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Spring School on Superstring Theory and Related Topics

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Wall-crossing formulae for BPS states & some applications.

Gregory W. Moore
NHETC and Department of Physics and Astronomy
Rutgers University
Piscataway, NJ 08855-0849
USA

WALL-CROSSING FORMULA FOR BPS STATES & SOME APPLICATIONS

TRIESTE SPRING SCHOOL LECTURE IV

APRIL 4, 2008

BASED ON WORK DONE WITH

F. DENEF (hep-th/0702146)

AND FURTHER RESULTS WITH

E. DIACONESCU (0706.3193)

E. ANDRIYASH

M. AGANAGIC + D. JAFFERIS

J. MANSCHOT (0712.0573)

1. INTRODUCTION

THE "SPACE OF BPS STATES" HAS BEEN A CENTRAL CONCEPT IN SUSY GAUGE THEORY & STRING THEORY FOR ALMOST 30 YEARS.

TODAY I'LL FOCUS ON RECENT PROGRESS IN UNDERSTANDING PHENOMENA ASSOCIATED TO MARGINAL STABILITY.

1. INTRODUCTION
2. WALL-CROSSING FORMULAE:
3. PHYSICAL DERIVATION
4. D6-D2-D0 SYSTEM
5. D4D2D0 SYSTEM: MODULAR GEN. FUNCTIONS
6. ROUTE TO OSV: ENTROPY ENIGMA & DEGENERACY DICHOTOMY
7. KONTSEVICH-SOIBELMAN FORMULA
8. OPEN PROBLEMS

A. DEFINING THE "SPACE OF BPS STATES"

FOR DEFINITENESS, WE FOCUS ON THEORIES WITH $d=4$, $\mathcal{N}=2$ SUSY IN (ASYMPTOTIC) MINKOWSKI SPACE $\mathbb{M}_4$

HILBERT SPACE OF ONE-PARTICLE STATES, $\mathcal{H}$, IS A REP. OF THE $d=4$, $\mathcal{N}=2$ ALGEBRA.

$\hat{Z}$: CENTRAL CHARGE OPERATOR

$$\{\hat{Q}_{i\alpha}, \hat{Q}_{j\beta}\} = \delta_{ij} (C\Gamma^\mu)_{\alpha\beta} \hat{P}_\mu + \epsilon_{ij} C_{\alpha\beta} \hat{Z}$$

DECOMPOSE $\mathcal{H} = \bigoplus_{z \in \mathbb{C}} \mathcal{H}_{\hat{Z}=z}$

LEMMA: $E \geq |Z|$ ON $\mathcal{H}_Z$

PROOF: $\mathcal{N} = 2 \Rightarrow$

$$\{ Q_{i\alpha}, Q_{j\beta} \} = \delta_{ij} (C\Gamma^{\mu})_{\alpha\beta} P_{\mu} + \epsilon_{ij} C_{\alpha\beta} Z$$

THIS IS A 6D SUSY ALGEBRA Q_A,

$$\{ Q_A, Q_B \} = (C\Gamma^M)_{AB} P_M$$

WITH $P_4 + i P_5 = Z$. BUT

$$M^2 = E^2 - \vec{P}^2 - |Z|^2 \geq 0. \qquad \blacksquare$$

$$\boxed{\text{DEF'N: } \mathcal{H}_{BPS} \text{ IS THE SUBSPACE OF } \mathcal{H} \text{ WHERE } E = |Z|.}$$

NOW – SPECIALIZE TO TYPE II STRING THEORY ON $M_4 \times X$.

- M_4 IS NONCOMPACT $\Rightarrow$ TO DEFINE THE HILBERT SPACE AS A REP. OF $\mathcal{N}=2$ WE MUST SPECIFY BOUNDARY COND'S FOR THE MASSLESS FIELDS:

$$\lim_{\vec{x}\to\infty} \left(g_{\mu\nu}, \phi, B_{\mu\nu}, RR \right) := \Phi_\infty \in \tilde{\mathcal{M}}$$

$\mathcal{H}_{\Phi_\infty}$: 1-PARTCLE HILBERT SPACE DEPENDS ON Φ_∞

- GENERALIZED MAXWELL THEORY $\Rightarrow$ $\mathcal{H}_{\Phi_\infty}$ IS GRADED BY ELECTRIC/MAGNETIC CHARGE SECTORS:

$$\mathcal{H}_{\Phi_\infty} = \bigoplus_{\Gamma} \mathcal{H}^{\Gamma}_{\Phi_\infty}$$

$\Gamma \in$ (TWISTED) K-THEORY(X)

K-THEORY TO COHOMOLOGY

PHYSICISTS USUALLY WORK WITH COHOMOLOGY

$$\mathcal{E} \in K^0(X) \longrightarrow ch(\mathcal{E})\sqrt{\hat{A}} \in H^{ev}(X, \mathbb{Q})$$

D-BRANES ARE SOURCES:

D6	D4	D2	D0
p^0	$\underline{P}$	Q	q_0
H_6	H_4	H_2	H_0
H^0	H^2	H^4	H^6

Often identify $H^6(X, \mathbb{Z}) \cong \mathbb{Z}$

$K^0(X)/\text{TORSION}$ = LATTICE Λ

$\mathrm{Ch}(\mathcal{E})\sqrt{\hat{A}} \Rightarrow$ CORRESPONDING LATTICE IN $H^{ev}(X,\mathbb{Q})$

Λ HAS A $\natural$ SYMPLECTIC FORM

$$\langle \mathcal{E}_1, \mathcal{E}_2 \rangle = \mathrm{Index}\, \not{D}_{\mathcal{E}_1 \otimes \bar{\mathcal{E}}_2}$$

$$= \int \left(\mathrm{ch}\,\mathcal{E}_1 \sqrt{\hat{A}}\right) \wedge \left(\mathrm{ch}\,\bar{\mathcal{E}}_2 \sqrt{\hat{A}}\right)$$

IN TERMS OF COHOMOLOGY

$$\langle \Gamma, \Gamma' \rangle = \int -p^0 q_0' + PQ' - QP' + q_0 p^{0\prime}$$

PHYSICALLY: DIRAC–SCHWINGER–ZWANZ..

DUALITY INVT. PRODUCT OF ELECTRIC AND MAGNETIC CHARGES.

NOW WE PUT THESE THINGS TOGETHER:

CONSIDER IIA STRINGS WITH

1. X = STATIC, COMPACT, CY 3-FOLD
2. FLAT B-FIELD: $B \in H^2(X, \mathbb{R})$
3. FLAT RR FIELDS

$\Longrightarrow$ $\mathcal{N}=2$, $d=4$ SUGRA

- EACH $\mathcal{H}^{\Gamma}_{\Phi_\infty}$ IS A REP OF $\mathcal{N}=2$
- CENTRAL CHARGE $\boxed{Z = Z(\Gamma; \Phi_\infty)}$

SO, WE STUDY THE BPS SPECTRUM

$$\mathcal{H}_{BPS} = \bigoplus_{\Gamma \in K^0(X)} \mathcal{H}^{\Gamma}_{\Phi_\infty, BPS}$$

FINITE DIMENSIONAL

B. DEPENDENCE ON MODULI

THE SPACES $\mathcal{H}^{\Gamma}_{\Phi_\infty, BPS}$ ARE LOCALLY CONSTANT BUT NOT GLOBALLY CONSTANT AS FUNCTIONS OF Φ_∞

MODULI SPACE $\widetilde{\mathcal{M}}$ IS A PRODUCT:

HYPERMULTIPLETS $\times$ VECTORMULTIPLETS

[CPLX STR., ϕ, RR FIELDS] [COMPLEXIFIED KÄHLER]

WE WORK AT A GENERIC HYPERMULTIPLET.

RECENT PROGRESS HAS BEEN CONCERNED WITH THE DEPENDENCE ON VECTORMULTIPLETS, IN THIS TALK,

$$\boxed{t = B + iJ}$$

- THE JUMPING LOCUS IS REAL CODIMENSION ONE

DEFINE AN INDEX

$$\Omega(\Gamma;\Phi_\infty) = -\frac{1}{2}\,\mathrm{Tr}_{\mathcal{H}^{\Gamma}_{\Phi_\infty,BPS}} (2J_3)^2(-1)^{2J_3}$$

(COMPARE A. SEN'S TALK: HE HAD 6TH HELICITY SUPERTRACE.)

- TECHNICAL POINT:

$$\mathcal{H}^{\Gamma}_{\Phi_\infty,BPS} = \underbrace{\mathcal{H}_{\frac{1}{2}HM}}_{\text{1/2 hyper}} \otimes \mathcal{H}(\Gamma,t_\infty)$$

$2(0)+(\frac{1}{2})$ as spin rep$^{\text{n}}$

$$\Omega(\Gamma;t_\infty) = \mathrm{Tr}_{\mathcal{H}(\Gamma,t_\infty)}(-1)^F$$

HENCEFORTH FOCUS ON $\mathcal{H}(\Gamma;t_\infty)$

- KEY POINT: Ω CHANGES ACROSS WALLS OF MARGINAL STABILITY

C. WHY DO WE CARE?

PHYSICS MOTIVATION

1. THE MAIN MOTIVATION FOR RECENT WORK IS THE PROGRAM, INITIATED BY STROMINGER–VAFA (1995) OF ACCOUNTING FOR BH ENTROPY VIA MICROSTATE COUNTING. THAT GOAL IS STILL NOT FULLY ACCOMPLISHED.

WE DON'T KNOW BPS DEGENERACY FOR CERTAIN NATURAL CHARGE REGIMES, FOR EXAMPLE:

$$\Gamma \longrightarrow \lambda\Gamma \qquad \lambda \longrightarrow \infty$$

2. OSV CONJECTURE:

RELATION BETWEEN

$\Omega(\Gamma)$ & GW/DT/GV INVARIANTS

$\Longrightarrow$ NONPTVE TOPOLOGICAL STRING?

MATH MOTIVATION

1. PHYSICAL STABILITY OF BPS STATES IS RELATED TO MATH. STABILITY IN THE BOUNDED DERIVED CATEGORY OF COHERENT SHEAVES ON A C.Y.: KONTSEVICH, DOUGLAS, BRIDGELAND, THOMAS, PANDHARIPANDE ....

