Basic Fluid Dynamics

S. Sridhar

Raman Research Institute, Bangalore 560080, India

ssridhar@rri.res.in

1. Introduction

- Continuum treatment of classical fluids is valid when the linear dimensions (L) of the system are so large that the volume can be partitioned into many cells, each of which contains many particles: $L \gg \Delta x \gg n^{-1/3}$, where n is the number density of particles. Then the *mass density*, $\rho(\boldsymbol{x}, t)$, is a smoothly varying function of space. In contrast to solids, fluids cannot maintain shear stress without yielding to it.

- If Δx is much larger than the mean free path for collisions, particles cannot free–stream out of cells. Rather, the whole cell can be thought of as moving with a common velocity. Then the *mass-weighted average* velocity, $\boldsymbol{v}(\boldsymbol{x}, t)$, is a smoothly varying function. *Streamlines* are integral curves of the velocity field at any instant of time.

- If we average over times much longer than the collision time, the particles in any cell may be assumed to be in *local thermodynamic equilibrium* (LTE). Then two thermodynamic variables determine all other thermodynamic quantities. The simplest example is a perfect gas, whose equation of state, $p = \rho k T/\mu m_p$ determines $p(\boldsymbol{x}, t)$ as a function of $\rho(\boldsymbol{x}, t)$ and $T(\boldsymbol{x}, t)$.

- **Mass conservation**:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 , \qquad \text{continuity equation} \tag{1}$$

  Define the *convective* derivative, $d/dt \equiv \partial/\partial t + \boldsymbol{v} \cdot \boldsymbol{\nabla}$. Then the continuity equation can also be written as

$$\frac{d\rho}{dt} = -\rho \, (\boldsymbol{\nabla} \cdot \boldsymbol{v}) \tag{2}$$

  Note that $(\nabla \cdot \boldsymbol{v})$ is the fractional rate of change of volume of a fluid element.

- **Internal stresses**: The forces acting on a fluid element can be external (e.g. gravity), as well as those due to the fluid outside of the element. The latter are usually surface forces, such as (i) pressure and (ii) viscous (frictional) forces in the case of a non–ideal fluid. The *stress tensor* makes precise the notion of one part of the medium acting on another part, by exerting a force across their common area of contact. Imagine a small plane of area ΔA oriented perpendicular to the x–axis. Suppose that the material to the left of the area element exerts force $\Delta\mathbf{F}$ on the material to the right. Resolve the force into its components, ΔF_x, ΔF_y, and ΔF_z. If the area element is small enough, the force will be proportional to ΔA. So it makes sense to define

$$S_{xx} \;=\; \frac{\Delta F_x}{\Delta A}\,; \qquad S_{yx} \;=\; \frac{\Delta F_y}{\Delta A}\,; \qquad S_{zx} \;=\; \frac{\Delta F_z}{\Delta A} \tag{3}$$

We call S_{xx} the *normal* component of the stress. S_{yx} and S_{zx} are the *tangential* components of the stress, also referred to as components of the *shear* stress. At any point in the material, we can evidently construct nine numbers, S_{xx}, S_{yx}, $\ldots$, S_{zz}. For convenience, we will organise them into a matrix, often denoted by S_{ij}, where the indices i, j take all possible values, $1, 2, 3$.

S_{ij} is the ith component of the force exerted, per unit area, across a small area element oriented with its normal in the jth direction. Some important properties of any stress tensor are (see § 31-6 of Feynman Lectures II):

1: S_{ij} is a *tensor* field: the ith component of the force per unit area on an area element with unit normal $\mathbf{n}$ is equal to $S_{ij}n_j$.

2: The stress tensor is symmetric: $S_{ij} = S_{ji}$, because of the conservation of angular momentum. Therefore only six of the nine components are independent.

3: The stress tensor may be diagonalised at any point: the stress is normal across area elements oriented perpendicular to the principal axes.

4: The force per unit volume is equal to the *negative of the divergence of the stress tensor*.

- **Momentum balance**: In the rest frame of a fluid element, for an inviscid (or *ideal*) fluid the stress tensor is *isotropic*, and independent of the velocity field. We write

$$S_{ij} \;=\; p\,\delta_{ij}\,; \qquad\qquad \text{ideal fluid} \tag{4}$$

where p is the *pressure*. The force *per unit volume* is

$$f_i \;=\; -\,\frac{\partial S_{ij}}{\partial x_j} \;=\; -\,\frac{\partial p}{\partial x_i} \tag{5}$$

Therefore, momentum balance for an ideal fluid gives

$$\rho\,\frac{d\boldsymbol{v}}{dt} \;\equiv\; \rho\left(\frac{\partial \boldsymbol{v}}{\partial t} \;+\; (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}\right) \;=\; -\boldsymbol{\nabla}p \;-\; \rho\boldsymbol{\nabla}\varphi_g\,, \qquad \text{Euler equation} \tag{6}$$

where $\varphi_g(\boldsymbol{x}, t)$ is the potential of an externally applied gravitational field.

- **Thermodynamics**: In an ideal fluid, the entropy per unit mass, $s(\boldsymbol{x}, t)$, is conserved:

$$\frac{ds}{dt} \equiv \frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} s = 0 \tag{7}$$

- **Boundary conditions**: A fluid cannot penetrate a solid boundary, so the normal component of the *relative* velocity must vanish on a boundary. However, for an *ideal* fluid, there is no constraint placed on the relative tangential velocity.

- **Comment**: Equations (1), (6) and (7) are 5 partial differential equations involving the 6 unknown quantities, $(\rho, \boldsymbol{v}, p, s)$. One more equation of the form $f(\rho, p, s) = 0$, due to LTE, is always assumed to be specified. Therefore, if we are given $(\rho, \boldsymbol{v}, p, s)$ as functions of $\boldsymbol{x}$ at some instant of time, we can, in principle, integrate the equations forward in time, to obtain $(\rho, \boldsymbol{v}, p, s)$ as functions of $\boldsymbol{x}$ at a later time.

