Spring School on Superstring Theory and Related Topics

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N=2 and the elliptic genus

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LECTURE II: $\mathcal{N}=2$ & THE ELLIPTIC GENUS

Now we are going to combine the constraints of modularity with extended supersymmetry.

1. Review $\mathcal{N}=2$ algebra, SF, unitarity.

2. Modular invce of partition fns

3. Elliptic genus

4. Jacobi forms

5. Physical interpretation of ⊕-decomp.

6. Polar states for the elliptic genus

7. Explicit reconstruction formula.

8. Rademacher

9. AdS_3/CFT_2 + Fareytail

1. $\mathcal{N}=2$ Superconformal Symmetry

$$T(z) = \sum L_n z^{-n-2}$$

$$J(z) = \sum J_n z^{-n-1} \quad U(1) \text{ current}$$

$$G^\pm(z) = \sum_r G_r^\pm z^{-r-3/2}$$

Main relation:

$$\boxed{\left\{ G_r^\pm, G_s^\mp \right\} = 2L_{r+s} \pm (r-s)J_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r+s,0}}$$

$$r \in \mathbb{Z} \pm a \quad \text{in a } \mathbb{Z} \text{ torsor}$$

$$a \in \mathbb{Z} + \tfrac{1}{2}: \quad \text{"NS algebra"}$$

$$a \in \mathbb{Z} \quad \text{"R algebra"}$$

SPECTRAL FLOW ISOMORPHISM $\quad a \to a + \theta$

$$G^\pm_{n \pm a} \longrightarrow G^\pm_{n \pm (a+\theta)}$$

$$L_n \longrightarrow L_n + \theta J_n + \frac{c}{6}\theta^2 \delta_{n,0}$$

$$J_n \longrightarrow J_n + \frac{c}{3}\theta \delta_{n,0}$$

Notation: $c = 3\hat{c} = 6m$

Susy σ-MODEL (X)

$\mathcal{N} = (2,2)$ $\hat{c} = d$ X Kähler, dim $= d$

$\quad\quad = (4,4)$ $m = d/2$ X Hyperkähler

Exercise: Show that

$$4mL_0 - J_0^2 \quad \text{spectral flow invt.}$$

$$\mathcal{H} = \bigoplus \; V_{h,q} \otimes \tilde{V}_{\tilde{h},\tilde{q}}$$

$V_{h,q}:$
$$\begin{cases} (G^{\pm}, J, L)_{>0} \, |h,q\rangle = 0 \\[2mm] L_0 \, |h,q\rangle = h \, |h,q\rangle \\[2mm] J_0 \, |h,q\rangle = q \, |h,q\rangle \end{cases}$$

Constraints of unitarity were worked
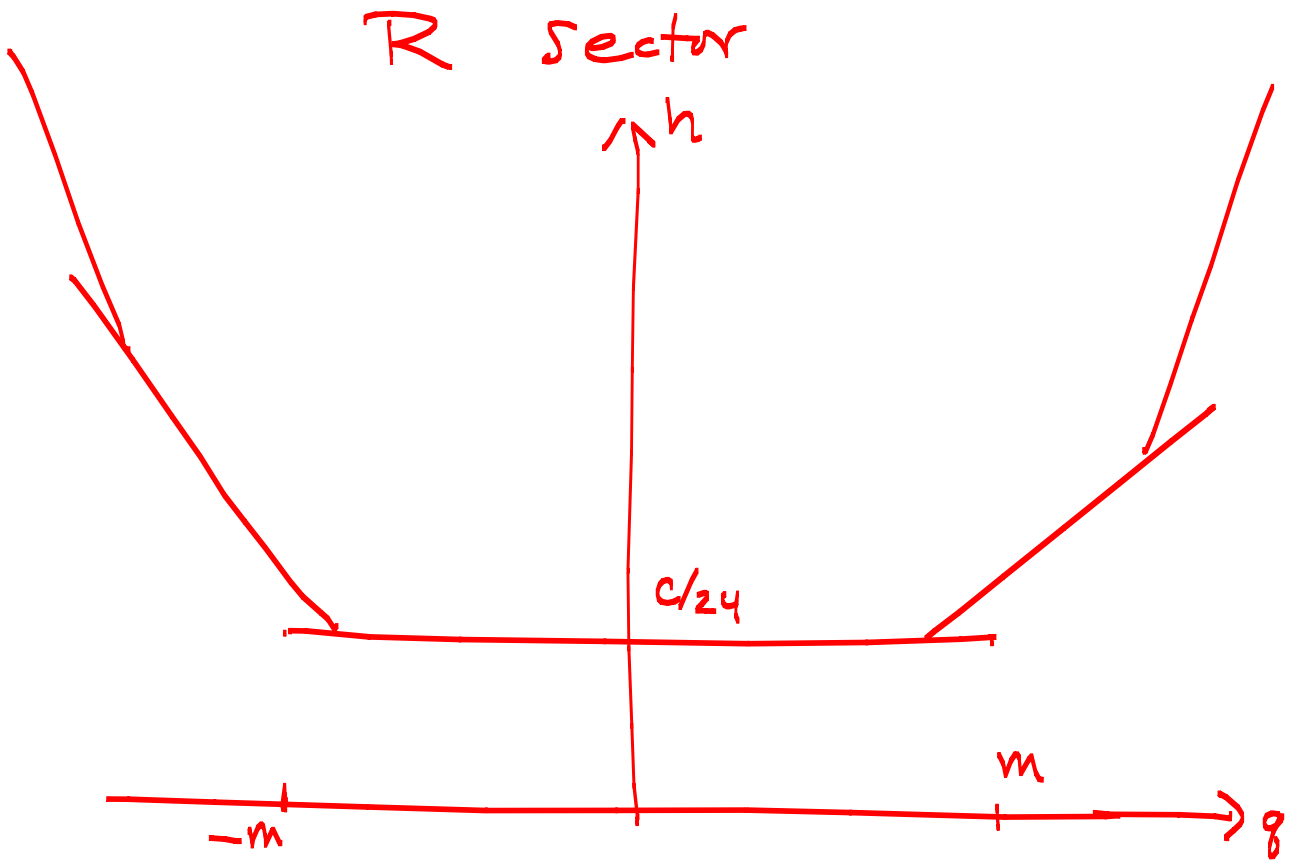out by Boucher-Friedan-Kent.
We just summarize the main facts.