PHYSICS $\Rightarrow$ PREDICTIONS/CONSTRAINTS ON WHAT WE EXPECT SHOULD BE TRUE.

2. MANY INTERESTING CONNECTIONS TO AUTOMORPHIC FORMS AND ANALYTIC NUMBER THEORY; SOME RELATIONS TO ARITHMETIC CY'S.

3. THERE ARE SEVERAL OTHER MORE SPECULATIVE APPLICATIONS, E.G. BPS ALGEBRAS: GENERALIZING NAKAJIMA'S WORK AND SUGGESTED BY TYPE II/HET DUALITY SHOULD BE CLOSELY RELATED.

2. WALL-CROSSING FORMULAE: STATEMENT

$N=2,\ d=4$ Algebra $\Longrightarrow$

- MODULI OF VACUA $\widetilde{\mathcal{M}}$
- LATTICE OF ELECTRIC/MAGNETIC CHARGES Λ
- CENTRAL CHARGE: $Z: \Lambda \times \widetilde{\mathcal{M}} \longrightarrow \mathbb{C}$

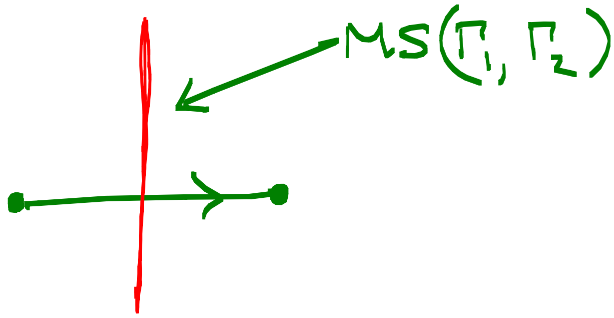
WALLS WHERE $\mathcal{H}_{BPS}$ MIGHT JUMP

$$MS(\Gamma_1,\Gamma_2) := \{ t \mid Z(\Gamma_1,t) = \lambda Z(\Gamma_2,t),\ \lambda \in \mathbb{R}_+ \}$$

$$|Z_1 + Z_2| = |Z_1| + |Z_2|$$

CECOTTI, INTRILIGATOR, VAFA ; SEIBERG & WITTEN:

A BOUNDSTATE OF PARTICLES WITH CHARGES Γ_1, Γ_2 CAN DECAY



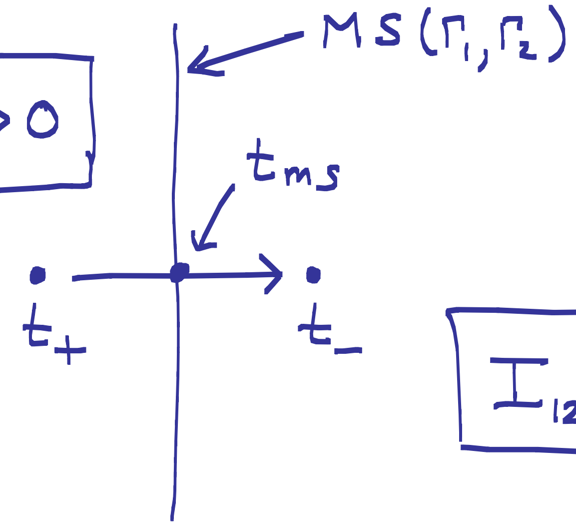
WE WANT TO SAY HOW MANY STATES DECAY.

PRIMITIVE WALL-CROSSING FORMULA:

Λ HAS SYMPLECTIC FORM $\langle \cdot , \cdot \rangle$

LET $I_{12} := \langle \Gamma_1 , \Gamma_2 \rangle$

$\boxed{I_{12}\,\mathrm{Im}(Z_1 Z_2^*) > 0}$



$\boxed{I_{12}\,\mathrm{Im}(Z_1 Z_2^*) < 0}$

Γ_1, Γ_2 PRIMITIVE, t_{ms} GENERIC $\Longrightarrow$

$$\boxed{\mathcal{H}_+ - \mathcal{H}_- = (J_{12}) \otimes \mathcal{H}(\Gamma_1; t_{ms}) \otimes \mathcal{H}(\Gamma_2; t_{ms})}$$

$$J_{12} = \frac{1}{2}\left(|I_{12}| - 1\right)$$

$$\Delta\Omega = (-1)^{I_{12}-1} |I_{12}|\, \Omega(\Gamma_1; t_{ms})\, \Omega(\Gamma_2; t_{ms})$$

SEMI-PRIMITIVE WALL-CROSSING FORMULA

IN ADDITION TO $\Gamma_1 + \Gamma_2$ BOUNDSTATES WE CAN ALSO FORM $N_1\Gamma_1 + N_2\Gamma_2$ BOUNDSTATES

$$MS(\Gamma_1, \Gamma_2) = MS(N_1\Gamma_1, N_2\Gamma_2) \quad N_1, N_2 \in \mathbb{Z}_+$$

CONSIDER $N_1 = 1$, $N_2 \geq 1$:

$$\bigoplus_{N_2} u^{N_2} \Delta\mathcal{H}\Big|_{\Gamma \to \Gamma_1 + N_2\Gamma_2}$$

CLAIM: THIS IS A $\mathbb{Z}_2$-GRADED FOCK SPACE

$$\mathcal{H}(\Gamma_1; t_{ms}) \bigotimes_{k=1}^{\infty} \mathcal{F}\left(u^k \underbrace{\left(J_{\Gamma_1, k\Gamma_2}\right) \otimes \mathcal{H}(k\Gamma_2; t_{ms})}_{\text{GRADED SPACE OF OSCILLATORS}} \right)$$

IN PARTICULAR:

$$\boxed{\begin{aligned} &\Omega_1 + \sum_{N>0} u^N \Delta\Omega(\Gamma_1 + N\Gamma_2) = \\ &= \Omega(\Gamma_1) \prod_{k>0} \left(1 - (-1)^{\langle \Gamma_1, k\Gamma_2 \rangle} u^k\right)^{|\langle \Gamma_1, k\Gamma_2 \rangle| \, \Omega(k\Gamma_2)} \end{aligned}}$$

3. PHYSICAL DERIVATION OF WCF

A. SUPERGRAVITY TOOLS

D-BRANES ARE OBJECTS IN A CATEGORY

IN TYPE IIA/CY, THE SUBCATEGORY OF SUSY BRANES IS PROBABLY THE BOUNDED DERIVED CATEGORY OF COHERENT SHEAVES.

BUT WE WANT TO DESCRIBE THE (PHYSICALLY) STABLE OBJECTS.

AT WEAK STRING COUPLING, AND $J \to \infty$ $\exists$ A BEAUTIFUL DESCRIPTION OF STABLE BPS STATES USING SUGRA.

IN THE SEMICLASSICAL LIMIT

$\psi \in \mathcal{H}_{BPS} \rightsquigarrow$ BPS SOLUTION OF SUGRA EQUATIONS

* SUPERGRAVITY ALLOWS ONE TO IDENTIFY MANY "STABLE OBJECTS" THANKS TO THE ATTRACTOR MECHANISM.

ATTRACTOR MECHANISM: (F.K.S. ; STROMINGER)

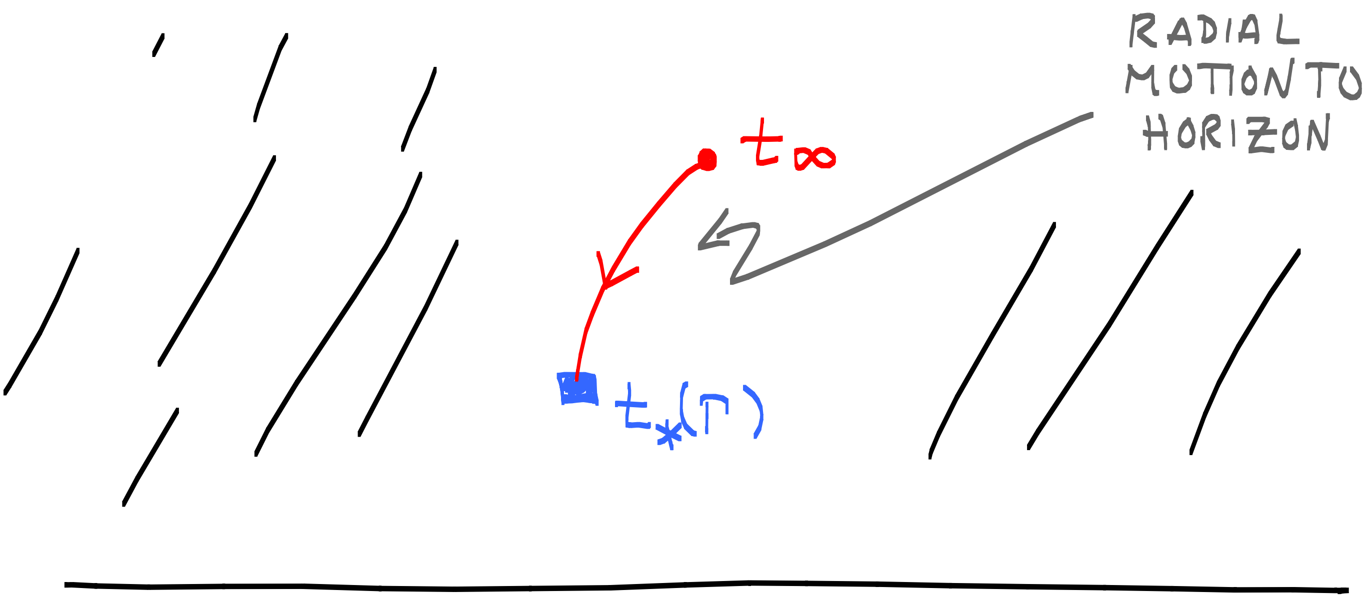
Γ, t_∞ & SPHERICAL SYMMETRY

$\Longrightarrow$ $\exists$ AT MOST ONE BPS SOLUTION OF SUGRA.

IF IT EXISTS....