2. Hydrostatic equilibrium

- Hydrostatic means $\partial/\partial t$ of all quantities vanish, and $\boldsymbol{v} = \boldsymbol{0}$. Hence

$$\frac{1}{\rho} \boldsymbol{\nabla} p = -\boldsymbol{\nabla} \varphi_g \tag{8}$$

Suppose that $\varphi_g(\boldsymbol{x})$ is some given gravitational potential. Then the solution exists only if the isocontours of $p(\boldsymbol{x})$ and $\rho(\boldsymbol{x})$ coincide. To solve this equation, we need to know something about the thermodynamic state of the fluid. Let us, for simplicity, consider a perfect gas.

- **Plane parallel atmosphere**: All quantities are functions of z, the height above ground–level. Then $\varphi_g = gz$, where g is the acceleration due to gravity. The equation of hydrostatic balance is

$$\frac{1}{\rho} \frac{dp}{dz} = -g \tag{9}$$

When $T =$ constant, the atmosphere is said to be *isothermal*. We can use the perfect gas equation of state, $p = \rho kT/\mu m_p$, to eliminate p from eqn. (9):

$$\frac{1}{\rho} \frac{dp}{dz} = \frac{kT}{m} \left(\frac{1}{\rho} \frac{d\rho}{dz} \right) = -g \tag{10}$$

whose solution is,

$$\rho(z) = \rho(0) \exp\left(-z/H\right) \tag{11}$$

where $H = kT/\mu m_p g$ is called the *scale height* of the atmosphere.

- **Problems**:

  1. Hydrostatic equilibrium of *isothermal* and *isentropic* ($s =$ constant) atmospheres.
  2. Archimedes' principle on hydrostatic equilibrium: buoyancy.

3. Steady flow of an ideal fluid

- A flow is steady if $\partial/\partial t$ of all quantities vanish, but $\boldsymbol{v} \neq \mathbf{0}$. In a steady flow, streamlines are the paths along which fluid elements move. A *Streamtube* is the surface spanned by all the streamlines that pass through a simple, closed curve.

- **Energy conservation**: The energy per unit mass in the fluid is,

$$\varepsilon(\boldsymbol{x}, t) \;=\; \frac{v^2}{2} + \varepsilon_{\rm int} + \varphi_g \tag{12}$$

  where $\varepsilon_{\rm int}$ is the internal energy per unit mass. Accounting for the "pdV" work done by pressure forces on a fluid element moving through a streamtube we can derive *Bernoulli's equation*:

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}B \;=\; 0\,, \qquad B = \frac{v^2}{2} + \varepsilon_{\rm int} + \frac{p}{\rho} + \varphi_g \tag{13}$$

  The above equation states that the quantity B is constant on streamlines. Note that the combination $\varepsilon_{\rm int} + p/\rho = h$, is the *enthalpy* per unit mass.

- Using equation (7), we can also prove that, $\boldsymbol{v}\cdot\boldsymbol{\nabla}s = 0$: i.e. the entropy per unit mass is also constant along streamlines.

- **Applications of Bernoulli's equation**

  (i) *Lift on a 2-dimensional aerofoil*: Consider a thin *aerofoil* inclined at a small angle to the flow, so that the spanwise direction is perpendicular to the flow direction ($\hat{x}$) everywhere. The upward force (per unit length in the spanwise direction) on element dx is $(p_b - p_t)\,dx$, where p_b and p_t are the pressure below and above the aerofoil. Bernoulli's equation gives,

$$p_b - p_t \;=\; \frac{\rho}{2}\left(u_t^2 - u_b^2\right) \;\simeq\; \rho U_0\left(u_t - u_b\right) \tag{14}$$

  where we have used $u_t \simeq u_b \simeq U_0$, the free–stream speed (which is appropriate for a thin aerofoil). Therefore, the total lift per unit span is

$$F_L \;=\; \rho U_0 \int_0^a \left(u_t - u_b\right)\,dx \tag{15}$$

(ii) *When can a steady flow be considered as nearly incompressible*? To answer this question let us consider flow in the absence of an external field. i.e. let us assume that $\varphi_g = 0$. Since $\Delta B = 0$ and $\Delta s = 0$ along a streamline, we have

$$\Delta\left(\frac{v^2}{2}\right) \;=\; -\,(\Delta h)_s \;=\; -\,\frac{1}{\rho}\,(\Delta p)_s \;=\; -\,\left(\frac{\Delta\rho}{\rho}\right)_s c^2 \qquad (16)$$

where c is the speed of sound. Therefore, $|\Delta\rho/\rho|_s \sim (v/c)^2$. For highly subsonic flows, $v \ll c$, and the density variations in the flow are very small. Then the continuity equation (1) implies that $\boldsymbol{\nabla}\cdot\boldsymbol{v} \simeq 0$. Most flows in the lab, or inside the earth, or in our atmosphere are nearly incompressible.

- **Problems**:

  1. Equations of motion in *conservation* form.
  2. The Schwarzschild criterion for the *local stability* of an atmosphere.

4. Vorticity

- Vorticity is a vector field, defined by

$$\boldsymbol{\omega} \;=\; \boldsymbol{\nabla}\times\boldsymbol{v} \qquad (17)$$

A small tracer placed in the fluid will move as a whole with velocity $\boldsymbol{v}$, and rotate with angular velocity $\boldsymbol{\omega}/2$.

- *Vortex lines* are integral curves of $\boldsymbol{\omega}(\boldsymbol{x}, t)$ at time t. Because $\boldsymbol{\nabla}\cdot\boldsymbol{\omega} = 0$, vortex lines are either closed or are infinitely long, or end on a solid boundary. A *vortex tube* is the surface spanned by all the vortex lines that pass through a simple, closed curve.