NS-SECTOR
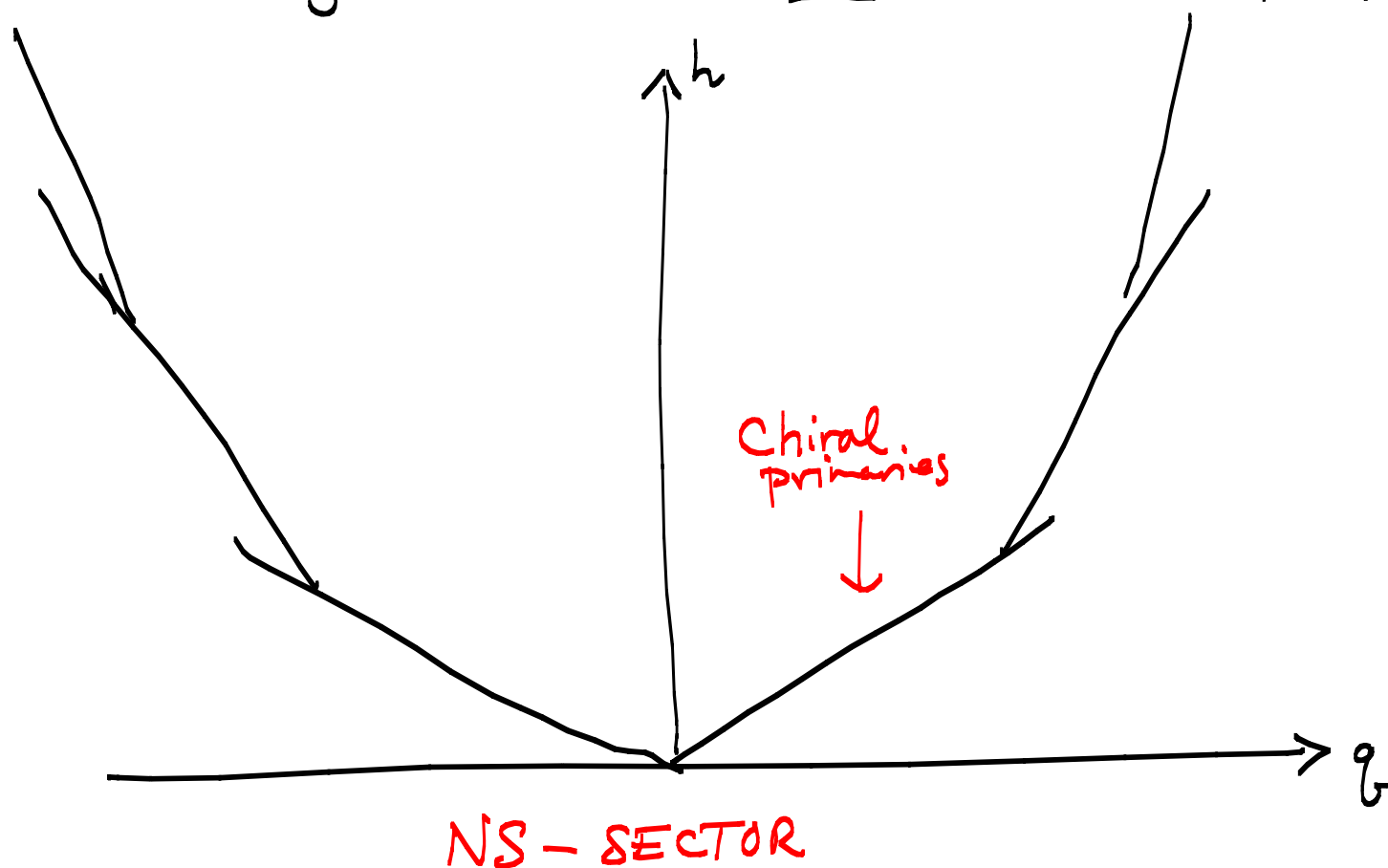
$$0 \leq \| G^{\pm}_{-1/2} |h,q\rangle \|^2 \Rightarrow 0 \leq \langle h,q | \{ G^{\mp}_{\frac{1}{2}}, G^{\pm}_{-\frac{1}{2}} \} | h,q \rangle$$

