Basic Fluid Dynamics: Assignments

Shiv Sethi

Raman Research Institute, Bangalore, India
sethi@rri.res.in

Assignment 1

1. *Hydrostatic equilibrium*: A plane-parallel atmosphere, composed of a perfect gas, is in hydrostatic equilibrium in an external field, $-\hat{z}g$.

   (a) Derive an expression for the entropy gradient if the atmosphere is isothermal.

   (b) Obtain expressions for $p(z)$ and $\rho(z)$ if the atmosphere is isentropic.

   (c) *Earth's atmosphere*: In the lower stratosphere, the air is isothermal. Use the condition of hydrostatic equilibrium to show that:

   $$p(z) \propto \exp(-z/H), \tag{1}$$

   where the scale height, $H = kT/(\mu m_p g)$. Estimate the scale height. (Use mean molecular weight $\mu = 29$ and $T = 300\,\mathrm{K}$). Now assuming that the air is isentropic, show that:

   $$\frac{dT}{dz} = -\left(\frac{\gamma - 1}{\gamma}\right)\frac{g\mu m_p}{k} \tag{2}$$

   Here $\gamma \simeq 1.4$ is the ratio of specific heats for gases like Nitrogen and Oxygen. Why does the above expression vanish for $\gamma = 1$?

2. *Archimedes' principle* states that, when a solid body is totally or partially immersed in a fluid, the total buoyant upward force of the liquid on the body is equal to the weight of the displaced fluid. Prove the law assuming conditions of hydrostatic equilibrium. Using this result, estimate how much more would one weigh in vacuum.

3. *Fluid equations as conservation laws*: Using continuity equation, the Euler equation, and the first law of thermodynamics, derive conservation laws for the momentum and energy of an ideal fluid. Hint: proving conservation means writing equations in the form:

$$\frac{\partial}{\partial t}(\text{mom. or energy density}) + \nabla\cdot(\text{mom. or energy current density}) = 0 \tag{3}$$

4. *Convective instability*: When a fluid is disturbed and it settles back into equilibrium it usually manages to reach mechanical equilibrium faster than thermal equilibrium.

   (a) Estimate these time scales for a parcel of air of size 1 m, and 1 km. The coefficient of thermal conductivity, $\kappa = 0.2\,\mathrm{cm^2 sec^{-1}}$ and speed of sound $c_s = 350\,\mathrm{msec^{-1}}$.

   (b) Earth's atmosphere could be used as an example of this kind: it is in approximate mechanical equilibrium but has temperature gradients in hydrostatic equilibrium. Derive the conditions (called Schwarzchild criterion) under which this equilibrium is stable.

5. Show that the lift on a 2-dim aerofoil, can be written as $F_L = -\rho U_0 \Gamma$, where Γ is the circulation around any loop enclosing the aerofoil. This is an example of an exact result for inviscid, potential flows, known as the *Kutta–Joukowski* theorem.

6. *Vorticity*:

   (a) Sketch the following (two-dimensional) velocity fields and calculate the vorticity vector fields: (i) Uniform rotation, $\mathbf{v} = \hat{\phi}\Omega R$, (ii) A flat rotation curve, $\mathbf{v} = V_0\hat{\phi}$, and (iii) $\mathbf{v} = V_0 y\hat{x}$. Here Ω and V_0 are constants.

   (b) A fluid fills the half space $y > 0$, which is bounded by a wall that may be taken as the x-z plane. A vortex line of strength $\hat{z}\Gamma$ is at distance d from the wall. Calculate the velocity of the vortex line.

   (c) Consider two straight vortex line with strengths $\pm\hat{z}\Gamma$ separated by some distance d. Determine the dynamics of these vortex lines.

Assignment 2

1. *Potential, Compressible flow: Linear theory of sound*: Assume the unperturbed medium is unbounded, static, uniform in its properties: $\rho = \rho_0$, $p = p_0$, and $v_0 = 0$. The medium is then perturbed:

   (i) Write down the linearised continuity and Euler equations satisfied by perturbations ρ_1, p_1, and v_1.

   (ii) What is the linearized equation satisfied by the perturbed vorticity.