- A *barotropic* fluid is one whose equation of state is $p = p(\rho)$. Taking Curl of Euler's equation (6), and using the continuity equation (1), we can derive an equation of motion for the vorticity field of a barotropic fluid:

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) \;\equiv\; \left(\frac{\partial}{\partial t} \,+\, \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\frac{\boldsymbol{\omega}}{\rho} \;=\; \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\boldsymbol{\nabla}\right)\boldsymbol{v} \qquad (18)$$

- The *separation*, $d\boldsymbol{x}$, between two nearby fluid elements satisfies the same equation as $(\boldsymbol{\omega}/\rho)$. Consider a vortex tube of infinetesimal length, $d\boldsymbol{x}$, and cross–sectional area, $d\mathbf{A}$, in a barotropic fluid. Let the density and vorticity in the tube be ρ and $\boldsymbol{\omega}$, respectively. Over time, the vortex tube moves to a new location, with new values

$(d\boldsymbol{x}', d\mathbf{A}', \rho', \boldsymbol{\omega}')$. Mass conservation and the fact that $(\boldsymbol{\omega}/\rho)$ behaves like $d\boldsymbol{x}$ imply that

$$\boldsymbol{\omega}' \cdot d\mathbf{A}' \;=\; \boldsymbol{\omega} \cdot d\mathbf{A} \tag{19}$$

This fact is sometimes stated as, "vorticity is frozen in an ideal fluid".

- **Kelvin's Circulation theorem**: Consider an imaginary simple closed curve, $C(t)$, in the fluid. Imagine that the curve is spanned by a surface, S, which is partitioned into many infinitesimal area elements $d\mathbf{A}_i$. If $\boldsymbol{\omega}_i$ be the vorticity of the $i^{\rm th}$ area element, then the sum, $\sum \boldsymbol{\omega}_i \cdot d\mathbf{A}_i$ is conserved in time as the imaginary curve $C(t)$ moves with the fluid. Using Stokes' theorem, we can see that the *circulation* around the moving loop $C(t)$, defined by

$$\Gamma \;=\; \oint_{C(t)} \boldsymbol{v} \cdot d\boldsymbol{x} \tag{20}$$

  is constant in time for a barotropic fluid.

- **Straight vortex line**: This is a steady flow of a barotropic fluid, in which vorticity is concentrated along a single, straight, infinite line:

$$\boldsymbol{\omega} \;=\; \hat{z}\,\Gamma\,\delta(x)\delta(y) \tag{21}$$

  where Γ is the (constant) circulation due to the vortex line. The velocity field has components (in cylindrical coordinates), given by

$$v_R \;=\; v_z \;=\; 0\,, \qquad v_\phi \;=\; \frac{\Gamma}{2\pi R} \tag{22}$$

- Two straight vortex lines, each of strength $\hat{z}\Gamma$, and separated by distance d rotate steadily around each other with angular velocity,

$$\boldsymbol{\Omega} \;=\; \hat{z}\frac{\Gamma}{\pi d^2} \tag{23}$$

- **Problems**

  1. Lift and circulation.
  2. Problems on vorticity.

5. Potential Flows

- In an ideal fluid, if the vorticity of a fluid element is zero at some initial time, we can see from equation (18), that it will be zero for all times. A flow in which $\boldsymbol{\omega} = 0$ is called a *potential* flow. In such a flow, we must have

$$\boldsymbol{v} = \boldsymbol{\nabla}\phi \tag{24}$$

where $\phi(\boldsymbol{x}, t)$ is called the velocity potential. Note that $\phi \to \phi + f(t)$ describes the same flow. Using equation (24) in Euler's equation (6) gives

$$B \;=\; \frac{\partial\phi}{\partial t} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + h + \varphi_g \;=\; \text{constant} \tag{25}$$

Notice that the Bernoulli function, B, takes a constant value in the entire fluid.

- The *boundary condition* on the flow now is that the normal component of the velocity, $\partial\phi/\partial n$, is equal to the normal velocity of the boundary. For steady, potential flows, equation (25) is equivalent to Bernoulli's equation.

- **Incompressible potential flows**: In an incompressible flow, $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$. If the flow is also potential, we have

$$\boldsymbol{\nabla}^2\phi \;=\; 0 \tag{26}$$

Once the boundary conditions have been specified, $\phi(\boldsymbol{x}, t)$ can be determined by solving Laplace's equation. For instance, if a solid object moves in a fluid with some velocity, $\mathbf{u}(t)$, and the flow around the object is potential, then we can conclude that ϕ must depend only on $\mathbf{u}(t)$, and not on the acceleration of the object. Equation (25) is needed only to determine the pressure. No "*pdV*" work is done in an incompressible fluid, so we may assume the internal energy to be constant, and replace h by p/ρ:

$$\frac{p}{\rho} \;=\; B' \;-\; \frac{\partial\phi}{\partial t} \;-\; \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 \;-\; \varphi_g \tag{27}$$

- **Problems**:

  1. Theory of linear sound waves.
  2. Incompressible, potential flow past a cylinder; "d'Alembert's paradox".

6. Gravity Waves and Instability

- An incompressible fluid of density ρ_2 lies on top of another incompressible fluid of density ρ_1. In the *unperturbed* state, both fluids are in hydrostatic equilibrium under the influence of an external gravitational field. Our goal is to describe the flow in both fluids, when their interface is *perturbed* by a small amount.

- Let the unperturbed interface be described by the plane $z = 0$, and the gravitational force per unit mass equal to $-g\hat{z}$. The unperturbed density and pressure fields are

$$\rho_0(z) = \begin{cases} \rho_2\,, & \text{if } z > 0 \\ \rho_1\,, & \text{if } z < 0 \end{cases} \qquad p_0(z) = \begin{cases} -\rho_2 g z\,, & \text{if } z > 0 \\ -\rho_1 g z\,, & \text{if } z < 0 \end{cases} \tag{28}$$

- The perturbed interface is described by a height function, $z = \zeta(x, y, t)$. If the perturbation does not introduce vorticity into the two fluids, the velocity field is potential: $\boldsymbol{v} = \boldsymbol{\nabla}\phi$, where

$$\phi(\boldsymbol{x}, t) = \begin{cases} \phi_2(\boldsymbol{x}, t)\,, & \text{if } z > \zeta(x, y, t) \\ \phi_1(\boldsymbol{x}, t)\,, & \text{if } z < \zeta(x, y, t) \end{cases} \tag{29}$$

- Incompressibility implies that $\boldsymbol{\nabla}\cdot\boldsymbol{v} = \boldsymbol{\nabla}^2\phi = 0$. Also, the influence of the perturbation must vanish for large $|z|$.

