiv \beta P$, then for the internal specific energy we have:

$$\varepsilon = A\frac{P}{\rho}\,, \quad \text{where} \quad A \equiv \frac{\beta}{\gamma-1} + 3(1-\beta) \qquad (6)$$

- Continuity equation:

$$\partial_t \rho + \nabla(\rho \vec{v}) = 0$$

in our case has the form:

$$\partial_t \rho + \frac{1}{r}\partial_r (r \rho v_r) + \partial_z (\rho v_z) = 0 \qquad (7)$$

integrating this equation over the z-direction:

$$\partial_t \int \rho \, dz + \frac{1}{r}\partial_r \left(\int r \rho v_r \, dz \right) = 0 \qquad (8)$$

Let us define the surface density $S(r,t)$ as:

$$S(r,t) \equiv \int \rho \, dz \qquad (9)$$

then from (8) we get:

$$\partial_t S + \frac{1}{r}\partial_r (r S v_r) = 0 \qquad (10)$$

Now, defining the accretion rate:

$$\dot{M}(r,t) = -2\pi r S v_r \qquad (11)$$

we write (10) as:

$$\partial_t S = \frac{1}{2\pi r}\partial_r \dot{M} \qquad (12)$$

For the steady disc we have:

$$\dot{M} = \mathrm{const}(r) \qquad (13)$$

Problem 3.2. Calculate components of the $(\vec{v}\cdot\nabla)\vec{v}$ vector in cylindrical coordinates.

(c) Angular momentum conservation:

$$\rho\left(D_t v_\varphi + \frac{v_r v_\varphi}{r}\right) = \frac{1}{r}\partial_r(r t_{\varphi r}) + \partial_z(t_{\varphi z}) + \frac{1}{r} t_{r\varphi} \qquad (14)$$

where $D_t \equiv \partial_t + v_r \partial_r + v_z \partial_z$.

Multiplying (14) by r we obtain:

$$\rho D_t(r v_\varphi) = \frac{1}{r}\partial_r(r^2 t_{r\varphi}) + \partial_z(r t_{\varphi z}) \qquad (15)$$

Now we multiply continuity equation by $r v_\varphi$ and add the resulting equation to (15). It leads to:

$$\partial_t(\rho r v_\varphi) + \frac{1}{r}\partial_r(r v_r \rho r v_\varphi) + \partial_z(v_z \rho r v_\varphi) = \frac{1}{r}\partial_r(r^2 t_{r\varphi}) + $$
$$+ \partial_z(r t_{z\varphi}) \qquad (16)$$

Integrating over z gives:

$$\partial_t\left(\int \rho r v_\varphi dz\right) + \frac{1}{r}\partial_r\left(\int v_r \rho r^2 v_\varphi dz\right) = \frac{1}{r}\partial_r\left(r^2 W_{r\varphi}\right) \qquad (17)$$

where

$$W_{r\varphi} \equiv \int t_{r\varphi}\, dz \qquad (18)$$

For thin discs (17) is approximately

$$\partial_t\left(S r v_\varphi\right) + \frac{1}{r}\partial_r\left(v_r S r^2 v_\varphi\right) = \frac{1}{r}(r^2 W_{r\varphi}) \qquad (19)$$

Using again the continuity equation (10) we have:

$$S\left\{\partial_t(r v_\varphi) + v_r \partial_r(r v_\varphi)\right\} = \frac{1}{r}\partial_r(r^2 W_{r\varphi}) \qquad (20)$$

Since $r v_\varphi$ is the specific angular momentum, essentially (20) is the conservation equation for it.

Note that: $\qquad t_{r\varphi} = \eta r \partial_r(v_\varphi/r) = \eta r\, d\Omega/dr \qquad (21)$

- Radial momentum conversation:

$$\rho\left(D_t v_r - \frac{v_\varphi^2}{r}\right) = -\rho\partial_r\Phi - \partial_r P + \frac{1}{r}\partial_r(r t_{rr}) + \partial_z(t_{rz}) - \frac{1}{r}t_{\varphi\varphi}$$

Neglecting all components of t_{ik} (except $t_{r\varphi}$ in previous case) and v_z we derive:

$$\rho\left(\partial_t v_r + v_r\partial_r v_r\right) = \rho\left(\frac{v_\varphi^2}{r} - \partial_r\Phi\right) - \partial_r P \qquad (22)$$

and after integration over z we have:

$$S\left(\partial_t v_r + v_r\partial_r v_r\right) = S\left(\frac{v_\varphi^2}{r} - \frac{GM}{r^2}\right) - \partial_r W \qquad (23)$$

where $W \equiv \int P\,dz$.