SCALAR FIELDS $t = t(r)$, AND EVOLUTION FROM $r=\infty$ to $r=0$ APPROACHES AN ATTRACTIVE FIXED POINT $t_*(\Gamma)$:

$\widetilde{\mathcal{M}}_{VM}$



ATTRACTOR FLOW = GRADIENT FLOW FOR

$$\log |Z(\Gamma; t)|^2$$

$$Z = \frac{\langle \Gamma, \omega \rangle}{\sqrt{\langle \omega, \omega^* \rangle}}$$

$$\langle \Gamma, \Gamma' \rangle = \int -p^0 q_0' + PQ' - QP' + q_0 p^{0\prime}$$

ω = PERIOD VECTOR

IN LARGE RADIUS APPROXIMATION:

$$\omega = -e^{t} = -e^{B+iJ}$$

$$\boxed{Z \approx \frac{\frac{1}{6}p^0 t^3 - \frac{1}{2}Pt^2 + Qt - q_0}{\sqrt{(\mathrm{Im}\, t)^3}}}$$

BASIC TRICHOTOMY

1. $t_*(\Gamma) \in \text{Interior}(\widetilde{\mathcal{M}})$

   and $Z(\Gamma; t_*(\Gamma)) \neq 0$

   "REGULAR ATTRACTOR POINT"

2. $\exists$ NONEMPTY SUBVARIETY $\subset \widetilde{\mathcal{M}}$

   $Z(\Gamma; t) = 0$

3. $t_*(\Gamma) \in \partial \widetilde{\mathcal{M}}$

(1.) $\exists$ SPHERICALLY SYMMETRIC BPS BLACK HOLES IN $\mathcal{H}_{BPS}(\Gamma; t)$ FOR ALL t

(2.) $\mathcal{H}_{BPS}(\Gamma; t) = \phi$ IN AN OPEN REGION OF THE ZERO LOCUS.

$\mathcal{H}_{BPS}$ MIGHT BE NONEMPTY FURTHER AWAY

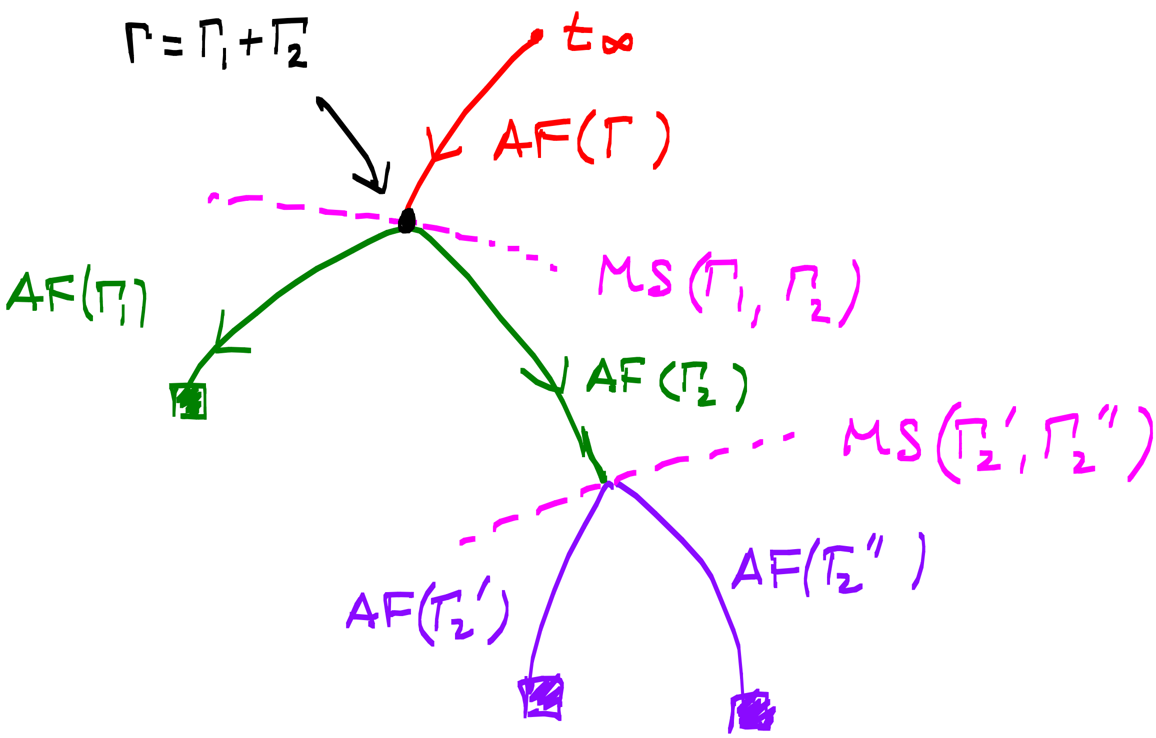
(3.) CANNOT USE SUGRA TO ESTABLISH EXISTENCE: MUST USE MICROSCOPIC ARGUMENTS.

B. SPLIT ATTRACTOR FLOWS

IF $Z(\Gamma;t)=0$ HAS SOLUTIONS IN THE INTERIOR OF MODULI SPACE THEN USE:

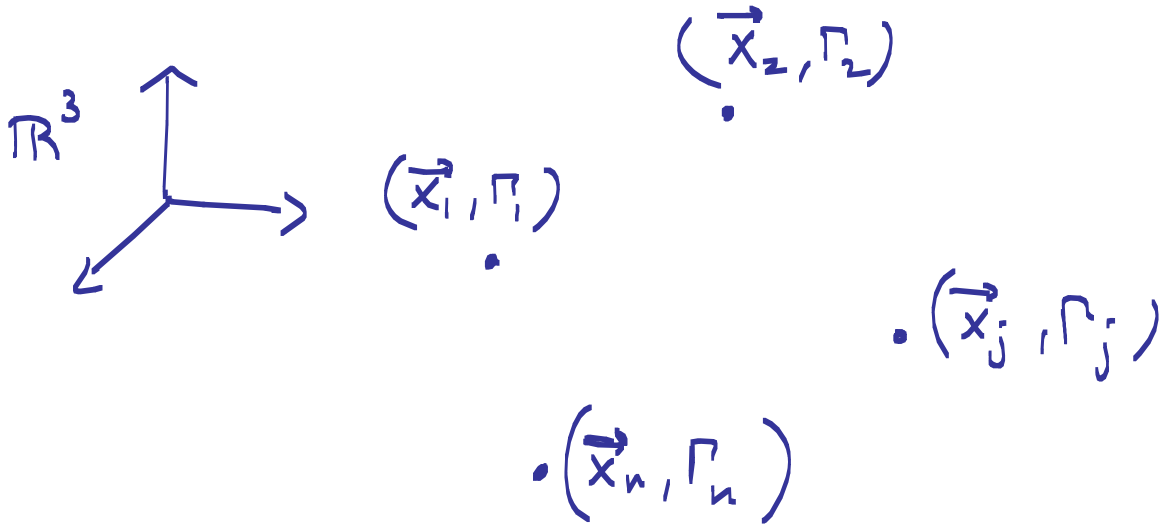
DENEF'S RULE: $\mathcal{H}(\Gamma;t) \neq 0 \Longleftrightarrow$ $\exists$ A SPLIT ATTRACTOR FLOW (S.A.F.)

S.A.F.: A PIECEWISE ATTRACTOR FLOW, JOINED ALONG WALLS OF M.S., CONSERVING CHARGE AT THE VERTICES, TERMINATING ON R.A.P.'S:



- IF SUCH ATTRACTOR FLOW TREES EXIST WE CAN CONSTRUCT A CORRESPONDING SOLUTION OF SUGRA.

- SPACETIME PICTURE:



- NEAR EACH POINT $\vec{x}_i$ THE SOLUTION LOOKS LIKE THE SINGLE-CENTERED SOLUTION: "BLACK-HOLE MOLECULES"

MULTICENTERED SOLUTIONS:

GENERAL BPS EQUATIONS

(1.) $ds^2 = -e^{2U}(dt+\oplus)^2 + e^{-2U} d\vec{x}^2$

$U = U(\vec{x})$, $\vec{x} \in \mathbb{R}^3$

(2.) CHOOSE A HARMONIC MAP

$H : \mathbb{R}^3 \longrightarrow H^{ev}(X, \mathbb{R})$

$$H(\vec{x}) = \sum_j \frac{\Gamma_j}{|\vec{x} - \vec{x}_j|} + H_\infty$$

$$\boxed{2e^{U} \operatorname{Im}(e^{-i\alpha}\omega) = -H(\vec{x})} \Rightarrow$$

(a.) $U(\vec{x})$ completely fixed,

(b.) $e^{-2U(\vec{x})} = S(H(\vec{x}))$

(3.) $\quad *_3 d\,Ⓗ = \langle dH, H\rangle$

$\Longrightarrow$ INTEGRABILITY CONDITION :

$$\boxed{\sum_{\substack{j \\ j\neq i}} \frac{\langle \Gamma_i, \Gamma_j\rangle}{|\vec{x}_i - \vec{x}_j|} = 2\,\mathrm{Im}\left(e^{-i\alpha} Z(\Gamma_i)\right)_\infty}$$

SUGRA SOLUTION EXISTS $\Longleftrightarrow$

$\forall\, \vec{x} \in \mathbb{R}^3 :$

$t(\vec{x}) \in \mathcal{M}_{VM}$ $\quad$ &

$$\pi e^{-2U(\vec{x})} = S(H(\vec{x})) \geq 0$$

(A VERY NONTRIVIAL CONDITION
TO CHECK ...)

SPLIT ATTRACTOR CONJECTURE (DENEF)

(a.) (COMPONENTS OF MODULI OF) MULTICENTERED SOLUTIONS ARE IN $1 \leftrightarrow 1$ CORRESPONDENCE WITH S.A.F.'s.