$$\begin{aligned} \boldsymbol{\nabla}^2\phi_2 &= 0\,, \quad \text{if } z > \zeta\,; & \boldsymbol{\nabla}\phi_2 &\to \boldsymbol{0}\,, \quad \text{when } z \to +\infty \\ \boldsymbol{\nabla}^2\phi_1 &= 0\,, \quad \text{if } z < \zeta\,; & \boldsymbol{\nabla}\phi_1 &\to \boldsymbol{0}\,, \quad \text{when } z \to -\infty \end{aligned} \tag{30}$$

- The equation defining the interface, $z = \zeta(x, y, t)$, must be satisfied at all times. This implies that, at the interface, the z–component of the velocities of both fluids must be equal to $d\zeta/dt$:

$$\begin{aligned} \frac{\partial\phi_2}{\partial z} &= \left.\frac{d\zeta}{dt}\right|_2 \equiv \frac{\partial\zeta}{\partial t} + \frac{\partial\phi_2}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi_2}{\partial y}\frac{\partial\zeta}{\partial y}\,, \qquad \text{at } z = \zeta \\ \frac{\partial\phi_1}{\partial z} &= \left.\frac{d\zeta}{dt}\right|_1 \equiv \frac{\partial\zeta}{\partial t} + \frac{\partial\phi_1}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi_1}{\partial y}\frac{\partial\zeta}{\partial y}\,, \qquad \text{at } z = \zeta \end{aligned} \tag{31}$$

  Separation (i.e. cavity formation) is forbidden, but slip is allowed at the interface.

- The normal stress must be continuous at the interface: i.e. the interfacial pressures in both fluid must be equal to each other. Using equation (27),

$$\rho_1\left[B_1 - \frac{\partial\phi_1}{\partial t} - \frac{1}{2}|\boldsymbol{\nabla}\phi_1|^2 - gz\right]_{z=\zeta} = \rho_2\left[B_2 - \frac{\partial\phi_2}{\partial t} - \frac{1}{2}|\boldsymbol{\nabla}\phi_2|^2 - gz\right]_{z=\zeta} \tag{32}$$

The constants, B_1 and B_2, can be fixed by requiring that the unperturbed flow satisfies equation (32) at $z = 0$:

$$\rho_1 B_1 \;=\; \rho_2 B_2 \tag{33}$$

Equations (29) through (33) pose the *nonlinear* problem of *stability.*

- **Linearisation**: In the limit of very small perturbations, we can drop terms involving the products of the small quantities, ϕ_1, ϕ_2 and ζ. Equations (29) through (33) reduce to

$$\begin{aligned} \boldsymbol{\nabla}^2\phi_2 &= 0\,, \quad \text{if } z > 0\,. \qquad \boldsymbol{\nabla}\phi_2 \;\rightarrow\; \mathbf{0}\,, \quad \text{when } z \rightarrow +\infty \\ \boldsymbol{\nabla}^2\phi_1 &= 0\,, \quad \text{if } z < 0\,. \qquad \boldsymbol{\nabla}\phi_1 \;\rightarrow\; \mathbf{0}\,, \quad \text{when } z \rightarrow -\infty \end{aligned} \tag{34}$$

$$\frac{\partial\phi_2}{\partial z} \;=\; \frac{\partial\phi_1}{\partial z} \;=\; \frac{\partial\zeta}{\partial t} \qquad \text{at } z = 0 \tag{35}$$

$$\rho_1\left[\frac{\partial\phi_1}{\partial t} + g\zeta\right]_{z=0} \;=\; \rho_2\left[\frac{\partial\phi_2}{\partial t} + g\zeta\right]_{z=0} \tag{36}$$

- Equations (34) through (36) constitute a set of linear PDEs with constant coefficients, so we proceed to Fourier–analyse them. Let us assume that

$$(\phi_1\,,\;\phi_2\,,\;\zeta\,) \;=\; \left(\hat{\phi}_1(z)\,,\;\hat{\phi}_2(z)\,,\;\hat{\zeta}\,\right)\,\exp\left(\sigma t + ik_x x + ik_y y\right) \tag{37}$$

where σ, k_x and k_y are constants. The functions, $\phi_1(z)$ and $\phi_2(z)$ are defined for $z < 0$ and $z > 0$, respectively. Our goal is to derive an equation relating these three constants.

- Use equation (37) in equations (34):

$$\hat{\phi}_1(z) \;=\; A_1\,\exp\left(kz\right); \qquad\qquad \hat{\phi}_2(z) \;=\; A_2\,\exp\left(-kz\right) \tag{38}$$

where $k = \sqrt{k_x^2 + k_y^2}$, and A_1 and A_2 are some constants.

- Use equations (37) and (38) in equations (35):

$$k\,A_1 \;=\; -k\,A_2 \;=\; \sigma\,\hat{\zeta} \tag{39}$$

- Use equations (37) and (38) in equations (36):

$$\rho_1\left[\sigma\,A_1 + g\hat{\zeta}\right] \;=\; \rho_2\left[\sigma\,A_2 + g\hat{\zeta}\right] \tag{40}$$

- Eliminating the amplitudes, A_1, A_2, and $\hat{\zeta}$ from equations (39) and (40) leads to a quadratic equation for σ:

$$\rho_1 \left[gk + \sigma^2\right] \;=\; \rho_2 \left[gk - \sigma^2\right] \tag{41}$$

which may be solved to obtain two modes:

$$\sigma \;=\; \pm\sqrt{gk\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)} \tag{42}$$

Note that σ can be real or imaginary. In the latter case, we must remember that all physical quantities must be real: this is achieved by taking the real part of the right side of equation (37).

- **Rayleigh–Taylor Instability**: From equation (42), σ is *real* when $\rho_2 > \rho_1$, and the flow amplitudes increase exponentially in time (when amplitudes are still small). The motions are restricted to a region of vertical extent $\sim 1/k$ about the interface. Note that our derivation is valid if the gravitational field was replaced by some other mechanism of acceleration.