$$\Rightarrow \quad h \geq \tfrac{1}{2}|q| \qquad \text{BPS BOUND}$$

$h = q/2 \qquad G^{+}_{-1/2} |h,q\rangle = 0$ ⎱ BPS STATE
$\qquad\qquad\qquad\qquad\qquad\qquad$ "CHIRAL PRIMARY"
$h = -q/2 \qquad G^{-}_{-1/2} |h,q\rangle = 0$ ⎰

R sector

$$\|\, G_0^{\pm} |h,q\rangle \,\|^2 \geqslant 0$$

$$\{ G_0^+, G_0^- \} = 2 \left(h - \frac{c}{24} \right)$$

SPECTRAL FLOW TAKES CHIRAL
PRIMARIES TO RAMOND GROUNDSTATES,
SO WE REFER TO THOSE AS BPS STATES.

2. PARTITION FUNCTIONS

NOW CONSIDER $\mathcal{C} = (2,2)$ CFT

WE DEFINE

$$Z_{RR} = \text{Tr}_{\mathcal{H}_{RR}} \, q^{L_0 - c/24} \, e^{2\pi i z J_0} \, \bar{q}^{\tilde{L}_0 - c/24} \, e^{2\pi i \bar{z} \tilde{J}_0}$$

$$\cdot \, e^{i\pi(J_0 - \tilde{J}_0)}$$

THIS HAS THE INTERPRETATION OF A PATH INTEGRAL ON THE TORUS

$$Z_{RR} = \left\langle e^{2\pi i \int_{E_\tau} \left(A^{0,1} J + A^{1,0} \tilde{J} \right)} \right\rangle_{E_\tau}$$

$$A = \frac{i}{2 \, \text{Im} \tau} \left(\bar{z} \, d\xi - z \, d\bar{\xi} \right) \qquad E_\tau$$

$$\implies \text{GOOD MODULAR PROPERTIES}$$

NEED HOLOMORPHY

3. ELLIPTIC GENUS

PUT $\bar{z} = 0$

Then, on the R-movers we are computing the Witten index: $\text{Tr } \bar{q}^{\tilde{L}_0 - c/24} (-1)^F$.

On highest weight rep^n's:

$$\text{Tr}_{V_{h,q}} q^{L_0 - c/24} e^{i\pi J_0} = \begin{cases} e^{i\pi q} & h = \frac{c}{24} \\ 0 & h > \frac{c}{24} \end{cases}$$

<u>Exercise</u>: Use $G_0^{\pm}$ to write states in B/F pairs.

DEF: THE (2,2) ELLIPTIC GENUS

$$\chi(\tau, z; \mathcal{C}) := Z_{RR}(\tau, z; \bar{\tau}, 0)$$

- HOLOMORPHIC
- COUNTS $(*, BPS)$

- MODULAR: $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$\gamma \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right)$$

$$\chi\left(\gamma \cdot (\tau, z)\right) = e^{2\pi i m \frac{c z^2}{c\tau + d}} \chi(\tau, z)$$

(AS EXPLAINED IN THE NOTES)

FOR SIMPLICITY:
1. $m \in \mathbb{Z}$
2. $\text{Spec}(J_0) \subset \mathbb{Z}$

THEN SPECTRAL FLOW $\Longrightarrow$

$$\chi(\tau, z + \theta\tau + \theta') = e^{-2\pi i m \left(\theta^2 \tau + 2\theta z\right)} \chi(\tau, z)$$

$$\theta, \theta' \in \mathbb{Z}$$

4. JACOBI FORMS

$$\phi\left(\gamma\cdot(\tau,z)\right) = \left(c\tau+d\right)^{w} e^{2\pi i m \frac{cz^2}{c\tau+d}} \phi(\tau,z)$$

$$\phi\left(\tau, z+\theta\tau+\theta'\right) = e^{-2\pi i m (\theta^2\tau + 2\theta z)} \phi(\tau,z)$$

$$\theta,\theta'\in\mathbb{Z}$$

Are the transformations of a Jacobi form of weight w and index m.