(b.) FOR A FIXED (t_∞, Γ) THERE ARE A FINITE NUMBER OF S.A.F.'s

- USEFUL BECAUSE CHECKING $S(H(\vec{x}_i)) > 0$ IS DIFFICULT

- $\mathcal{H}_{BPS}$ IS PARTITIONED BY SPLIT ATTRACTOR FLOWS

- $\exists$ SOME INTERESTING OPEN PROBLEMS HERE ....
  - * QUANTUM MIXING BETWEEN DIFFERENT AF TREES
  - * USEFUL EXISTENCE CRITERION FOR SCALING SOLUTIONS.

C. DERIVATION OF PRIMITIVE WCF:

CONSIDER BOUNDSTATE OF TWO PRIMITIVE CHARGES:

$$\overset{\Gamma_1}{\bullet} \xrightarrow{\quad \vec{R} \quad} \overset{\Gamma_2}{\bullet}$$

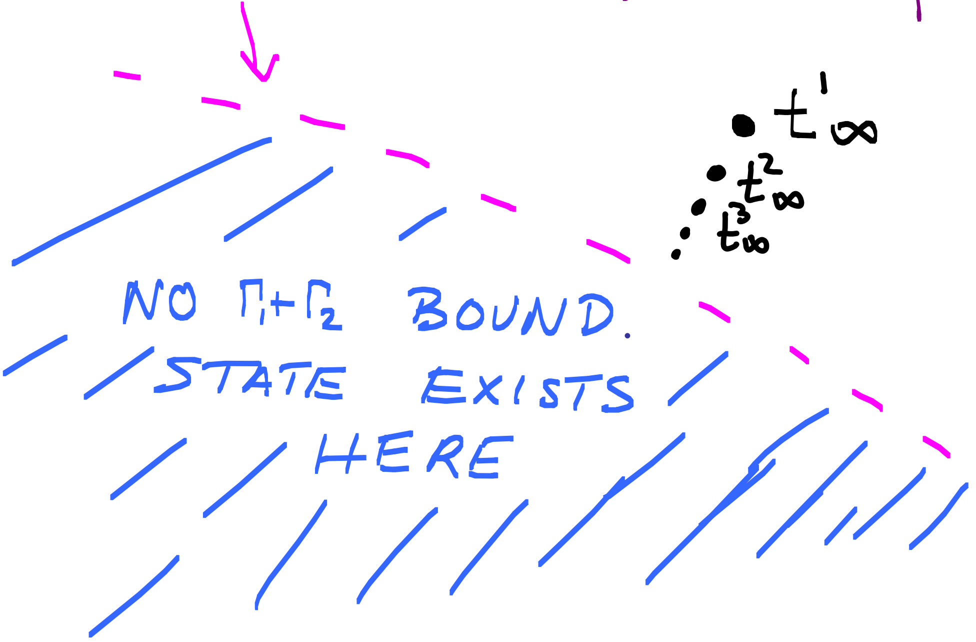
$$R = \frac{1}{2}\langle \Gamma_1, \Gamma_2 \rangle \frac{|Z_1 + Z_2|_\infty}{\mathrm{Im}(Z_1 \bar{Z}_2)_\infty}$$

- NOTE: $\langle \Gamma_1, \Gamma_2 \rangle \mathrm{Im}(Z_1 \bar{Z}_2)_\infty > 0$

- NOTE THAT BY CHANGING t_∞ WE CAN MAKE $\mathrm{Im}(Z_1 \bar{Z}_2)\big|_{t_\infty} \to 0$ WHILE $|Z_1 + Z_2|_{t_\infty} \neq 0$

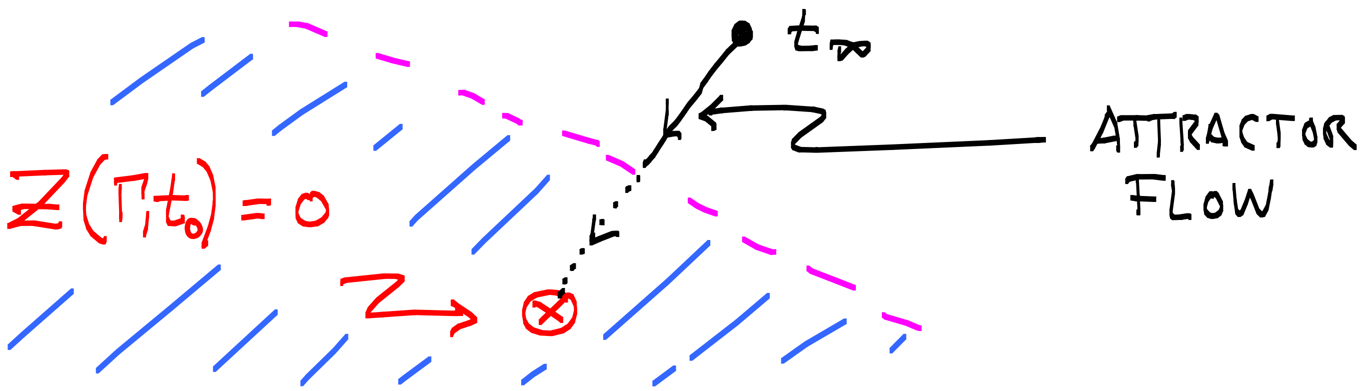
ILLUSTRATES THE KEY POINT OF MARGINAL STABILITY:

$$MS(\Gamma_1,\Gamma_2) := \left\{ t \in \mathcal{M}_{VM} \;\middle|\; \frac{Z_1}{Z_2} \in \mathbb{R}_+ \right\}$$



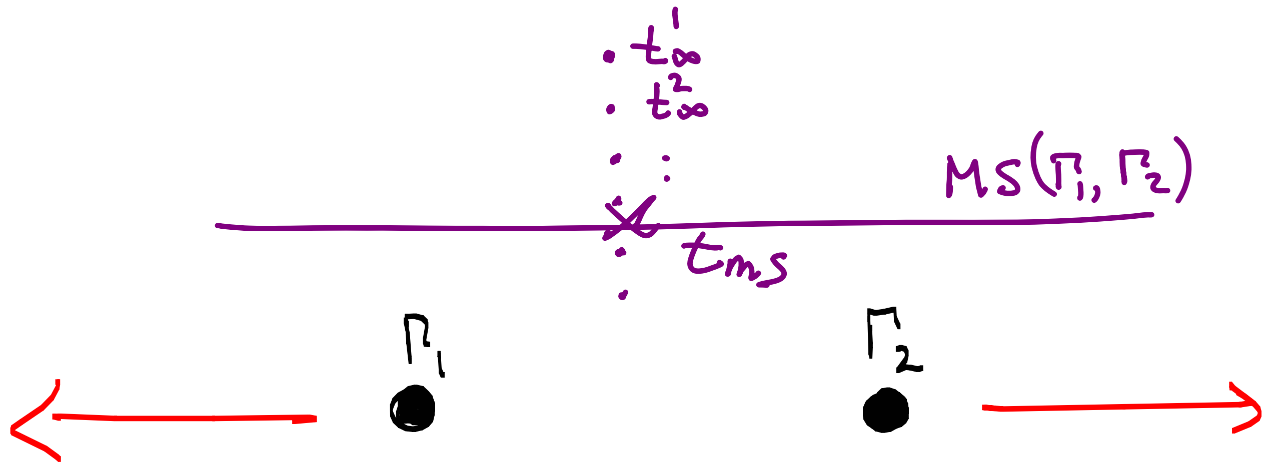
CHANGE BC's @ $r=\infty \Rightarrow$
$R_{12} \longrightarrow \infty$

IF $Z(\Gamma;t)$ HAS A ZERO THEN THERE IS NO BOUNDSTATE OF TYPE $\Gamma_1+\Gamma_2$ IN THE BLUE REGION.



MACROSCOPIC ARGUMENT FOR WCF:

$$R_{12} = \frac{1}{2}\langle \Gamma_1, \Gamma_2 \rangle \frac{|Z_1 + Z_2|_\infty}{\mathrm{Im}(Z_1 \bar{Z}_2)_\infty}$$



ELECTROMAGNETIC FIELD OF TWO DYONS HAS SPIN:

$$J_{12} = \frac{1}{2}\left(|\langle \Gamma_1, \Gamma_2 \rangle| - 1 \right)$$

quantum correction

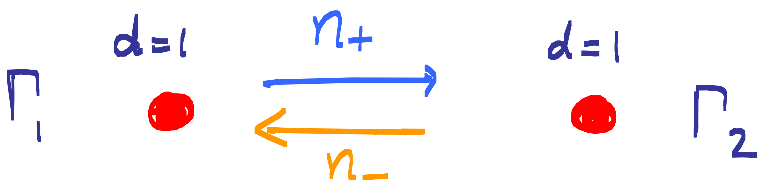
LOCALITY $\Rightarrow$ FOR Γ_1, Γ_2 PRIMITIVE:

STATES LOST FROM $\mathcal{H}(\Gamma; t_\infty)$ ARE

$$(J_{12}) \otimes \mathcal{H}(\Gamma_1; t_{MS}) \otimes \mathcal{H}(\Gamma_2; t_{MS})$$

MICROSCOPIC ARGUMENT FOR WCF:

WHEN $\vartheta = \arg Z_2/Z_1 \longrightarrow 0$, MODEL LIGHT D.O.F. BY A QUIVER GAUGE THRY:



TRANSLATION TO SUPERGRAVITY:

STABILITY DATA: $(\vartheta, -\vartheta)$

$$n_+ - n_- = I_{12}$$

GENERICALLY $n_+ = 0$ or $n_- = 0$.