- **Internal gravity waves**: From equation (42), σ is *imaginary* when $\rho_2 < \rho_1$, and the fluid motion is oscillatory.

- **Surface gravity waves**: When $\rho_2 = 0$, we have $\sigma = \pm i\sqrt{gk}$, describing stable oscillations. Let us write $\sigma = -i\omega$ and choose the negative square–root. Then

$$\phi_1 \;=\; A_1 \exp{(kz)} \cos{(k_x x + k_y y - \omega t)} \tag{43}$$

where

$$\omega(k_x, k_y) \;=\; \sqrt{gk} \tag{44}$$

is the (isotropic) *dispersion relation* for surface gravity waves (also called deep water waves).

- The *phase speed*, $V_{\rm ph}$, and the *group velocity*, $\mathbf{V}_g$ are

$$V_{\rm ph} \;=\; \frac{\omega}{k} \;=\; \sqrt{\frac{g}{k}}\,; \qquad\qquad \mathbf{V}_g \;=\; \frac{\partial\omega}{\partial\boldsymbol{k}} \;=\; \frac{\hat{\boldsymbol{k}}}{2}\sqrt{\frac{g}{k}} \tag{45}$$

Hence individual crests travel at twice the speed of a wavepacket: crests seem to appear at the back of a wavepacket and diappear at the front.

- **Problems**

  1. Surface gravity waves: shallow water waves; effect of *surface tension*.
  2. Ship waves.
  3. Kelvin–Helmholtz instability.

7. Viscous Fluids

- Elastic solids (but not fluids) at rest can possess internal *shear stresses*. However, when they flow, real fluids develop shear stresses, which we have ignored until now. These stresses give rise to *frictional forces* between neighbouring fluid elements and cause *dissipation* of the kinetic energy of the flow. Moreover, there are shear forces between a fluid and a solid boundary. It is a non–trivial (and not self–evident) fact that the relative velocity between the fluid and solid is *zero*.

- **Viscous Stress**: We have already come across one constituent of the stress tensor, the pressure, which contributes to the normal stress in a fluid at rest. As noted earlier, the movement of a real fluid gives rise to additional stresses. In the rest frame of a fluid element, the stess tensor is

$$S_{ij} \;=\; p\,\delta_{ij} \;+\; T_{ij} \tag{46}$$

where T_{ij} is the *viscous* stress tensor. Galilean invariance implies that T_{ij} can depend only on the *gradients* of the velocity field, not on the velocity field itself.

- **Rate of Strain Tensor**: This is equal to the velocity gradient, $\partial v_i/\partial x_j$, at any point in the fluid. Split the velocity gradient into symmetric and anti–symmetric components. The symmetric component is itself split into a divergence–free (shear) part and a pure divergence part:

$$\frac{\partial v_i}{\partial x_j} \;=\; \sigma_{ij} \;+\; \frac{1}{3}\,\theta\delta_{ij} \;+\; r_{ij} \tag{47}$$

where

$$\sigma_{ij} \;=\; \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right)\;; \qquad \text{rate of shear} \tag{48}$$

$$\theta \;=\; \frac{\partial v_k}{\partial x_k}\;; \qquad \text{rate of expansion} \tag{49}$$

$$r_{ij} \;=\; \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \;=\; -\frac{1}{2}\epsilon_{ijk}\omega_k\;; \qquad \text{rate of rotation} \tag{50}$$

- **Stress–Strain relation**: In a *Newtonian* fluid, the viscous stress is proportional to the velocity gradient. However, the stress cannot depend on r_{ij}, because this term describes a local motion in which relative distances between fluid particles do not change. Therefore, in a *homogeneous* and *isotropic* fluid, we must have

$$T_{ij} \;=\; -2\eta\sigma_{ij} \;-\; \zeta\theta\delta_{ij} \tag{51}$$

where η and ζ are the coefficients of *dynamic* and *bulk* viscosities, respectively. In many cases, these can be treated as constants (and we shall do so, in the interests of simplicity of treatment).

8. The Navier–Stokes equation

- Adding the contribution of the force per unit volume, due to viscous stresses, the equation of momentum balance is

$$\rho\,\frac{dv_i}{dt} \;=\; -\frac{\partial p}{\partial x_i} \;-\; \rho\frac{\partial \varphi_g}{\partial x_i} \;-\; \frac{\partial T_{ij}}{\partial x_j} \tag{52}$$

  where T_{ij} is given by equation (51): this is the *Navier–Stokes* (NS) equation. At a solid boundary, the relative velocity between fluid and solid must vanish. Mass conservation is described by the continuity equation (1). However, the entropy is not conserved, because viscous forces dissipate kinetic energy into heat. Therefore equation (7) is no longer true.

- **Problems**:

  1. Molecular origins of viscosity.
  2. Entropy (i.e. heat) production due to viscosity.

- The NS equations are applicable to *subsonic* as well as *supersonic* flows. Many astrophysical flows are supersonic. However, it is important to understand subsonic flows, because they are (i) simpler than supersonic flows; (ii) ubiquitous in the air and water that surrounds us. We saw earlier that subsonic flows could be considered as very nearly incompressible. Our aim is to understand flows, rather than density stratification. *Henceforth we only consider incompressible flows of a constant density fluid.*

- The NS equation for an incompressible fluid:

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} \;+\; (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} \quad &= \quad -\boldsymbol{\nabla}\left(\frac{p}{\rho}\right) \;-\; \boldsymbol{\nabla}\varphi_g \;+\; \nu\boldsymbol{\nabla}^2\boldsymbol{v} \\ \boldsymbol{\nabla}\cdot\boldsymbol{v} \quad &= \quad 0 \\ \text{On solid boundaries, } \boldsymbol{v} \quad &= \quad \text{velocity of the boundary} \end{aligned} \tag{53}$$

  where $\nu = \eta/\rho$ is the *kinematic* viscosity. We note that equations (53) are *complete* in themselves. The continuity equation is trivially satisfied and can be dropped. The entropy equation was needed to specify the local thermodynamical state of the fluid. However, the pressure is now determined by the condition of incompressibility, rather than thermodynamics.