   (iii) Assume that the perturbation gives rise to a pure potential flow, $\mathbf{v_1} = \nabla\phi$. Use this in the linearised Euler equation, and express p_1 in terms of ϕ_1.

   (iv) Assume that the flow is barotropic, with sound speed defined by $c_s = \sqrt{dp_0/d\rho_0}$. Derive a wave equation for ϕ_1.

   (v) Write down the general solution for ϕ_1, corresponding to plane waves travelling in the $\pm x$ directions.

   (vi) What are the corresponding expressions for $\mathbf{v}_1$ and p_1? Are the waves transverse or longitudinal?

   (vi) Consider plane waves travelling only along the positive x direction. How is p_1 related to the Mach number v_1/c_s?

   (vii) Let us define the intensity of sound by $\mathbf{I} = p_1\mathbf{v}_1$. Obtain an equation for the transport of the wave energy density, W, in the conservation form,

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{I} = 0 \tag{4}$$

   (viii) The human ear is responsive to frequencies between 0.02–20 kHz. Verify that the carrier of these waves (earth's lower atmosphere) can be treated as a fluid for these considerations.

2. Consider steady, potential, incompressible flow past a cylinder of radius a. Show that it can be formulated in terms of the velocity potential $\phi(\theta, r)$ which satisfies:

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{5}$$

If the free-stream speed is U, we may choose boundary conditions: $\phi \to Ur\cos\theta$ as $r \to \infty$ and $\partial\phi/\partial r = 0$ at $r = a$. Show that it gives:

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta \tag{6}$$

3. *Surface Gravity waves*:

   (i) Compute the velocity field for surface gravity waves on deep water. Argue that fluid elements move in circles, with radii that decrease with depth as $\exp(kz)$.

   (ii) Derive the dispersion relation for surface gravity waves in water of finite depth h. What is the trajectory of fluid elements in this case?

   (iii) *Shallow-water gravity waves: Tsunami*: Using the dispersion relation derived in (ii), examine the special case of shallow water waves $kh \ll 1$. Tsunami is an important example of shallow water gravity waves. A Tsunami is a large harbour wave that is generated an earthquake in deep ocean; its presence is hardly noticeable in deep ocean but it can cause major destruction upon reaching the shore. Estimate the speed of a Tsunami in deep ocean. Why don't deep water gravity waves turn into Tsunamis?

   (iv) *Surface tension: capillary waves*: At an interface between two fluids, the effect of surface tension could be important. Show that, when the surface tension is included, the dispersion relation of deep water gravity waves is

   $$\omega^2 = gk + \frac{\gamma}{\rho}k^3. \tag{7}$$

   Here γ is the coefficient of surface tension. Show that in this case there is a minimum value of the group velocity. Using $\gamma = 70\,\mathrm{dyne\,cm^{-1}}$ for water-air interface, compute this minimum velocity and the wavelength associated with it. (water density, $\rho = 1\,\mathrm{g\,cm^{-3}}$).

4. *Ship Waves*: Consider a boat moving on a deep water body with uniform velocity $\mathbf{u}$. We are interested in studying the stationary pattern of surface gravity and capillary waves it induces on water.

(i) In the rest frame of water, the dispersion relation of waves is given by Eq. (1). Compute the dispersion relation in the rest frame of the boat. (Hint: the phase of the wave $\mathbf{k}.\mathbf{x} - \omega t$ is invariant under this transformation). Show that it is given by:

$$\omega(k) = \omega_0 - \mathbf{k}.\mathbf{u} \tag{8}$$

Here $\omega_0(k)$ is the dispersion relation for waves in the rest frame of water.

(ii) Prove that the stationary wave pattern in any direction θ (θ is the angle between $\mathbf{u}$ and $\mathbf{V}_{g0} - \mathbf{u}$) is composed of waves whose wave vectors (characterized by k and ϕ; ϕ is the angle between $\mathbf{k}$ and $\mathbf{u}$) satisfy

$$\tan(\theta) = \frac{V_{g0}(k)\sin\phi}{u + V_{g0}\cos\phi} \tag{9}$$

$$\omega_0(k) = -uk\cos\phi \tag{10}$$

Here $\mathbf{V}_{g0}$ is the group velocity in the rest frame of water. (Hint: The only waves that contributes to a stationary pattern are the ones corresponding to $\omega = 0$.) Using this result, argue that for $u < c_m$, where c_m is the minimum velocity computed in the last problem, no such stationary pattern exists.