SUPPOSE $n_- = 0$:

$$\vartheta > 0 \qquad \mathcal{M} = \mathbb{C}P^{n_+ - 1}$$

$$\vartheta < 0 \qquad \mathcal{M} = \emptyset$$

$$\Delta \mathcal{H} = H^*\left(\mathbb{C}P^{n_+ - 1}\right)$$

$$\mathrm{spin}(3) \approx \text{Lefshetz}$$

QUIVER QUANTUM MECHANICS

0+1 SUSY QED WITH

1 VM $(A_0, \vec{x}, \lambda)$

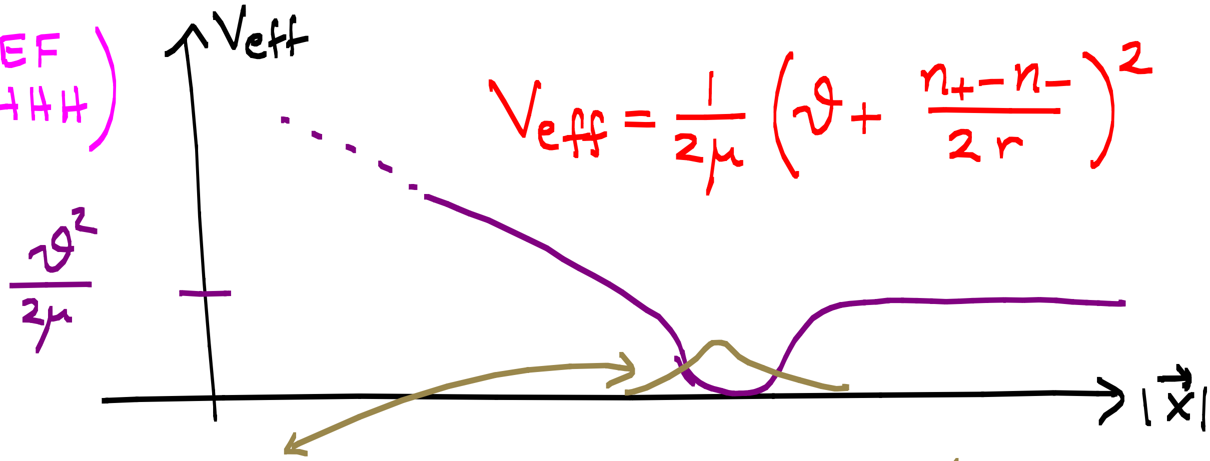
$n_\pm$ CM's $(\vec{\Phi}_\pm, \vec{\Psi}_\pm)$ CHARGE ± 1

SMALL $|\langle\vec{x}\rangle| \Rightarrow$ HIGGS BRANCH = MODULI OF STABLE QUIVER REP'S

LARGE $|\langle\vec{x}\rangle| \Rightarrow$ INTEGRATE OUT $\vec{\Phi}_\pm \Rightarrow$

(DENEF QQHHH)

$$V_{eff} = \frac{1}{2\mu}\left(\vartheta + \frac{n_+ - n_-}{2r}\right)^2$$



$(n_+ - n_-)$ BPS STATES OF SPIN $\frac{1}{2}(n_+ - n_- - 1)$

$\vartheta < 0$	$\vartheta = 0$	$\vartheta > 0$
n_+ HIGGS BR. BPS STATES	$(n_+ - n_-)$ COULOMB BR. $\to \infty$	n_- HIGGS BR. BPS STATES

D. DERIVATION OF SEMI-PRIMITIVE WCF

HALO STATES

SUPPOSE $\langle \Gamma_1, \Gamma_2 \rangle \neq 0$,

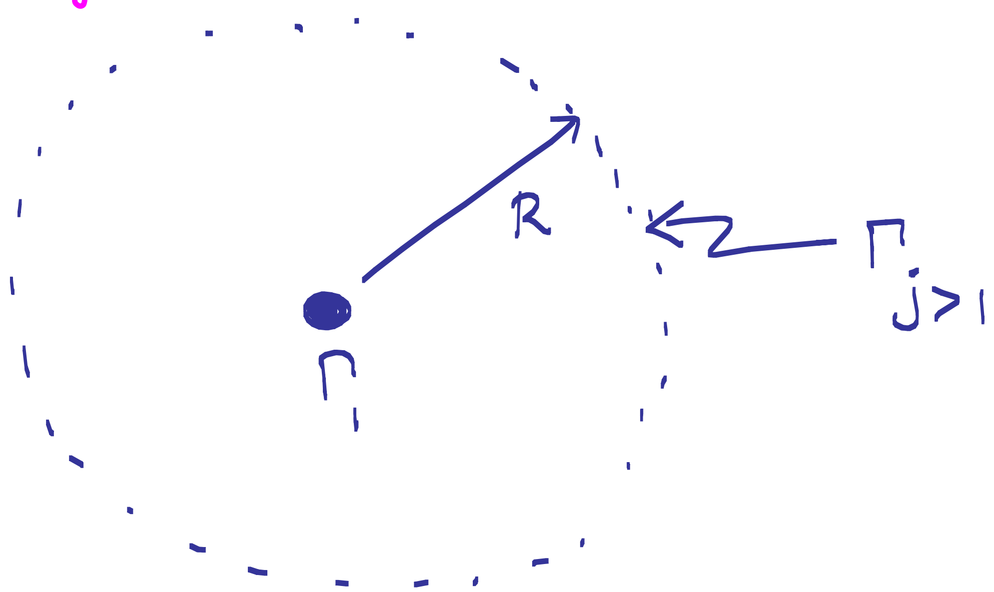
$$\Gamma_j = \lambda_j \Gamma_2 \qquad \lambda_j > 0, \; j = 2, \ldots, N$$

ARE ALL MUTUALLY LOCAL

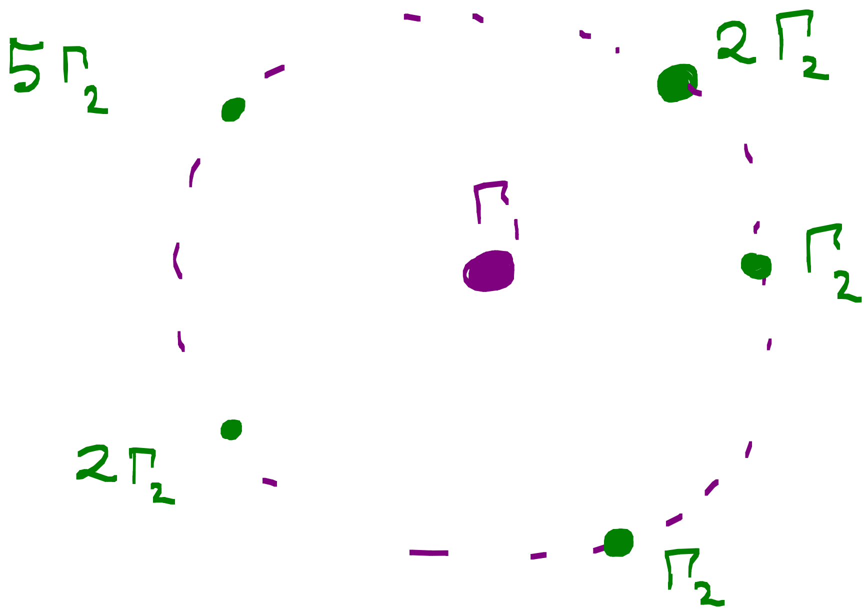
INTEGRABILITY CONDITIONS SAY

$$j \geq 2: \qquad \frac{\langle \Gamma_j, \Gamma_1 \rangle}{|\vec{x}_j - \vec{x}_1|} = 2 \frac{\mathrm{Im}\left(Z(\Gamma_j) \overline{Z(\Gamma)} \right)}{|Z(\Gamma)|}$$

$\Rightarrow$ ALL $|\vec{x}_j - \vec{x}_1|$ ARE EQUAL



CROSS $MS(\Gamma_1, \Gamma_2)$: HALO RADIUS $\nearrow \infty$



THE PARTICLES IN THE HALO GENERATE A FOCK SPACE WITH

$$\left(J_{\Gamma_1, k\Gamma_2}\right) \otimes \mathcal{H}(k\Gamma_2 ; t_{ms})$$

CREATION OPERATORS OF CHARGE $k\Gamma_2$

ALL WALLS $W(\Gamma_1, N\Gamma_2)$ COINCIDE $\Longrightarrow$ CROSSING A WALL WE LOSE ENTIRE FOCK SPACE:

$$\Omega(\Gamma_1) + \sum_{N \geq 1} \Delta\Omega(\Gamma \to \Gamma_1 + N\Gamma_2)\, u^N$$

$$= \Omega(\Gamma_1) \prod_{k>0} \left(1 - (-1)^{k\langle \Gamma_1, \Gamma_2 \rangle} u^k \right)^{|\langle \Gamma_1, k\Gamma_2 \rangle| \Omega(k\Gamma_2)}$$

4. D6D2D0 SYSTEM

AN IMPORTANT AND USEFUL EXAMPLE IS THE SYSTEM OF 1 D6 BRANE WRAPPING X, BOUND TO D2 & D0 BRANES IN X.

$$\underset{D6}{H^0} \oplus \underset{D4}{H^2} \oplus \underset{D2}{H^4} \oplus \underset{D0}{H^6} \ni \Gamma = (p^0, P, Q, q_0)$$

CONSIDER: $\Gamma(\beta, n) := \Gamma = (1, 0, -\beta, n)$

$\beta = P.D.[\sigma] \quad \sigma \subset X$ HOLOMORPHIC CURVE

CHARGE OF (THE DUAL OF) AN IDEAL SHEAF:

$$ch\, \mathcal{I} \sqrt{\hat{A}} = 1 - \beta + n\, dV$$

CONSIDER BINDING THESE TO D2D0 PARTICLES WITH CHARGE:

$$\Gamma_h = (0, 0, -\beta_h, n_h)$$

PLOT MARGINAL STABILITY CURVE

$$Z(\Gamma(\beta,n);t) = \lambda Z(\Gamma_h;t) \qquad \lambda \in \mathbb{R}_+$$

$$Z(\Gamma,t) = \frac{\langle \Gamma, \omega \rangle}{\sqrt{\langle \omega, \omega^* \rangle}}$$