- Energy conversation:

We start again from general equation:

$$\rho\left(\partial_t + v_r\partial_r\right)\left[\frac{1}{2}v_r^2 + \frac{1}{2}v_\varphi^2 + h + \Phi\right] = \partial_t P + \frac{1}{r}\partial_r(r t_{r\varphi} v_\varphi) - \partial_z F$$

where F is the vertical energy flux density ($F = q_z$). If we integrate over z. In the thin disc approximation we have:

$$S(\partial_t + v_r\partial_r)\left[\frac{1}{2}v_r^2 + \frac{1}{2}v_\varphi^2 + (A+1)\frac{W}{S} + \Phi\right] = \partial_t W + \frac{1}{r}\partial_r(r W_{r\varphi} v_\varphi) - Q^- \qquad (24)$$

where Q^- is the energy flux per unit area emitted at the disc surface: $Q^- = 2F$ (surface).

Note also that the dissipation function is defined as:

$$\hat{T} = 2 t_{r\varphi}\theta_{r\varphi} = t_{r\varphi}\, r\partial_r\Omega \qquad (25)$$

where $\theta_{ik} \equiv (v_{i,k} + v_{k,i})/2$ is the rate-of-deformation tensor.

and consequently:

$$Q^+ \equiv \int T\,dz = W_{r\varphi}\, r\, \partial_r \Omega \qquad (26)$$

is the energy produced per unit area. ("heating rate")

– Vertical structure:

The z-component of the momentum equation contains small components of the viscosity tensor and $D_t v_z$, all these terms are neglected and we reduce the equation to the condition of hydrostatic equilibrium in the z-direction.

$$\partial_z P = -\rho \frac{GM}{r^2}\frac{z}{r} = \qquad (27)$$

from here we can see that:

$$\frac{z_0}{r} \simeq \frac{c_s}{v_\varphi} << 1 \qquad (28)$$

indicating that the circular flow is highly supersonic. This poses restrictions on the disc interior temperature, which are satisfied if the gas is cooling sufficiently fast.

Usually it is assumed that the energy dissipated into heat is radiated totally in the vertical direction. Then we write:

$$\partial_z F = T = t_{r\varphi}\, \partial_r \Omega \qquad (29)$$

Equations considerably simplify if v_φ is approximated by the circular Keplerian velocity:

$$v_\varphi = \Omega r \;, \quad \Omega = \left(\frac{GM}{r^3}\right)^{1/2} \qquad (30)$$

In this case Eq. (20) reduces to:

$$\frac{\dot{M}\Omega r}{2} = -2\pi \partial_r (r^2 W_{r\varphi}) \qquad (31)$$

where

$$W_{r\varphi} = r\frac{d\Omega}{dr}\int \eta dz \qquad (32)$$

and we arrive to the following relation:

$$\dot{M} r^2 \Omega + 2\pi r^3 \frac{d\Omega}{dr}\int \eta dz = I = const(r) \qquad (33)$$

— Where the Keplerian approximation ($\Omega \approx \Omega_K$) follows from?

It follows from (23) if one neglects inertia and pressure gradient terms.

(29) and (26) take the following form:

$$\partial_z F = \frac{9}{4}\eta\frac{GM}{r^3} \qquad (34)$$

$$Q^+ = \frac{9}{4}\frac{GM}{r^3}\int \eta dz \qquad (35)$$

(!) Nonstationary equations are too tedious! First, let us steady ($\partial_t = 0$), Keplerian disc:

— Continuity:

$$\dot{M} = -2\pi r \Sigma v_r = const \qquad (36)$$

— Angular momentum conversation:

$$\dot{M} r^2 \Omega = -2\pi r^2 W_{r\varphi} + I \qquad (37)$$

here I is the net inward flux of angular momentum. Its value is taken to be $\dot{M} r_0^2 \Omega(r_0)$ (r_0 being the inner edge of the disc). In this case for the specific angular momentum $l(r) \equiv r^2\Omega(r)$ we have:

$$\dot{M}\left[I(r) - I(r_0)\right] = -2\pi r^2 W_{r\varphi} \qquad (38)$$

but the torque $2\pi r^2 W_{r\varphi}$ is determined by , i.e.:

$$W_{r\varphi} = -\frac{3}{2}\Omega \int \eta \, dz \qquad (39)$$

In the energy equation (24) we neglect derivatives of W, S, and v_r:

$$\frac{d}{dr}\left\{\dot{M}\left[\frac{1}{2}v_\varphi^2 - \frac{GM}{r}\right] + 2\pi r^2 W_{r\varphi}\Omega\right\} = 2\pi r Q^- \qquad (40)$$

taking into account (38) we get:

$$Q^- = \frac{3GM\dot{M}}{4\pi r^3}\left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right] \qquad (41)$$

this also follows from $Q^- = Q^+$ and (26) and (38).