(iii) For capillary waves show that:

$$\tan\theta = \frac{3\tan\phi}{1 - 2\tan^2\phi}. \tag{11}$$

Argue that the capillary wave pattern is present for all values of θ. For gravity waves show that:

$$\tan\theta = \frac{-\tan\phi}{1 + 2\tan^2\phi}. \tag{12}$$

Demonstrate that the gravity wave pattern is confined to a trailing wedge, whose opening angle is $\theta_{\mathrm{gw}} = 2\sin^{-1}(1/3)$.

5. *Waves and instabilities*: Consider the flow of incompressible, inviscid fluids in two horizontal and parallel streams of different velocities and

densities, with one stream above the other in an external gravitational field. The undisturbed flow is given by:

$$\mathbf{U} = (U_1, U_2)\hat{x} \quad ; \quad \rho = (\rho_1, \rho_2) \tag{13}$$

$$P = p_0 - g\rho_2 z \; (z > 0) \quad ; \quad P = p_0 - g\rho_1 z \; (z < 0) \tag{14}$$

This pattern of flow is disturbed. Analyse the dynamics of the resultant flow in linear perturbation theory, for perturbed quantities $\propto f(z)\exp(\sigma t + i\mathbf{k}.\mathbf{x})$, for Fourier modes $\mathbf{k} = (k_x, k_y)$.

(i) Show that for appropriate boundary conditions across the interface between two fluids is

$$\rho_1 \left[kg + (\sigma + ik_x U_1)^2\right] = \rho_2 \left[kg - (\sigma + ik_x U_2)^2\right] - \gamma k^3. \tag{15}$$

Show that this results in the following equation for σ:

$$\sigma = -ik_x \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[\frac{k_x^2 \rho_1 \rho_2 (U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} - \frac{kg(\rho_1 - \rho_2) + \gamma k^3}{\rho_1 + \rho_2}\right]^{1/2} \tag{16}$$

(ii) *Kelvin-Helmholtz instability*: Eq. (10) gives the general conditions for this instability. One application of this instability is the amplification of ocean waves in the presence of a strong breeze across its surface. Compute the minimum velocity of ocean breeze for causing this instability (use $U_1 = 0$ or assume stationary ocean for this estimate)

(iii) *Rayleigh-Taylor instability*: Using Eq. (10) it can readily be shown that for $\gamma = U_1 = U_2 = 0$, the flow is unstable when heavier fluid overlies lighter fluid i.e. $\rho_2 > \rho_1$. Eq. (10) generalizes this concept in the presence of shear velocity fields. Use Eq. (10) to show that even when lighter fluid overlies heavier fluid, the flow can become unstable. Argue that this also generalizes the criterion of convective instability in stratified atmosphere.

Assignment 3

1. *Molecular origin of shear viscosity*: For most fluids, the shear viscosity coefficient is only determined experimentally. For an ideal gas the shear viscosity can be determined using kinetic theory of gases. Consider a flow in x direction with shear .i.e. $\partial v_x/\partial y$ is non-zero. In this case, random motion of the gas molecules moving with typical thermal velocities v_t with a mean free path ℓ ($\ell = 1/(n\sigma)$, n is the number density and σ is the cross-section of collision) deposit different amounts of x-component of momentum across a plane $y =$ const. Show that the (x-component of) momentum deposited per unit time per unit volume of the fluid is given by:

$$F_x \sim \frac{\partial}{\partial y}\left(\eta \frac{\partial u_x}{\partial y}\right), \tag{17}$$

with the coefficient of shear viscosity $\eta = mv_t/\sigma$.

