Astrophysical Fluids

Hydrostatics

By setting time derivatives to zero in the Euler equation, one finds to condition of Hydrostatic Equilibrium. In a spherically symmetric case of fluid in gravitational field this looks like

$$\frac{dP}{dr} = -\frac{d\Phi(r)}{dr}\rho(r) = -\frac{GM(r)\rho(r)}{r^2}$$

Where $\Phi(r)$ is the gravitational potential and $M(r)$ is the mass included within radius r. Either $\Phi(r)$ or $M(r)$ needs to be specified: if the fluid's self gravity is dominant then

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

for example in the case of baryonic fluid in stars. On the other hand, if the gravitational potential is determined by a different source (e.g. Dark Matter, in case of baryonic fluid falling into a cluster halo), appropriate specification for $\Phi(r)$ has to be given. If the Dark Matter density distribution $\rho_{\mathrm{DM}}(r)$ is known, then

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho_{\mathrm{DM}}(r)$$

To obtain a solution one needs to also specify a relation between Pressure and Density: $P(\rho)$. This is called the "equation of state". In some cases this relation will involve other variables (e.g. thermal pressure depends both on temperature and density). One additional equation for each such variable is needed to solve the Hydrostatic Equilibrium equation.

A simple example is that of a highly degenerate white dwarf. In this case the temperature is unimportant, the pressure-density relation goes from

$$P \propto \rho^{5/3}$$

at low density, to

$$P \propto \rho^{4/3}$$

at high density where electrons become relativistic. Hydrostatic equilibrium for such "polytropic" equations of state:

$$P \propto \rho^{\gamma}$$

can be rewritten in the form

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{dw}{dz}\right) + w^n = 0$$

where $n = 1/(\gamma - 1)$, z is a scaled (dimensionless) radius and w is the gravitational potential in units of that at $r = 0$. One finds that the radius R of the configuration is finite if $n < 5$, and the mass-radius scaling goes as $M \propto R^{(n-3)/(n-1)}$. For a white dwarf, the solution gives $R \propto M^{-1/3}$ for low mass ($\gamma = 5/3$), but as $\gamma \to 4/3$, the Mass of the configuration approaches a unique value

$$M_{\rm crit} = \frac{5.836}{\mu_e^2} M_\odot$$

which is known as the Chandrasekhar limit. Here μ_e is the mean molecular weight per electron.

Another special case of a polytropic equation of state is that of an "isothermal sphere": T = constant, $P \propto \rho$, for which the solution turns out to be

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}$$

which is often used as a simple model for the density profile of Dark Matter halos, star clusters etc. The mass in this configuration increases linearly with radius:

$$M(r) = \frac{2\sigma^2}{G} r$$

If the pressure depends on temperature, then the temperature stratification needs to be simultaneously solved for. Information about this comes from energy transport. Energy flows down a temperature gradient, so one can relate the luminosity of the object to the temperature profile. In a star, where the energy is generated by nuclear burning in a small core, The amount of energy crossing any spherical shell outside the core per unit time is the same, and is equal to the total luminosity L.

Much of the heat transport in stars is radiative. The radiation flux interacts with matter and exerts a force which equals the radiation pressure gradient

$$\frac{L}{4\pi r^2 c}\kappa\rho = -\frac{d}{dr}\left(\frac{1}{3}aT^4\right) = -\frac{4}{3}aT^3\frac{dT}{dr}$$

where κ is the opacity which gives

$$\frac{dT}{dr} = -\frac{3\kappa\rho}{4acT^3}\frac{L}{4\pi r^2}$$

as the temperature gradient necessary to transport the flux radiatively. Combining this with the Hydrostatic equilibrium equation, one may write

$$\nabla \equiv \frac{d\ln T}{d\ln P}$$

$$\nabla_{\rm rad} = \frac{3}{16\pi acG}\frac{\kappa LP}{M(r)T^4}$$

When the luminosity or the opacity is large, however, the necessary temperature gradient becomes so large that convection can set in.

Convection

The criterion for convective instability can be worked out as follows.

Let us consider a matter element at a radius r in the star, and displace it upwards to $r + dr$. The element would come to pressure equilibrium with the new surroundings, but its density and temperature would not necessarily be the same as those of the surrounding material (see fig. 1). If its density is smaller than the surrounding material then the element