Note that the transformation laws imply that there is a Fourier expansion:

$$\phi(\tau,z) = \sum_{n,\ell\in\mathbb{Z}} c(n,\ell)\, q^n y^\ell$$

$$y := e(z) \qquad q := e(\tau).$$

As with modular forms, there is a growth condition.

WEAK JACOBI FORMS: $c(n,\ell) = 0$ IF $n < 0$: $\widetilde{J}_{w,m}$

$\Big($ Jacobi forms : $c(n,\ell) = 0$ if $4mn - \ell^2 < 0$ $\Big)$
and don't concern us.

$n = L_0 - c/24$: SO UNITARITY $\Rightarrow$
WK JAC. FRM. CONDITION

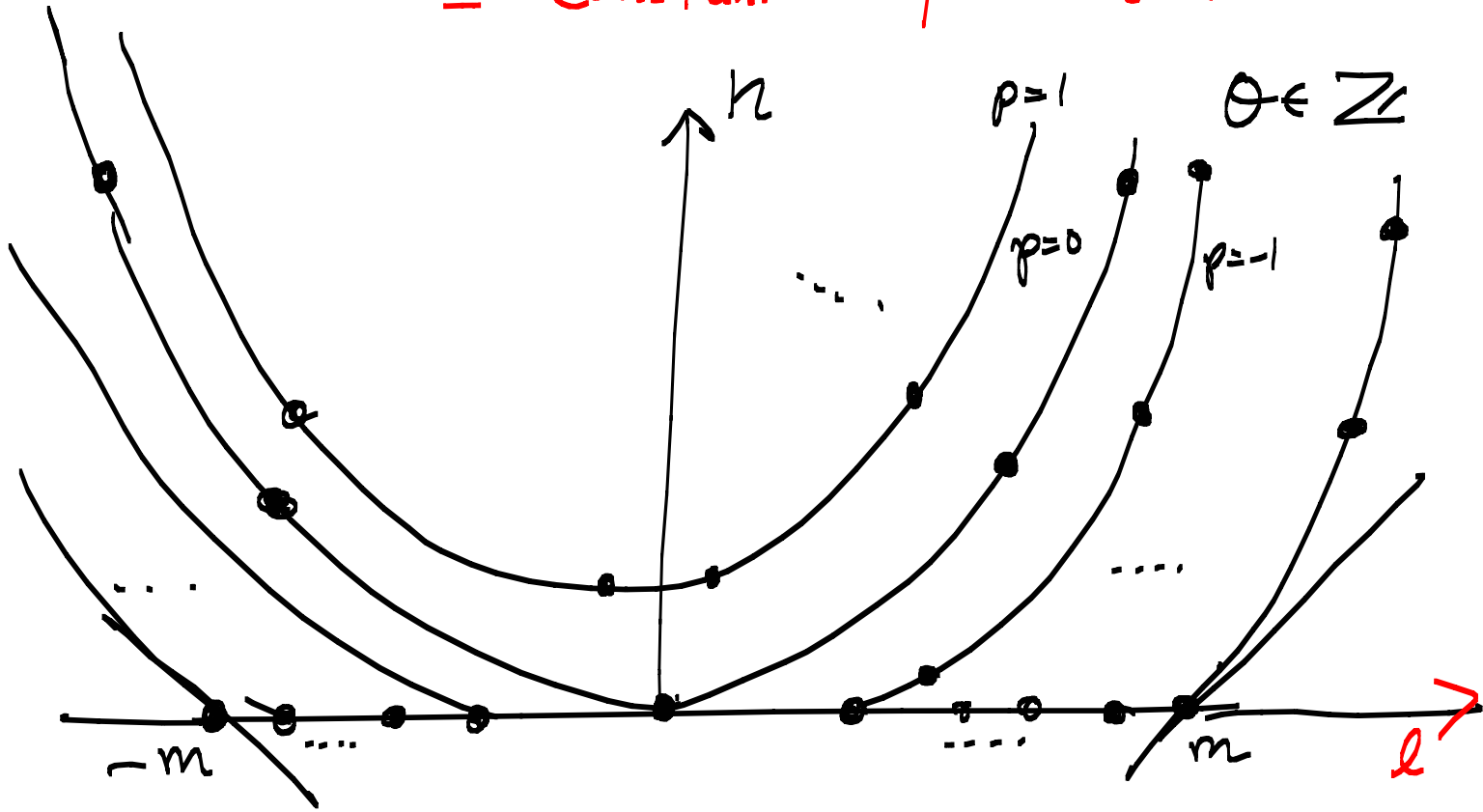$\boxed{N = (2,2) \text{ ELLIPTIC GENUS} \in \widetilde{J}_{0,m}}$