– Vertical structure equations:

Here we have the following equations:

$$\frac{dP}{dz} = -\rho\,\Omega^2 z \qquad (42)$$

$$\frac{dF}{dz} = \frac{9}{4}\Omega^2 \eta \qquad (43)$$

$$P = \frac{k}{\mu m_H}\rho T + \frac{a}{3}T^4 \qquad (44)$$

and energy transport equations. If radiation transport dominates and the disc is optically thick

$$\frac{dT}{dz} = -\frac{3\kappa\rho F}{4acT^3} \qquad (45)$$

but if the energy transport is dominated by convection:

$$\frac{dT}{dz} = \rho \Omega^2 z \frac{T}{p} \nabla_{conv} \qquad (46)$$

The opacity is dominated by electron scattering and free-free transitions. These contributions to the total opacity are:

$$\frac{1}{\varkappa} \simeq \frac{1}{\varkappa_{es}} + \frac{1}{\varkappa_{ff}} \qquad (47)$$

where:

$$\varkappa_{es} = 0{,}40 \ cm^2 \cdot g^{-1} \qquad (48)$$

$$\varkappa_{ff} = (0{,}645 \times 10^{23} \ cm^2 \cdot g^{-1}) \frac{Z^2}{A} \bar{G} \left(\frac{\rho_0}{g/cm^3} \right) T_K^{\ 7/2} \qquad (49)$$

If the disc is optically thin, the energy is mainly lost by free-free emission:

$$Q^- = \int \varepsilon_{ff} \, dz \quad (\tau < 1) \qquad (50)$$

$$\varepsilon_{ff} = (1{,}4 \times 10^{-27} erg \cdot s^{-1} cm^{-3}) T^{1/2} n_e n_i Z^2 \bar{G}(T) \qquad (51)$$

(*) Note! Some qualitative conclusions about the disc accretion efficiency can be made independently from the actual viscosity law, which we have yet to specify.

For example, from (41) we immediately obtain the total luminosity of the disc from the both sides:

$$L_D = 2\pi \int_{r_0}^{\infty} Q^- r dr = \frac{1}{2} \frac{GM}{r_0 c^2} \dot{M} c^2 = \frac{GM\dot{M}}{2r_0} \qquad (52)$$

In other words the efficiency of the Kepler disc is:

$$\varepsilon = \frac{1}{2} \frac{GM}{r_0 c^2} \qquad (53)$$

(*) It means that half of the potential energy is radiated away.

The other half is on the form of kinetic energy, situated just outside the boundary layer. Note that for Neutron stars and/or white dwarfs this part of the energy will be radiated away when the accreted gas ultimately falls on the stellar surface.

In those parts of the disc where the radiation is blackbody, the effective surface temperature is determined by the Stefan-Boltzmann law:

$$F = \frac{1}{2} Q^{-} = \sigma T_{eff}^{4} \qquad (54)$$

from (41) we can see that:

$$T_{eff}(r) = \left\{ \frac{3GM\dot{M}}{8\pi r^3 \sigma} \left[1 - \left(\frac{r_0}{r} \right)^{1/2} \right] \right\}^{1/4} \qquad (55)$$

The same can be written as:

$$T_{eff} = T_{*} \left(\frac{r}{r_0} \right)^{-3/4} \left[1 - (r_0/r)^{1/2} \right]^{1/4} \qquad (56)$$

where T_* is the following characteristic temperature:

$$T_* = \left(\frac{3GM\dot{M}}{8\pi r_0^3 \sigma} \right)^{1/4} = 1{,}4 \times 10^7 \, K \left(\frac{3 r_g}{r_0} \right)^{3/4} \left(\frac{M}{M_\odot} \right)^{-1/2} \dot{M}_{17}^{1/4} \qquad (57)$$

where $\dot{M}_{17} = \dot{M} / 10^{17} \, g \cdot s^{-1}$.

(*) Note! The maximum temperature is reached for $r = (49/36) r_0$.