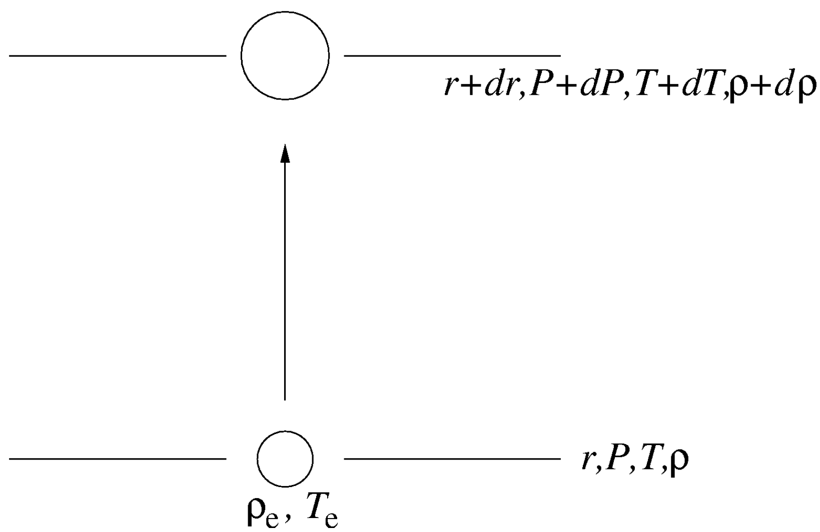

Figure 1: Perturbation of matter element to test for convective instability

would rise due to buoyancy. If the density is higher than the surroundings, it would sink back. The situation will be stable against convection if

$$\left(\frac{d\rho}{dr}\right)_{\rm e} - \left(\frac{d\rho}{dr}\right)_{\rm s} > 0 \tag{1}$$

where the subscript 'e' refers to the element and 's' to the surroundings. Since $P = \rho kT/\mu m_p$, one has

$$\frac{d\rho}{\rho} = \frac{dP}{P} - \frac{dT}{T} + \frac{d\mu}{\mu}$$

Ignoring composition gradient for the time being, we can rewrite eq. (1) as

$$\left(\frac{1}{P}\frac{dP}{dr}\right)_{\rm e} - \left(\frac{1}{T}\frac{dT}{dr}\right)_{\rm e} - \left(\frac{1}{P}\frac{dP}{dr}\right)_{\rm s} + \left(\frac{1}{T}\frac{dT}{dr}\right)_{\rm s} > 0$$

The terms containing pressure gradient cancel due to the pressure equilibrium established between the element and the surroundings, leaving a stability condition in terms of the temperature gradients:

$$-\left(\frac{d\ln T}{dr}\right)_{\rm e} > -\left(\frac{d\ln T}{dr}\right)_{\rm s}$$

Writing in terms of derivatives w.r.t. pressure P instead of r,

$$\left(\frac{d\ln T}{d\ln P}\right)_{\mathrm{s}} < \left(\frac{d\ln T}{d\ln P}\right)_{\mathrm{e}}$$

or

$$\nabla < \nabla_{\mathrm{e}}$$

since $d\ln P/dr < 0$. If the element evolves adiabatically then

$$\nabla_{\mathrm{e}} = \nabla_{\mathrm{ad}} = \frac{\gamma - 1}{\gamma}$$

where γ is the ratio of specific heats. For monatomic gases with $\gamma = 5/3$ the value of ∇_{ad} is 0.4, except in regions of partial ionisation where addition of energy causes an increase in number of particles and hence temperature increases slower than normal, depressing ∇_{ad} below its standard value of 0.4.

If indeed all the transport does take place via radiation then

$$\nabla = \nabla_{\mathrm{rad}}$$

We can then write the condition for stability against convection as

$$\nabla_{\mathrm{rad}} < \nabla_{\mathrm{ad}} \tag{2}$$

This is called the *Schwarzschild criterion* for dynamical stability. If a composition gradient is present, then the stablility criterion is modified to

$$\nabla_{\mathrm{rad}} < \nabla_{\mathrm{ad}} + \nabla_{\mu} \tag{3}$$

where $\nabla_{\mu} = (d\ln\mu/d\ln P)_{\mathrm{s}}$. This is called the *Ledoux criterion* for dynamical stability. If these conditions are violated then convection sets in to transport energy and the temperature gradient ∇ is no longer given by ∇_{rad}. ∇_{rad} now stands for the temperature gradient that would have been necessary to transport the whole flux by radiation.

Convective motion present in the outer layers of solar-type stars is the main contributor to the generation of strong magnetic fields, sunspots (and starspots), and mechanical injection of energy into the atmosphere, producing hot coronae.

Stellar Wind

The hot corona of the Sun tends to expand and gives rise to the solar wind. Parker (1958) constructed a model for the solar wind assuming that it is steady, spherically symmetric and isothermal. The basic equations then are:

$$\dot{M} = -4\pi r^2 \rho v$$

giving

$$\frac{1}{\rho}\frac{\partial \rho}{\partial r} = -\frac{1}{vr^2}\frac{\partial}{\partial r}(vr^2)$$

The Euler equation gives

$$\rho v \frac{\partial v}{\partial r} = -\frac{\partial P}{\partial r} - \frac{GM\rho}{r^2}$$

Writing $c_s^2 = \partial P/\partial \rho$, one can rearrange this to obtain

$$v\frac{\partial v}{\partial r} - \frac{c_s^2}{vr^2}\frac{\partial}{\partial r}(vr^2) + \frac{GM}{r^2} = 0$$

or

$$\frac{1}{2}\left(1 - \frac{c_s^2}{v^2}\right)\frac{\partial (v^2)}{\partial r} = -\frac{GM}{r^2}\left(1 - \frac{2c_s^2 r}{GM}\right)$$

we will see later that this equation has use beyond the solar wind.