NOW WE EXPLAIN AN IMPORTANT FACT

WEAK JACOBI FORM $\Longleftrightarrow$ V.V. N.H. MOD. FORM

To see this one translates the spectral flow condition to one on the Fourier coefficients:

$$c(n, \ell) = c(n + \ell\theta + m\theta^2, \ell + 2m\theta)$$

= Constant on parabolas $4mn - l^2 = \text{cnt.}$

$p=1$ $\theta \in \mathbb{Z}$

$p=0$

$p=-1$

$\uparrow n$

$-m$ m l

ON THE OTHER HAND, WE CAN ALWAYS BRING l INTO A FUNDAMENTAL DOMAIN FOR THE $\mathbb{Z}$-ACTION

$$l \longrightarrow l + 2m\theta \qquad \theta \in \mathbb{Z}$$

$$\Longrightarrow \quad c(n,l) = c_\mu(p)$$

$$\mu = l \bmod 2m$$

$$p := 4mn - l^2 \quad \text{"polarity"}$$

SO NOW LOOK AT THE FOURIER EXP. OF
ϕ. SUM OVER (n, ℓ) BY FIXING μ,
THEN p, THEN SUM OVER POINTS
ON THE PARABOLA — THAT SUM IS
A $\oplus$-FUNCTION:

$$\sum c(n, \ell) q^n y^\ell =$$

$$= \sum_{\mu \bmod 2m} \sum_{p} C_\mu(p) q^{\frac{p}{4m}} \underbrace{\sum_{\ell = \mu \bmod 2m} q^{\frac{\ell^2}{4m}} y^\ell}_{:= \oplus_{\mu, m} (\tau, z)}$$

$$\Rightarrow \quad \phi(\tau, z) = \sum_{\mu \bmod 2m} h_\mu(\tau) \oplus_{\mu, m} (\tau, z)$$

$$h_\mu(\tau) = \sum_{p = -\mu^2 \bmod 2m} C_\mu(p) q^{\frac{p}{4m}}$$

NOW WE CAN COMPLETE THE
PROOF BECAUSE ONE CAN SHOW
BY POISSON SUMMATION:

$$\Theta_{\mu,m}\left(\gamma\cdot(\tau,z)\right) = (c\tau+d)^{1/2}\, e\left(m\,\frac{cz^2}{c\tau+d}\right)\cdot$$

$$\cdot\, M_{\mu\nu}(\gamma)\, \Theta_{\nu,m}(\tau,z)$$

$$\Longrightarrow\ h_\mu(\tau)\ \text{is a nearly holomorphic}$$
$$\text{vector-valued modular form of wt } w-1/2$$

with mult. system $M^{tr,-1}$

5. PHYSICAL INTERPRETATION

THE THETAFUNCTION DECOMPOSITION
OF THE ELLIPTIC GENUS
HAS A NICE PHYSICAL INTERPRETATION

U(1) CURRENT $\Rightarrow$ CHIRAL BOSON

$$J = i\sqrt{2m}\,\partial\phi \quad , \quad R^2 = m$$

$$SF = \text{MULT. BY } e^{i\sqrt{2m}\,\phi}$$

WE ARE SEPARATING OUT THE
CONTRIBUTIONS OF THIS FIELD

$$\chi(\tau, z) = \sum_{\mu} \tilde{h}_{\mu}(\tau)\, \frac{\vartheta_{\mu,m}(z,\tau)}{\eta}$$

MOREOVER

IF $\mathcal{C}$ HAS A HOLOGRAPHIC DUAL...

$$AdS_3 \times K_7$$

THEN THE U(1) CURRENT COUPLES

TO A GAUGE FIELD A ON AdS_3
WITH A CHERN-SIMONS TERM

$$S = \int \frac{1}{e^2} F * F + m \int A dA + \cdots$$

AT LONG DISTANCES CS-TERM
LEADS TO

SINGLETON MODES / EDGE STATES

AND THESE ARE DESCRIBED BY
THE CHIRAL SCALAR ϕ

$$\chi = \sum_{\mu \bmod 2m} \tilde{h}_\mu(\tau) \frac{\overset{\oplus}{\Theta}_{\mu,m}(z,\tau)}{\eta}$$

SINGLETON

BULK MODES IN

CHARGE SECTOR
$\mu \bmod 2m$

6. POLAR STATES FOR ELLIPTIC GENUS

LET US RETURN TO THE MAIN
THEME OF LECTURE I: THE
POLAR TERMS OF A V-V. N.H.
MOD. FORM DETERMINE THE
WHOLE FORM.