— Standard "α-model" of accretion discs:

— Detailed disc models can be built only if we know the viscosity law.

— But the medium is extremely complex: accretion disc consists of fluid which is:

(a) highly supersonic, (b) strongly shearing, (c) radiative, (d) large Reynolds number.

– Usual (molecular) viscosity is not sufficient to generate x-ray radiation of accretion discs.

– It is believed that ~~...~~ on accretion discs angular momentum transport is governed by turbulent viscosity!

– Shakura and Sunyaev (1973) introduced a phenomenological parametrization (so called "α-model") that became the standard model of thin, quasi Keplerian accretion disc.

$$\eta_{turb} \sim \rho v_{turb} \ell_{turb} \lesssim \rho c_s z_0 \qquad (58)$$

For the only nonzero component of the turbulent viscosity tensor $t_{r\varphi}$ we have:

$$t_{r\varphi} = t_{r\varphi}^{turb} + t_{r\varphi}^{mag} \qquad (59)$$

$$-t_{\varphi r}^{turb} \simeq \eta_{turb}\, \Omega \lesssim \rho c_s z_0 \Omega \simeq \rho c_s^2 \qquad (60a)$$

$$-t_{\varphi r}^{mag} < P_{mag} \left(= \frac{B^2}{8\pi}\right) \lesssim P_{therm} \simeq \rho c_s^2 \qquad (60b)$$

Therefore Shakura and Sunyaev propose:

$$t_{r\varphi} = -\alpha P \qquad (61)$$

where $\alpha = const \lesssim 1$. Models based on this parametrization are called "α-discs"

Clearly from (61) and (21) we have:

$$\eta = \frac{2\alpha}{3} P \left(\frac{GM}{r^3}\right)^{-1/2} \qquad (62)$$

Problems

(1.) Let $\gamma \equiv C_p/C_v$, $P_g \equiv \beta P$. Using thermodynamic relation between C_p and C_v show that

$$\varepsilon = A\frac{P}{\rho}, \quad \text{where} \quad A = \frac{\beta}{\gamma-1} + 3(1-\beta)$$

(2) Starting from the azimuthal (φ) component of the equation of motion

$$\rho\left(\partial_t v_\varphi + \frac{v_r v_\varphi}{r}\right) = \frac{1}{r}\partial_r(r\, t_{r\varphi}) + \frac{1}{r}t_{r\varphi} + \partial_z(t_{\varphi z})$$

show that taking into account the continuity equation and integrating over z one can derive the angular momentum conservation equation:

$$S\left[\partial_t(r v_\varphi) + v_r \partial_r(r v_\varphi)\right] = \frac{1}{r}\partial_r(r^2 W_{r\varphi})$$

where $S \equiv \int \rho\, dz$ and $W_{r\varphi} = \int t_{r\varphi}\, dz$.

(3) Show that in Keplerian accretion disc

$$Q^+ = Q^- = \frac{9}{4}\eta\,\Omega^2$$

(4) Suppose that Keplerian, steady accretion disc radiates as a black body. ~~Evaluate~~ Calculate the effective surface temperature of the disc and find out at what radial distance it is the hottest.

(5) Assume that for a Keplerian, steady disc polytropic equation of state $P(z) = K\rho(z)^{1+1/N}$ is valid for fixed r. Using this equation solve the disc hydrostatic balance equation in vertical direction (27) and calculate the

ratio P_c/ρ_c (central plane) as a function of r. Express surface density $S \equiv \int \rho dz$ and vertically averaged pressure $W \equiv \int P dz$ as functions of P_c, ρ_c, z_0 and N.

6. Using the "α-law" in the form $W_{r\varphi} = \alpha W$ calculate radial structure of vertically averaged pressure function $W(r)$. Calculate also functions $P_c(r)$, $\rho_c(r)$, $T_c(r)$ and $v_r(r)$ as functions of r and disc half-thickness $z_0(r)$.

Further reading:

1. Shapiro S.L., Teukolsky S.A. "Black holes, White dwarfs and neutron stars: The physics of compact objects" (Wiley, 1983)
2. Pringle J.E. "Accretion discs in astrophysics" ARAA, 19, 137 (1981)
3. Straumann N. "General Relativity and Relativistic Astrophysics" (Springer, 1984)
4. Frank J., King A., Raine D. "Accretion power in astrophysics" (Cambridge, 2002)
5. Pringle J.E., King A. "Astrophysical flows" (Cambridge, 2007)