SUGRA REGIME: $\Omega = -e^{t}$

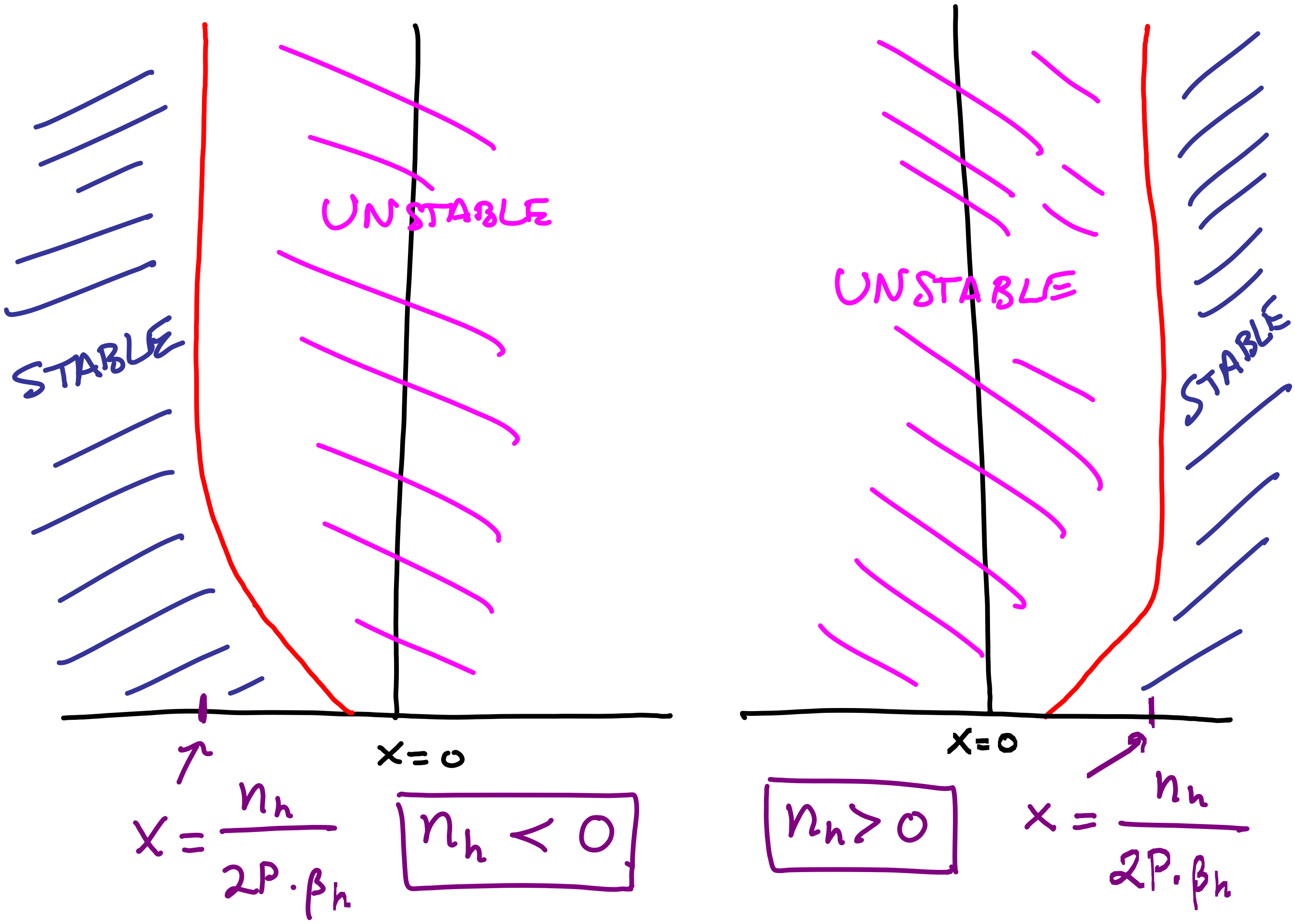
$$t = B + iJ$$

$$\boxed{\frac{t^3}{6} - \beta \cdot t - n = \lambda\left(-\beta_h \cdot t - n_h\right)} \quad \lambda \in \mathbb{R}_+$$

THESE WALLS EXTEND TO ∞ IN THE KÄHLER CONE!

SET $t = z\underline{P}$ $\quad$ $\underline{P} \in \mathcal{K}$

$\underline{z = x + iy}$



CONSIDER THE HALO BOUNDSTATES WITH CENTRAL PARTICLE $\Gamma(\beta, n)$ AS WE INCREASE THE B-FIELD

$$B = xP \qquad x \text{ INCREASES}$$

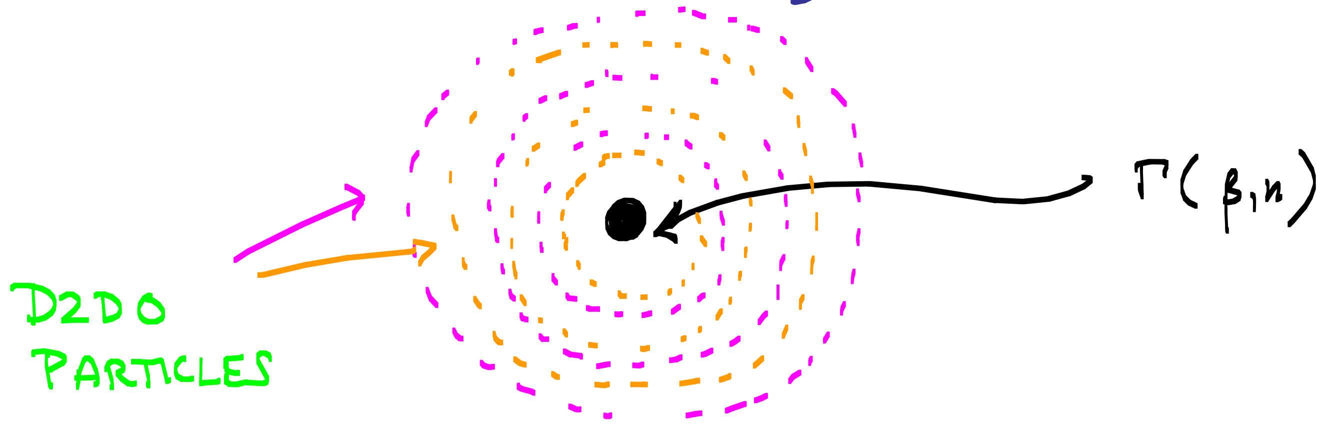
HALOS OF D2D0 PARTICLES $(0, 0, -\beta_h, n_h)$. APPEAR & DISAPPEAR.

FOR $x > 0$

ALL $n_h < 0$ STATES HAVE DECAYED.

AS $x \longrightarrow +\infty$ WE MOVE INTO THE STABLE REGION FOR ALL $n_h > 0$, AND EVER LARGER "ATOMS" BECOME STABLE

GENERAL PICTURE: BOHR MODEL

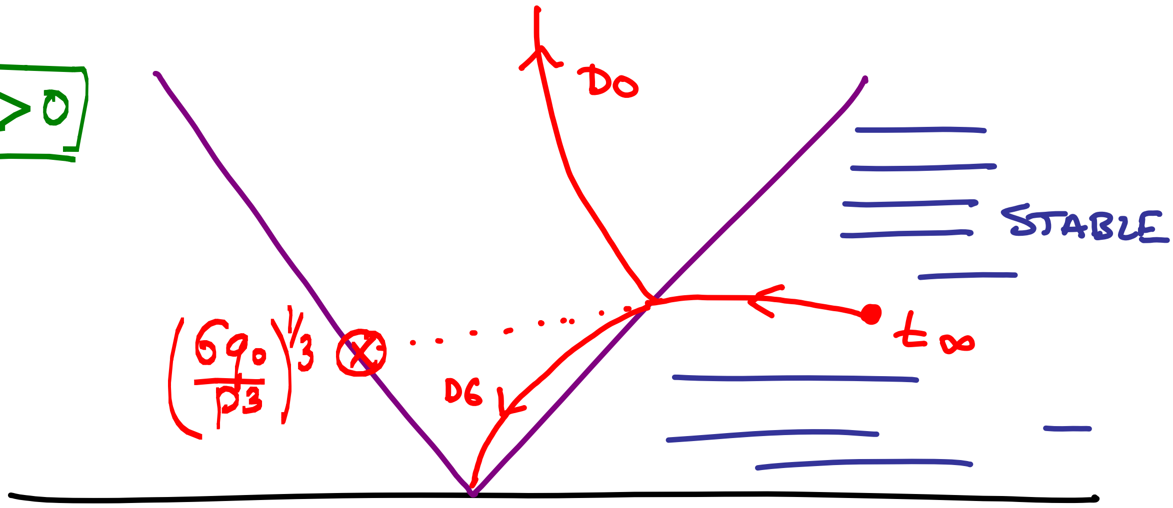


WHEN $\beta_h = 0$ WALLS LOOK DIFFERENT

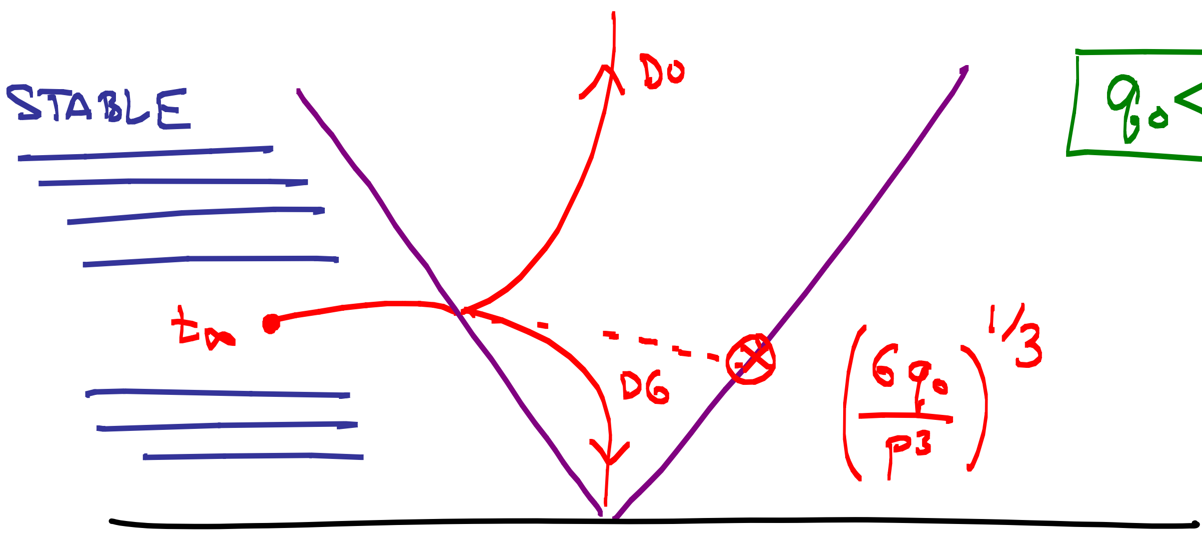
$$\Gamma = \underbrace{1}_{\Gamma_1} + \underbrace{q_0\, dV}_{\Gamma_2} \qquad Z = \frac{t^3}{6} - q_0$$