Introducing a critical radius $r_c = GM/2c_s^2$, we see that at $r = r_c$ either $v = c_s$ or $dv/dr = 0$. On the other hand if $v = c_s$ then either $r = r_c$ or $dv/dr = \infty$

This allows five different types of solution of the differential equation. One can integrate the equation to give

$$\left(\frac{v}{c_s}\right)^2 - \ln\left(\frac{v}{c_s}\right)^2 = 4\frac{r_c}{r} + 4\ln\left(\frac{r}{r_c}\right) + C$$

where C is an integration constant. Different values of C may choose different branches of solutions.

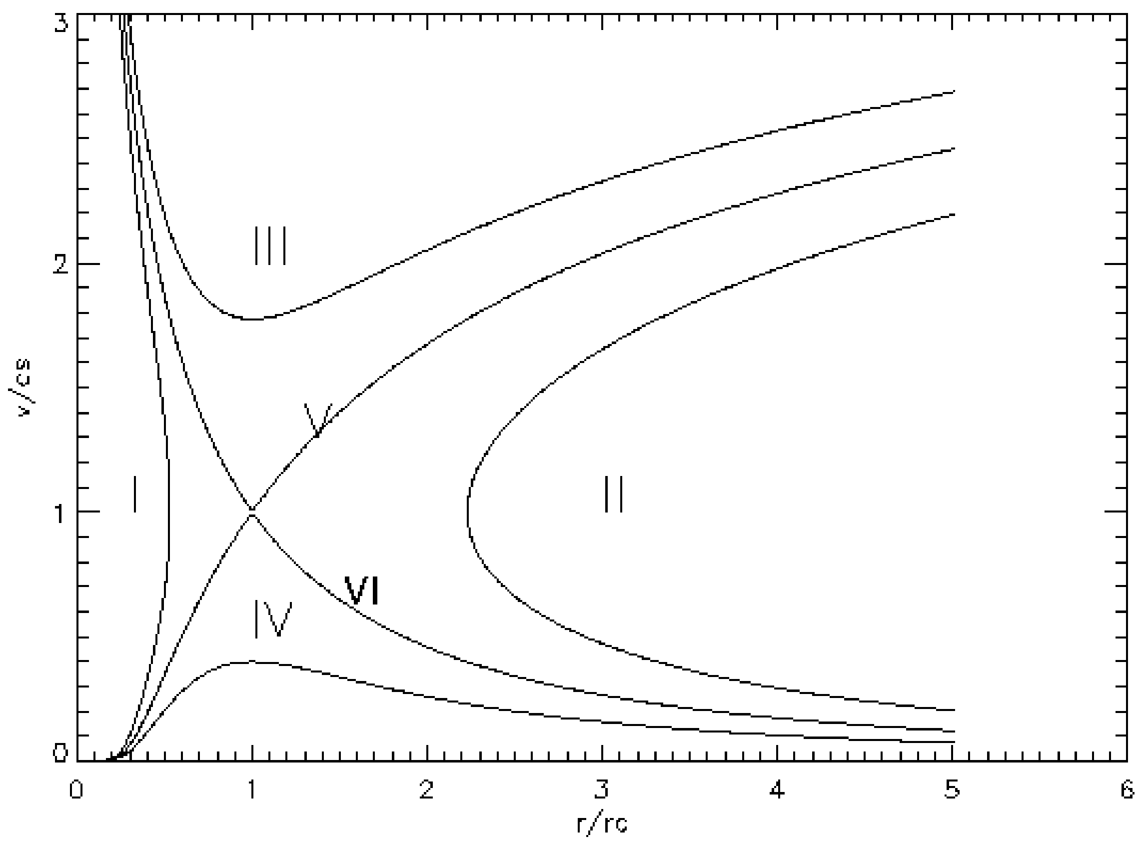

Figure 2: Different solution classes for the Parker wind equation

For solar wind, the velocity is finite at large r, while the wind is subsonic at small r. So the flow is transonic, i.e. $v = c_s$ at $r = r_c$. This is realised for C = -3 and corresponds to the solution V drawn in figure 2. At large r this gives

$$\frac{v}{c_s} \approx 2\left(\ln\frac{r}{r_c}\right)^2$$

and hence

$$\rho \propto \frac{1}{r^2\sqrt{\ln(r/r_c)}}$$

The other solution (VI) passing though the critical point has, at large r, $v \propto 1/r^2$, suggesting a constant density (and hence constant pressure). This can not be contained by ISM pressure and is thus considered unphysical for the solar wind. Also this solution would need the wind to start with a very large speed at the solar surface, which is unlikely.

The time-reversed version of solution VI is, however, reasonable and corresponds to spherical accretion onto a central mass.

Solution IV is referred to as the ”Solar Breeze” solution, which would correspond to a subsonic wind from the sun. Satellite measurements, however, reveal that the solar wind at large r is indeed supersonic and hence solution V provides the appropriate escription.