- Take dot product of $\boldsymbol{v}$ with equation (53) and integrate over space, to obtain the rate at which the *kinetic energy* of the fluid is dissipated:

$$\frac{d}{dt}\int d^3x\,\frac{v^2}{2} \;=\; -\frac{\nu}{2}\int d^3x\,\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 \tag{54}$$

- **Problems**

  1. From equation (53), derive a Poisson equation for the pressure.
  2. Potential flows and viscosity.
  3. Flow down an inclined plane with gravity.
  4. Poiseuille flow.
  5. Couette flow.

9. Viscous diffusion of Vorticity

- Take Curl of the NS equations (53):

$$\frac{d\boldsymbol{\omega}}{dt} \equiv \frac{\partial\boldsymbol{\omega}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} + \nu\boldsymbol{\nabla}^2\boldsymbol{\omega} \tag{55}$$

In addition to advection and stretching of vortex lines, vorticity diffuses through the fluid by viscous action.

- **The impulsively pulled plate**: A fluid at rest fills the region $y > 0$. The lower boundary is suddenly jerked at time $t = 0$, and attains velocity $\hat{x}U_0$ (which condition, we assume, is maintained for all time). If the fluid was non-viscous, it would continue to remain at rest, while the lower boundary slips past it. However, when $\nu \neq 0$, the fluid will be set into motion, and this happens by the diffusion of vorticity. For $t > 0$, the velocity field in the fluid must be of the form $\boldsymbol{v} = \hat{x}u(y,t)$. Hence the vorticity field is $\boldsymbol{\omega} = \hat{z}\omega(y,t)$, where $\omega = -\partial u/\partial y$. In equation (55), the advective and vortex stretching terms drop out, and we are left with a diffusion equation for ω:

$$\frac{\partial\omega}{\partial t} = \nu\boldsymbol{\nabla}^2\omega \tag{56}$$

This initial–value problem requires us to specify $\omega(y, 0_+)$. We know that

$$u(y,0_+) = \begin{cases} U_0\,, & \text{if } y = 0 \\ 0\,, & \text{if } y > 0 \end{cases} \tag{57}$$

Hence

$$\omega(y,0_+) = -\frac{\partial u}{\partial y} = U_0\,\delta(y)\,; \qquad \text{vortex sheet at } y = 0 \tag{58}$$

and the required solution to equation (56) is

$$\omega(y,t) = \frac{U_0}{\sqrt{\pi\nu t}}\exp\left(-\frac{y^2}{4\nu t}\right) \tag{59}$$

The velocity field is

$$u(y,t) \;=\; U_0 \;-\; \int_0^y dy'\omega(y',t) \tag{60}$$

After an interval of time, t, fluid in the region $0 < y < \Delta y \sim \sqrt{\nu t}$ has been set in motion.

- **Boundary Layers**: Consider flow past a thin plate, which occupies the region $y = 0, x > 0$. For $x \to -\infty$, the velocity field is $\hat{x}U_0$, where $U_0 > 0$ is a constant. If the fluid were inviscid, it would slip past the plate, and the velocity field would be $\hat{x}U_0$ everywhere outside of the plate. However, when $\nu \neq 0$, the fluid elements encountering the front of the plate (at $x = 0, y = 0$) decellerate to zero velocity, because of the no-slip boundary condition. The steep velocity gradient is responsible for the creation of a sharp spike of vorticity. As the fluid flows past the plate, this vorticity diffuses into the bulk of the fluid. Over an interval of time t, a fluid element (which is not in contact with the plate) travels a distance $x \sim U_0 t$ down the plate. From our experience with the previous problem, we may guess that vorticity should have diffused a distance $\Delta y \sim \sqrt{\nu t}$ perpendicular to the plate. The region, $x > 0, y < \Delta y \sim \sqrt{\nu x/U_0}$ is called the *boundary layer*: at any $x > 0$, the fluid velocity increases sharply, from zero at $y = 0$ to about U_0 for $y \sim \Delta y$. For $y > \Delta y$, the flow is nearly unaffected by the presence of the plate.

- **Problem**: Spin down of a vortex line.

10. Scaling in the NS equations

- Consider two geometrically similar bodies moving in fluids with different densities, and viscosities. Assume steady flows in both cases. Under what conditions are the flows related to each other through some simple scaling? To answer this question, let us begin by casting the NS equations (53) in a dimensionless form.

- Let U and L be typical scales of speed and length of the flow in any one of the two cases considered above. Let

$$\boldsymbol{x} \;=\; L\boldsymbol{x}'\,, \qquad\qquad \boldsymbol{v} \;=\; U\boldsymbol{v}' \tag{61}$$

where $\boldsymbol{x}'$ and $\boldsymbol{v}'$ are dimensionless. The quantity, (p/ρ) has dimensions of velocity-squared. So we write $(p/\rho) = U^2(p/\rho)'$, where $(p/\rho)'$ is dimensionless. Then the steady state NS equation can be written in the dimensionless form,

$$(\boldsymbol{v}'\cdot\boldsymbol{\nabla}')\,\boldsymbol{v}' \;=\; -\boldsymbol{\nabla}'\left(\frac{p}{\rho}\right)' \;+\; \frac{1}{\mathrm{Re}}\boldsymbol{\nabla}'^2\boldsymbol{v}' \tag{62}$$

where

$$\mathrm{Re} \; = \; \frac{UL}{\nu} \, ; \qquad\qquad \text{Reynolds number} \tag{63}$$

Note that the Reynolds number can be thought of as the ratio of *inertial* to *viscous* forces acting on typical fluid elements. The velocity field must be of the form,

$$\boldsymbol{v}' \; = \; \boldsymbol{f}(\boldsymbol{x}', \mathrm{Re}) \, , \qquad \text{i.e.} \qquad \boldsymbol{v} \; = \; U \, \boldsymbol{f}(\boldsymbol{x}/L, UL/\nu) \tag{64}$$

where $\boldsymbol{f}$ is some divergence–free vector function. All other physical quantities, like pressure, can now be constructed. Auto/Aircraft manufacturers use this scaling property of the NS equations by conducting tests on scale–models in wind tunnels.