SET $t = (x+iy)P \implies$ ZERO @ $z = \left(\frac{6q_0}{P^3}\right)^{1/3} P$

$\boxed{q_0 > 0}$



$\boxed{q_0 < 0}$



INTRODUCE GENERATING FUNCTION

$$Z_{D6D2D0}(u,v;t) := \sum_{n,\beta} \Omega(\Gamma(\beta,n);t)\, u^n v^\beta$$

SEMI-PRIMITIVE WALL-CROSSING FORMULA:

CONTRIBUTION OF FOCK SPACE GENERATED BY $\Gamma_h = -\beta_h + n_h dV$ CROSSING INTO STABLE REGION:

$$Z_{D6D2D0} \to \left(1-(-u)^{n_h} v^{\beta_h}\right)^{|n_h|\, n^0_{\beta_h}} Z_{D6D2D0}$$

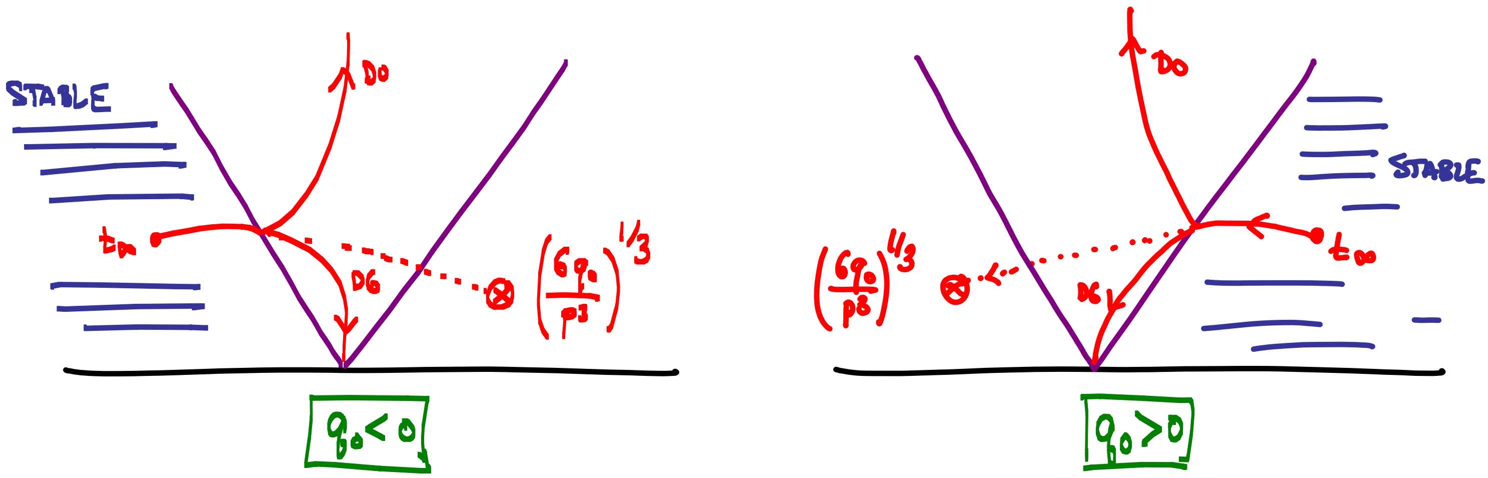
$$\Omega(-\beta_h + n_h dV) = \sum_{m_L, m_R} (-1)^{2m_L+2m_R} N^{m_L m_R}_{\beta_h}$$

$$= n^0_{\beta_h}$$

"SPIN ZERO GV INVARIANT" $(\beta_h \neq 0)$

EXAMPLE: D6D0

$$Z_{D6D0}(u) = \sum \Omega(1+q_0 dV; t)\, u^{q_0}$$



$$\Omega(q_0 dV) = -\chi(X)$$

$$Z_{D6D0}(u) = \begin{cases} \left(M(-u)\right)^{\chi(X)} & \arg z < \frac{\pi}{3} \\ 1 & \frac{\pi}{3} < \arg z < \frac{2\pi}{3} \\ \left(M(-u^{-1})\right)^{\chi(X)} & \frac{2\pi}{3} < \arg z \end{cases}$$

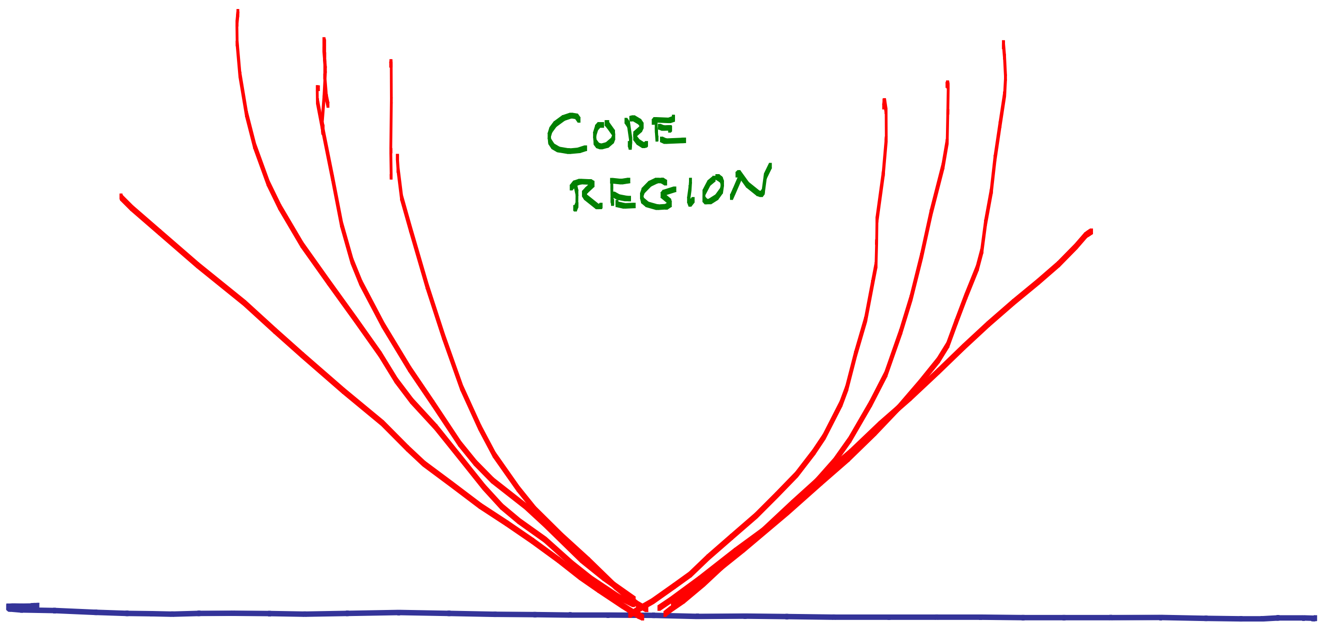
$$M(u) := \prod_{k\geq 1} (1-u^k)^{-k}$$

SIMILARLY, WALL-CROSSINGS FOR THE FULL Z_{D6D2D0} AS $x \to \infty$ BUILD UP AN INFINITE PRODUCT SIMILAR TO THE INFINITE PRODUCT FORM OF $Z_{DT}(u,v)$

ON THE OTHER HAND, AN ARGUMENT FROM M-THEORY [Dijkgraaf, Verlinde, Vafa; Denef, Moore] IMPLIES:

$$\lim_{x \to +\infty} Z_{D6D2D0}(u,v;z^P) = Z_{DT}(u,v)$$

$$\lim_{x \to -\infty} Z_{D6D2D0}(u,v;z^P) = Z_{DT}(u^{-1},v)$$



- STATES IN CORE REGION ARE COMPLICATED BOUNDSTATES

- PRODUCT OF WALL-CROSSINGS $\Rightarrow$

$$Z_{DT}^{1,r=0}(u,v) = \prod_{\beta>0,k>0}\left(1-(-u)^k v^\beta\right)^{k n_\beta^0}$$

- LIMIT FOR $x \longrightarrow +\infty$:

$$Z'_{DT}(u,v) = \underbrace{Z_{DT}^{1,r=0}(u,v)}_{\text{HALOS}} \; \underbrace{Z_{DT}^{1,r>0}(u,v)}_{\text{CORES}}$$

$$Z_{DT}^{1,r>0}(u,v) = \prod_{\substack{\beta>0,k>0\\ r>0}} \prod_{\ell=0}^{2r-2}\left(1-(-u)^{r-\ell-1}v^\beta\right)^{(-1)^{r+\ell}\binom{2r-2}{\ell} n_\beta^r}$$

5. THE D4-D2-D0 SYSTEM: MODULARITY

NOW CONSIDER $p^0 = 0$

$$\Gamma = P + Q + q_0 dV$$

REGULAR ATTRACTOR POINT:

$$\boxed{\begin{array}{c} P \text{ IN KÄHLER CONE } \& \quad \hat{q}_0 < 0 \\ \hat{q}_0 := q_0 - \frac{1}{2}\left(D_{ABC}P^C\right)^{-1} Q_A Q_B \end{array}}$$

THESE ARE BLACK HOLES:

HORIZON AREA $= 4\, S(\Gamma) = 4\pi \, |Z_*(\Gamma)|^2$

$$S(\Gamma) = \frac{2\pi}{\sqrt{6}} \cdot \sqrt{-\hat{q}_0\, \chi(P)}$$

$\chi(P) := P^3 + c_2 \cdot P > 0$ FOR $P \in$ KÄHLER CONE

EXPECT: $\log \Omega(\Gamma; t) \sim S(\Gamma)$
FOR "LARGE" Γ AND "LARGE" $\mathrm{Im}\, t$