DEF: A POLAR STATE IN e IS
AN EIGENSTATE of $\left(L_0 - \frac{c}{24}, J_0\right) = (n, \ell)$
WITH

$$P = 4mn - \ell^2 < 0.$$

The states of negative polarity
are precisely the states which
contribute to the polar terms of h_μ.

Thus, we are interested in
determining the polar degeneracies
because we want to know nonpolar deg's.

INDEPENDENT POLAR DEG's ?

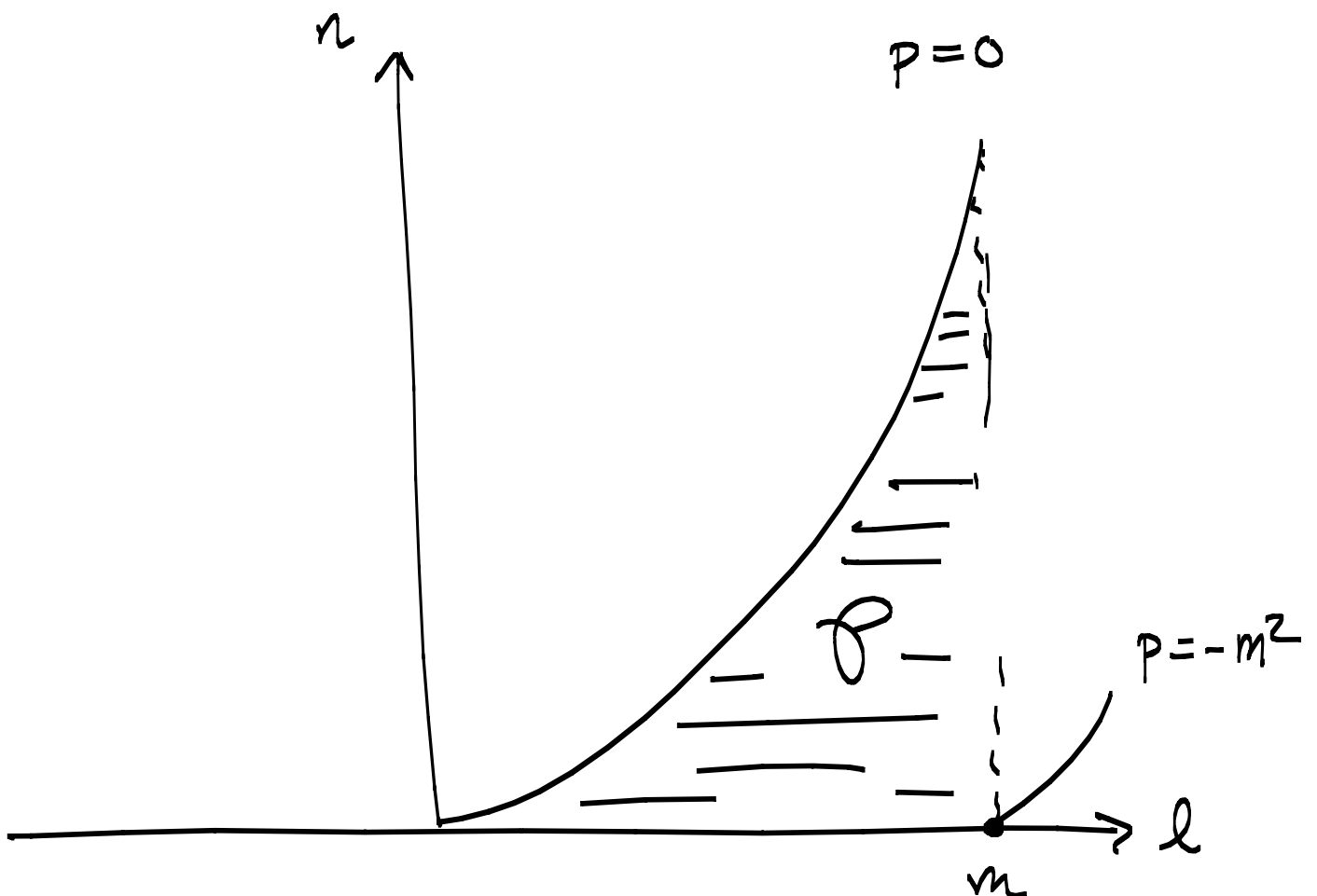$$\triangle = \left\{ (n, \ell) \mid -m^2 \leq p < 0 \right\}$$

$\uparrow$
UNITARITY

S.F. = $\mathbb{Z}$- ACTION PRESERVING P.