- **Problems**

  1. Estimate Re for various flows.
  2. Scaling used in wind-tunnel simulations.

11. Flow past obstacles

- **Stokes flow**: Consider flow past a spherical obstacle of radius a. If the fluid velocity far upstream is $\hat{x}U_0$, the Reynolds number for the flow may be defined as $\mathrm{Re} = U_0 a/\nu$. The flow is called Stokes flow when $\mathrm{Re} \ll 1$. The inertial term in the NS equation is much smaller than the viscous term, so force balance gives

$$\boldsymbol{\nabla} p \; = \; \eta \boldsymbol{\nabla}^2 \boldsymbol{v} \tag{65}$$

The problem is to look find $\boldsymbol{v}(\boldsymbol{x})$, such that $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, with $\boldsymbol{v} = 0$ on the surface of the sphere, and $\boldsymbol{v} \to \hat{x}U_0$ for $|\boldsymbol{x}| \to \infty$. Stokes solved this problem, and determined that the *drag force* acting on the sphere is

$$\mathbf{F} \; = \; \hat{x} \, 6\pi\eta a U_0 \tag{66}$$

where 2/3 of the force is due to viscous stress, and 1/3 is due to pressure. We can understand this result by making an order–of–magnitude estimate: The difference in fluid pressure between the front and back of the sphere is $\Delta p \sim \eta U_0/a$. The pressure acts on a surface area $\sim a^2$, so we expect a drag force $\sim \eta a U_0$. Viscous stresses acting on the sides of the sphere contribute similarly.

- **Problem**: Application of Stokes flow to sedimentation of small particles in the atmosphere.

- **Flow past a cylinder**: Consider flow past a cylindrical obstacle of diameter d, with its axis oriented along the z–axis. If the fluid velocity far upstream is $\hat{x}U_0$, the Reynolds number for the flow may be defined as $\mathrm{Re} = U_0 d/\nu$. If the flow is stationary, we know, from the scaling argument in § 10, that the velocity field must be of the form

$$\boldsymbol{v} \;=\; U_0\, \boldsymbol{f}(\boldsymbol{x}/d, U_0 d/\nu) \tag{67}$$

Figures 1 and 2 provide some idea of this steady flow pattern, for quite small values of Re. However, the flow is unstable at higher Re, and becomes unsteady.

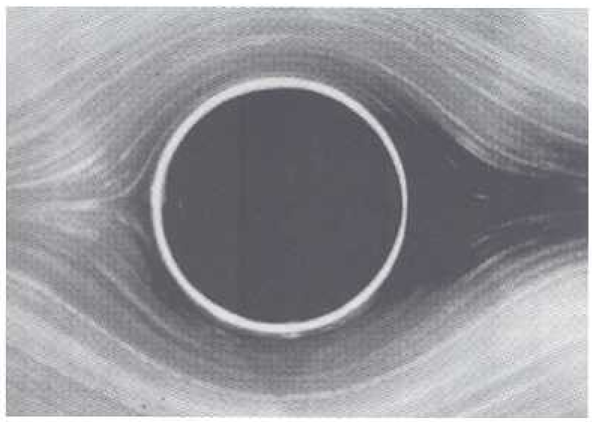

Fig. 1.— Flow at $\mathrm{Re} = 0.16$. When $\mathrm{Re} \ll 1$, there is creeping flow past the cylinder, similar to the case of Stokes flow. The flow is steady, two–dimensional, and has up–down symmetry (left–right symmetry is obtained only when $\nu = 0$).

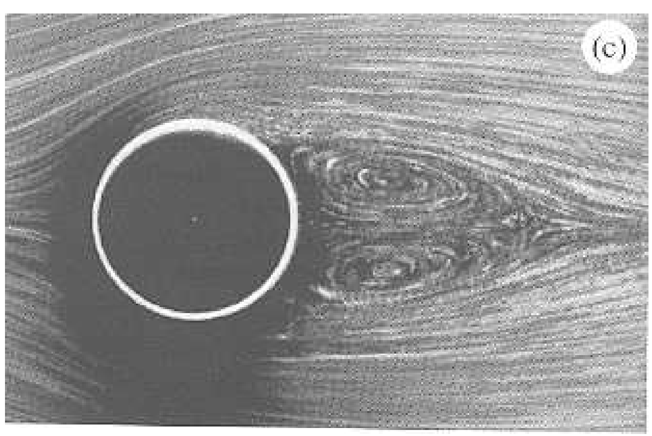

Fig. 2.— Flow at $\mathrm{Re} = 26$. The boundary layer has separated behind the cylinder for $\mathrm{Re} > 5$. Topology of flow changes due to formation of recirculating eddies.

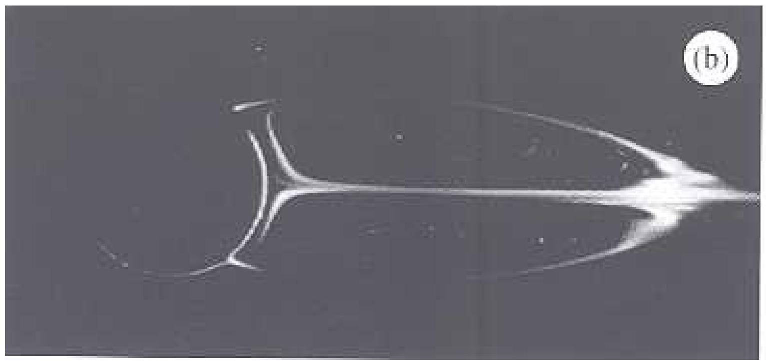

Fig. 3.— Flow at Re = 41 is time–periodic.