2. *Navier-Stokes equation*:

   (a) Using Navier-Stokes and continuity equations along with the first law of thermodynamics, show that:

$$\begin{aligned} \frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2 + u\right)\right] &+ \nabla.\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + h\right) - \zeta\theta\mathbf{v} - \eta\sigma.\mathbf{v}\right] \\ &= \rho T\frac{ds}{dt} - \zeta\theta^2 - \eta\sigma_{ij}\sigma^{ij} \end{aligned} \tag{18}$$

   Interpret different terms of this equation and show that it means that both the coefficients of viscosity must be positive.

   (b) For an incompressible flow, show that the pressure is determined entirely by the velocity field and derive a Poisson equation for the pressure.

   (c) Show that for an incompressible and non-rotational flow, there is no viscous dissipation in the fluid unless the coefficient of viscosity has space dependence.

3. *Simple viscous flows*: Here we study some examples of viscous, incompressible flows that can be exactly solved.

(a) Determine the flow of a viscous fluid of thickness h on a inclined plane due to the force of gravity.

(b) *Poiseuille flow*: Consider fluid flow in a pipe of radius a. The flow is such that the only component of velocity that is non-zero is along the pipe and its variation is only along the cross-section .i.e. the flow can be determined by a velocity field: $v_z(r)$. Show that for a given pressure gradient along the tube, dp/dz, the velocity field is given by:

$$v_z(r) = \frac{dp}{dz}\frac{1}{4\eta}(a^2 - r^2). \tag{19}$$

Compute the tangential force (per unit area) on the walls of the pipe.

(c) *Couette flow*: Consider a flow bounded by two infinite cylinders of radii a_1 and a_2 rotating with angular velocities Ω_1 Ω_2. In this case, the velocity field is given by functional dependence $v_\phi(r)$. Compute the velocity field in this case. Calculate the tangential force on the cylinders and show that if $\Omega_1 = \Omega_2$ this force vanishes.

4. *Spin down of a vortex line*: Consider a line vortex:

$$v_\theta(r) = \frac{\Gamma}{2\pi r}\delta(r) \tag{20}$$

at $t = 0$. Here Γ is a constant and $\delta(r)$ is the Dirac delta function. Using Navier-Stokes equations, show that its dynamical evolution is given by:

$$v_\theta(r,t) = v_\theta(r,0)\left[1 - \exp(-r^2/(4\nu t)\right] \tag{21}$$

Interpret this solution.

Assignment 4

1. *Scaling in the Navier-Stokes equations*:

   (i) Estimate the Reynolds number for: (a) flow past the wing of a jumbo jet at $150\,\mathrm{m\,sec}^{-1}$, (b) a thick layer of sugar syrup draining off a spoon, (c) a spermatozoan with tail length of $10^{-3}\,\mathrm{cm}$ swimming at $10^{-2}\,\mathrm{cm\,sec}^{-1}$.

   (ii) We wish to simulate the flow past a vehicle, 4 m long, travelling at $50\,\mathrm{km\,hr}^{-1}$, using a small scale model, located in a wind tunnel. What is the relationship between the length of the model, and the speed of the air flow in the wind tunnel? How small can we make the model, before the density fluctuations in the wind tunnel, $\Delta\rho/\rho \simeq 0.1$

2. An application of Stokes flow is to the problem of sedimentation of small spherical particles (.i.e. soot particles) in the earth's atmosphere. Show that the terminal velocity reached by the particles is:

$$V = \frac{2\rho_s a^2 g}{9\eta} \tag{22}$$

   Here $\rho_s \simeq 2000 kg\,\mathrm{m}^{-3}$ is the mass density of soot particles; a is the radius of the particles (assume $a \simeq 10^{-7}$–$10^{-4}\,\mathrm{m}$); $\eta = \rho_a \nu \simeq 10^{-5}\,\mathrm{kg\,m}^{-1}\,\mathrm{sec}^{-1}$ is the coefficient of viscosity of air. Compute the range of velocities of the soot particle. Are the assumptions made in deriving the Stokes flow valid in the entire range?

3. *Turbulence*: Verify that the rate of energy dissipation (per unit mass), $\varepsilon_{vis} \sim \nu \left\langle (\partial v_i/\partial x_j)^2 \right\rangle$, is independent of ν, and equal to ε, the rate of energy input.