CHARGE CONJ. $(n, \ell) \longrightarrow (n, -\ell)$

$\gamma = -1 \implies c(n, \ell) = c(n, -\ell)$

SO $D_\infty = \mathbb{Z}_2 \ltimes \mathbb{Z}$ ACTS ON $\triangle$

LATTICE POINTS IN $\mathcal{P}$ = INDPT
POLAR DEG'S

χ HAS wt $= 0 \Rightarrow h_\mu$ HAS wt $= -\frac{1}{2}$

$\Rightarrow$ FINITE # OF POLAR DEG'S
DETERMINE ALL $c(n, \ell)$

EXPLICIT FORMULA ?

_____

7. EXPLICIT FORMULA.

FOR ANY V.V. N.H. MOD FORM
WITH $w < 0$ WE CAN ASK

RECONSTRUCT f_μ FROM f_μ^- ?

YES! FOR SIMPLICITY TAKE $M(\gamma) = 1$

NAIVE GUESS

$$f(\tau) = \frac{1}{2} \sum_{\Gamma_\infty \backslash \Gamma} j(\gamma, \tau)^{-w} \bar{f}(\gamma \tau)$$

- FORMALLY TRANSFORMS AS MODFORM
- HAS CORRECT POLAR PART.

- BUT $w < 0 \Rightarrow$ ~~CONVERGE~~

$$\bar{f}(\tau) = \sum_{n - \Delta < 0} \hat{f}(n) \, e^{2\pi i (n - \Delta) \tau}$$

Typical term:

$$\left(c\tau + d \right)^{|w|} \underbrace{\exp\left(2\pi i (n - \Delta) \frac{a\tau + b}{c\tau + d} \right)}_{e^{2\pi i (n-\Delta)\frac{a}{c}} \, e^{-2\pi i (n-\Delta)\frac{1}{c(c\tau+d)}}}$$

SO MUST REGULATE:

$$f(\tau) = \frac{1}{2} \sum_{\Gamma_\infty \backslash \Gamma} \left\{ j(\delta,\tau)^{-w} \bar{f}(\gamma\tau) + \text{REG.} \right\} \circledast$$

(LEAVE ON THE BLACKBOARD !)

IF WE START WITH A RANDOM f^-
THEN THE REGULATOR SPOILS MOD. INVCE.

BUT! IF $f^- =$ POLAR PART OF A
MODULAR FORM SOMETHING MAGICAL
HAPPENS: ∃ REG. PRESERVING
MOD. INVCE :   J. MANSCHOT + G.M.

3 APPLICATIONS

1. RADEMACHER
2. ADS$_3$/CFT$_2$ PARTITION FN. "FAREYTAIL"
3. OSV CONJECTURE

8. RADEMACHER EXPANSION

$\hat{f}(n)$ FOR $n-\Delta > 0$ IN TERMS OF $\hat{f}(n)$ FOR $n-\Delta < 0$.

$$\hat{f}(n) = \int_0^1 d\tau \, e^{-2\pi i (n-\Delta)\tau} f(\tau)$$

SUBST. EQUATION ⊛ AND EXCHGE SUM + INTEGRAL.

DO THE INTEGRALS & GET BESSEL FUNCTIONS. RESULT:

FOR $n - \Delta > 0$:

$$\hat{f}(n) = 2\pi \sum_{m-\Delta < 0} \hat{f}(m) \sum_{c=1}^{\infty} \frac{1}{c} K_c(m-\Delta, n-\Delta)$$

$$\left(\frac{|m-\Delta|}{n-\Delta} \right)^{\frac{1-w}{2}} I_{1-w} \left(\frac{4\pi}{c} \sqrt{|m-\Delta|(n-\Delta)} \right)$$

$I_\nu(x) = $ Bessel function.

$$K_c(A, B) = i^{-w} \sum_{\substack{-c < d < 0 \\ (c,d)=1}} e \left(A \cdot \frac{a}{c} + B \cdot \frac{d}{c} \right)$$

= SUM OF PHASES : KLOOSTERMAN SUM.

- CONVERGENT SUM FOR $W < 0$

- GENERALIZES TO V-V FORMS.
  (PUT IN μ, ν)

- COROLLARY: HARDY-RAMANUJAN-CARDY:

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}} e^x, \quad \text{Re } x \longrightarrow +\infty$$

SO WE HAVE A "CONVERGENT SUM
OF EXPONENTIALS"

DOMINANT TERM FROM MOST
NEGATIVE $m - \Delta < 0$.