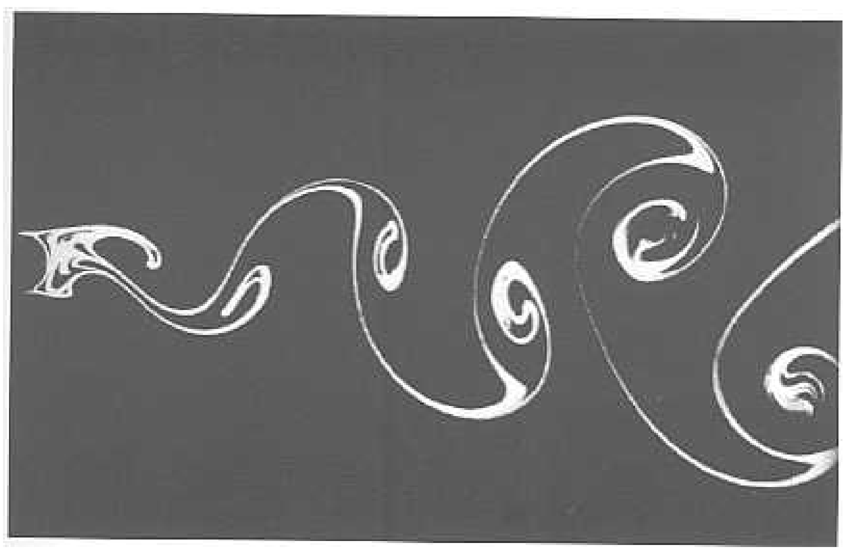

Fig. 4.— Flow for Re = 140. For Re $\sim$ 100, the shedding of the recirculating eddies leads to the formation of alternating vortices, known as a *von Karman street.*

- When Re > 1000, eddies are no longer visible, and the velocity field is irregular on all scales, and the flow is called *turbulent*. At very high Re the flow has *statistically* regular properties, and is called *fully developed turbulence*. A remarkable feature of fully developed turbulence is that its statistical properties (on suitable length and time scales) seem to be independent of the particular manner of its generation. For example, far enough downstream, the flow behind a cylinder is statistically similar to the flow behind a grid; see Figure 5 below.

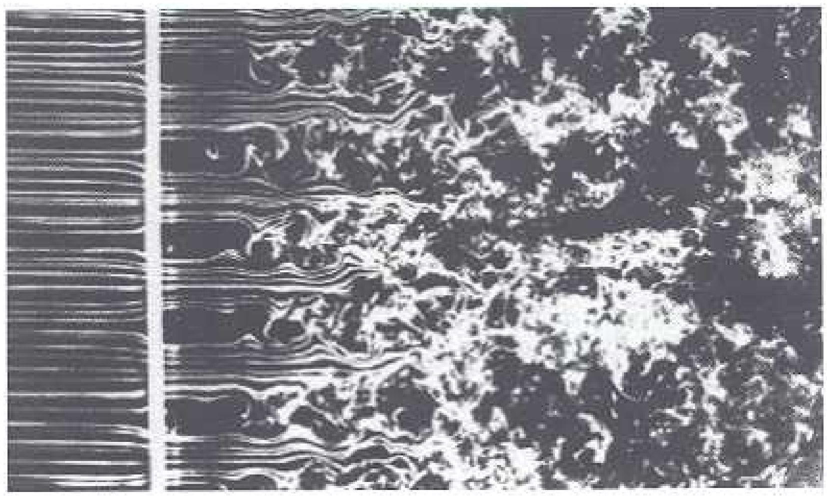

Fig. 5.— Grid generated turbulence.

12. Homogeneous Isotropic Turbulence

- Imagine that an incompressible fluid is stirred *randomly*, with random (subsonic) r.m.s. velocities, v_L, which are correlated on length scale L. The Reynolds number is assumed to be very large:
$$\mathrm{Re} \equiv \frac{L v_L}{\nu} \sim \frac{\text{inertial force}}{\text{viscous force}} \gg 1$$
so that the turbulence may be thought of as fully developed. In a steady state, over length scales much smaller than L, the velocity field will be statistically invariant under translations and rotations. Hence this flow is also known as *homogeneous, isotropic turbulence*.

- The mean energy input rate (per unit mass) is
$$\varepsilon \sim \frac{v_L^3}{L} \tag{68}$$

In a steady state, ε must be equal to the mean rate of viscous dissipation of kinetic energy into heat, $\varepsilon_{vis} \sim \nu \left\langle (\partial v_i/\partial x_j)^2 \right\rangle$ (see equation 54). The velocity gradient cannot be set equal to v_L/L. For large Re, the gradient must be taken on some appropriate length scale, that is much smaller than L.

- *Kolmogorov 1941*: Velocity fluctuations are created on small scales through the *non-linear* interactions provided by the $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\, \boldsymbol{v}$ term in the NS equation. The energy transfer rate through scale r is also

$$\varepsilon \;\sim\; \frac{v_r^3}{r}\,; \qquad\qquad \text{Kolmogorov cascade} \tag{69}$$

Therefore

$$v_r \;\sim\; (\varepsilon r)^{1/3} \;\sim\; v_L \left(\frac{r}{L}\right)^{1/3} \tag{70}$$

- Eddies of size r turn over in time

$$t_r \;\sim\; \frac{r}{v_r} \;\sim\; t_L \left(\frac{r}{L}\right)^{2/3} \tag{71}$$

The time for diffusion of momentum over scale r, due to viscosity,

$$t_r^{vis} \;\sim\; \frac{r^2}{\nu} \tag{72}$$

decreases more rapidly, as r decreases, than t_r. Hence the *cascade* of kinetic energy is dissipated as heat on the (viscous) scale, ℓ, at which $t_\ell \sim t_\ell^{\rm vis}$:

$$\ell \;\sim\; \frac{L}{\mathrm{Re}^{3/4}} \;\ll\; L \tag{73}$$

The range of scales, $\ell \ll r \ll L$ is called the *inertial-range*.

- The 3-dim power spectrum of velocity fluctuations on scale $k \sim 1/r$ is

$$E(k) \;\sim\; \frac{v_r^2}{k^3} \;\propto\; \frac{1}{k^{11/3}}\,; \qquad \frac{1}{L} < k < \frac{1}{\ell} \tag{74}$$

which is also known as the *Kolmogorov* spectrum.

- **Problem**: Verify that the rate of energy dissipation (per unit mass), $\varepsilon_{vis} \sim \nu \left\langle (\partial v_i/\partial x_j)^2 \right\rangle$, is indeed independent of ν, and equal to ε, the rate of energy input.

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