UNITARY THEORY $\boxed{\Delta = + c/24}$

$$\Rightarrow \quad \log \hat{f}(n) \sim 4\pi \sqrt{\Delta n} + \mathcal{O}(\log n)$$

$$= 2\pi \sqrt{\frac{c}{6} n} + \mathcal{O}(\log n)$$

WARNING: ONLY VALID FOR
$$n - \Delta \gg 1$$

AS DeBoer mentioned in his
lecture There are physical
situations — e.g. The "entropy
enigma" configurations where
we need

$$n - \Delta = \mathcal{O}(1) \quad , \quad n \longrightarrow \infty$$

For example, let

$$\eta^{-\chi}(\tau) = q^{-\chi/24} \sum_{n=0}^{\infty} P_\chi(n) \, q^n$$

Then it turns out that

$$P_\chi\left(\frac{\chi}{24} + \ell\right) \sim \chi^{-1/2} \exp\left(\frac{\chi}{2}\left(1 + \log\frac{\pi}{6}\right) + \frac{\pi^2}{3}\ell\right)$$

$$\chi \to \infty, \quad \ell \text{ fixed.}$$
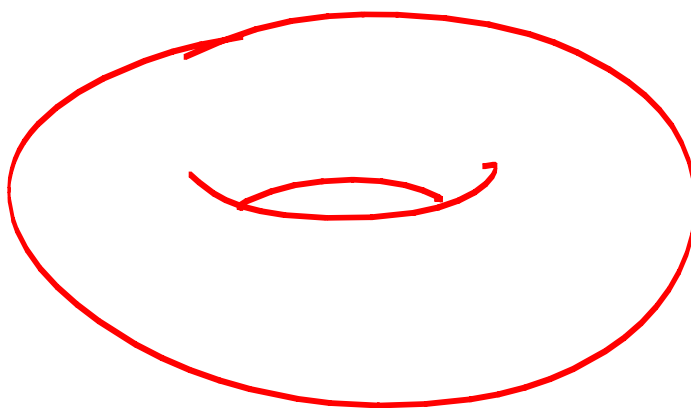
9. AdS_3/CFT_2 PARTITION FUNCTION

NOW LET US RETURN TO THE ELLIPTIC GENUS χ OF OUR (2,2) THEORY $\mathcal{C}$
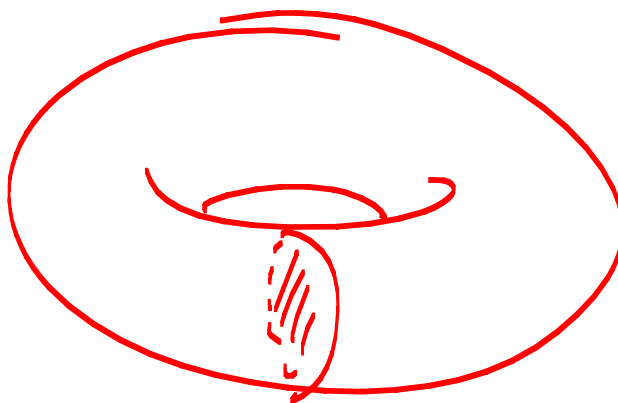
SUPPOSE IT HAS A DUAL

$$AdS_3 \times K$$

$$\chi(\tau, z) = P.F. \text{ ON}$$

$$E_\tau =$$

WE MUST FILL THIS IN WITH A HYPERBOLIC GEOMETRY.

EUCL. BTZ

$$= \mathbb{H}_3/\mathbb{Z}$$

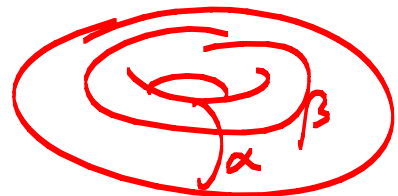$$AdS_3/CFT_2 \Rightarrow \quad Z_{CFT} = Z_{STRING}$$

APPLYING OUR RECONSTRUCTION FORMULA TO $h_\mu(\tau)$ FOR χ WE GET A FORMULA LIKE

$$\chi(z,\tau) = \sum_{\mathcal{P}} C_\mu(\mathcal{P}) \sum_{\substack{\Gamma_\infty \backslash \Gamma \\ = \{(c,d)\}}} \left(q^{\frac{\mathcal{P}}{4m}} \oplus \right)_{\mu,m} \Bigg|_{\gamma_{c,d}}$$

INTERPRETATION :

1. TO WRITE ACTION OF A BH WE MUST CHOOSE CONTRACTIBLE CIRCLE OF EUCL. TIME

$$= \quad c\alpha + d\beta$$

2. SUM OVER $\mathcal{P}$?

TRANSLATING FROM CFT TO
GRAVITY QUANTITIES

$$4mn - \ell^2 = 4mM - J^2$$

COSMIC CENSORSHIP BOUND $4mM - J^2 \geq 0$!

NONPOLAR TERMS $\sim$ BLACK HOLES

POLAR TERMS $\sim$ PARTICLES
IN ADS NOT SUFFICIENTLY
MASSIVE TO COLLAPSE INTO
A BLACK HOLE.

THIS RELATION BETWEEN NON-POLAR
TERMS + BLACK HOLES HAS
RECENTLY PLAYED A ROLE IN
WITTEN'S DISCUSSION OF 2+1
QUANTUM GRAVITY.