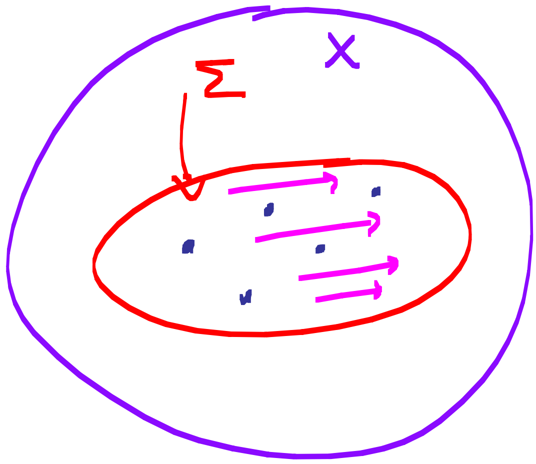
A. ROUGH MICROSCOPIC DESCRIPTION

FOR LARGE J: SINGLE D4 WRAPS $\Sigma \in |P|$

$\chi(P) = P^3 + c_2 \cdot P$ = EULER CHARACTER OF Σ

FLUX $F \in H^2(\Sigma, \mathbb{Z})$

AND N $\overline{D0}$'s



COMPUTE INDUCED RR CHARGES:

D2: $Q = (i_\Sigma)_*(F)$

D0: $\hat{q}_0 = \frac{\chi(P)}{24} + \frac{1}{2}(F^-)^2 - N$

SUSY $\Rightarrow$ $N \geq 0$, $F^{2,0} = 0 \Rightarrow (F^-)^2 \leq 0 \Rightarrow$

$$\hat{q}_0 \leq (\hat{q}_0)_{max} = \frac{\chi(P)}{24}$$

$\mathcal{M}(P,F,N)$:= MODULI OF SUCH D4's

ROUGHLY:

$$\begin{tikzcd} \mathrm{Hilb}^N(\Sigma) \arrow[r, hook] \arrow[d] & \mathcal{M}(P,F,N) \arrow[d] \\ \underset{\text{smooth}}{\Sigma} \arrow[r, hook] & \{\Sigma \in |P| \,\big|\, F \in H^{1,1}(\Sigma)\} \end{tikzcd}$$

$\parallel$ "MODULI OF STABLE OBJECTS $\mathcal{E}$
IN THE DERIVED CATEGORY
WITH SPECIFIED CHERN CLASSES"

$$\mathrm{ch}\,\mathcal{E}\sqrt{\hat{A}} = P + Q + q_0 \quad (*)$$

$$= \bigcup_{\substack{F,N \\ \text{s.t.} \\ (*)}} \mathcal{M}(P,F,N)$$

B. INDEX OF BPS STATES

$$\text{“}\,\Omega(\Gamma)_\infty := \lim_{J\to\infty} \Omega(\Gamma; B+iJ)\,\text{”}$$

$$d(F,N) := (-1)^{\dim \mathcal{M}} \chi\big(\mathcal{M}(P,F,N)\big)$$

$$\Omega(\Gamma)_\infty = \text{FINITE SUM OF } d(F,N)$$

SURPRISE: WHEN $h^{1,1}(X) > 1$ THERE ARE SPLITTINGS @ ∞ :

$$\Gamma = P + Q + q_0 dV$$

$$= (P' + Q' + q_0' dV) + (P'' + Q'' + q_0'' dV)$$

WITH: $\sqrt{-\hat{q}_0''(P'')^3} > \sqrt{-\hat{q}_0 P^3}$

$\Rightarrow$ EVEN THE LEADING ORDER ENTROPY IS CHAMBER DEPENDENT

[E. ANDRIYASH + G.M.]

- FOR $\Gamma = P + Q + q_0 \, dV$, $P \in$ KÄHLER CONE, $\exists$ DISTINGUISHED CHAMBER:

$$\boxed{\Omega(\Gamma)_\infty := \lim_{\lambda \to \infty} \Omega(\Gamma; B + i\lambda P)}$$

CLAIM: LIMIT EXISTS & IS B-INDEPENDENT

(FINITENESS OF ATTRACTOR FLOW TREES)

HENCEFORTH WORK IN THIS CHAMBER.

C. MODULARITY

$$\tau \in \mathcal{H} \quad \& \quad C \in \iota^*_\Sigma(H^2(X,\mathbb{C}))$$

$$Z(\tau,\bar\tau,C) :=$$

$$\sum_{F,N} d(F,N)\exp\left\{-2\pi i\tau\hat q_0 - 2\pi i\bar\tau\tfrac{1}{2}(F^+)^2 - 2\pi i F\cdot\left(C+\tfrac{P}{2}\right)\right\}$$

SUSY PARTITION FUNCTION OF D3 INSTANTON

U-DUALITY $\Longrightarrow$

$Z(\tau,\bar\tau,C)$ IS A JACOBI FORM $\Longrightarrow$

$$Z(\tau,\bar\tau,c) = \sum_{\mu\in L^*/L} H_\mu(\tau)\,\underbrace{\Theta_{\mu,L}(\tau,\bar\tau,C)}_{\text{SIEGEL-NARAIN}}$$

$$L := \iota^*_\Sigma(H^2(X,\mathbb{Z})) \subset \underbrace{H^2(\Sigma;\mathbb{Z})}_{\text{SELF-DUAL}}$$

$\ell \in L$ IS ALWAYS IN $H^{1,1}(\Sigma) \Rightarrow$

$$d(F+\ell,N) = d(F,N) \quad \forall\, \ell\in L$$

- $H_\mu(\tau)$ IS A VECTOR-VALUED NEARLY HOLO. MODULAR FORM OF WEIGHT $w = -1 - \frac{h^{1,1}(X)}{2}$ AND MULTIPLIER SYSTEM M^* DUAL TO THAT OF $\Theta_{\mu,L}$

- $w < 0 \Rightarrow H_\mu$ IS DETERMINED BY ITS POLAR TERMS.

SUPPRESS μ-INDEX FOR SIMPLICITY:

$$H(\tau) = \sum_{\hat{q}_0} \Omega(\Gamma)_\infty \, e^{-2\pi i \hat{q}_0 \tau}$$

$$= \underbrace{\sum_{0 < \hat{q}_0 \leq \frac{\chi(P)}{24}} (\cdots\cdots)}_{\text{POLAR}} + \underbrace{\sum_{-\infty < \hat{q}_0 \leq 0} (\cdots\cdots)}_{\text{NONPOLAR}}$$

D. MACROSCOPIC POLAR STATES

IF $\Gamma = (0, P, Q, q_0) = P + Q + q_0 dV$

IS POLAR: $0 < \hat{q}_0 \leq (\hat{q}_0)_{max}$

THEN $Z(\Gamma; t)$ HAS A ZERO.

INDEED $S(\Gamma) = \frac{2\pi}{\sqrt{6}} \cdot \sqrt{-\hat{q}_0 \, X(P)}$

SO NO SINGLE-CENTERED SOLUTION

BUT $H(\tau)$ HAS $w<0 \Rightarrow$ SOME POLAR DEGENERACIES ARE NONZERO

$\Rightarrow$ THESE MUST BE REALIZED AS SPLIT ATTRACTOR STATES.

SIMPLE EXAMPLE

PURE D4 : $\Gamma = P + q_0 dV$

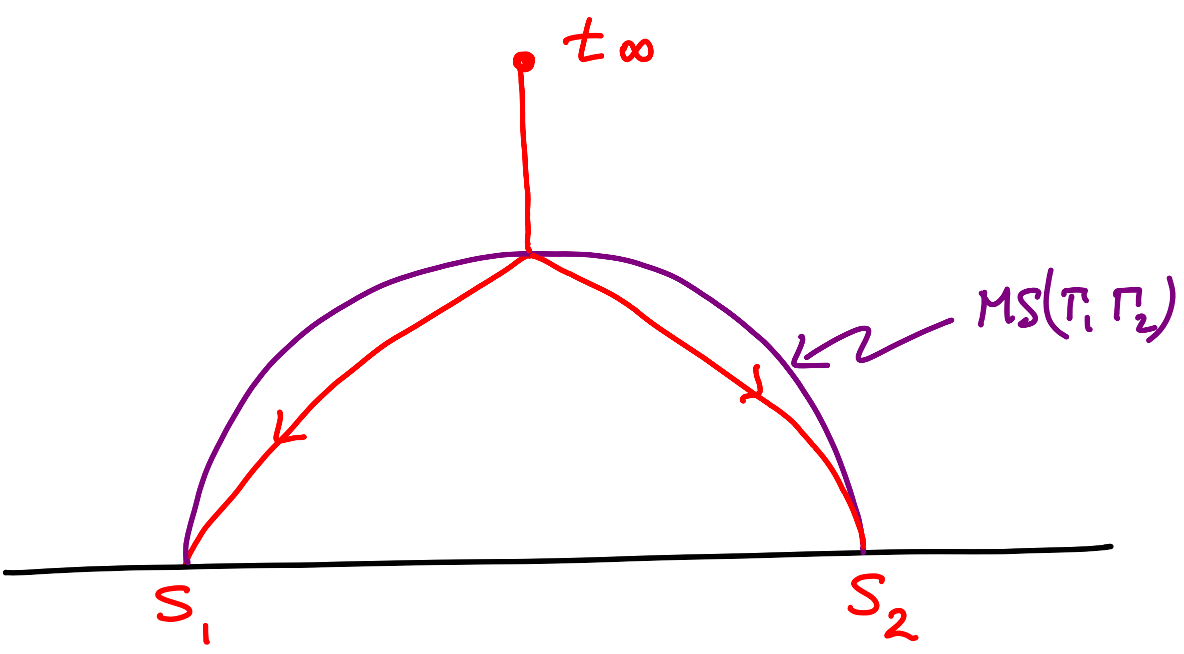
WITH $q_0 = \hat{q}_0 = (\hat{q}_0)_{max} = \frac{\chi(P)}{24}$

FIND ONLY ONE SPLITTING

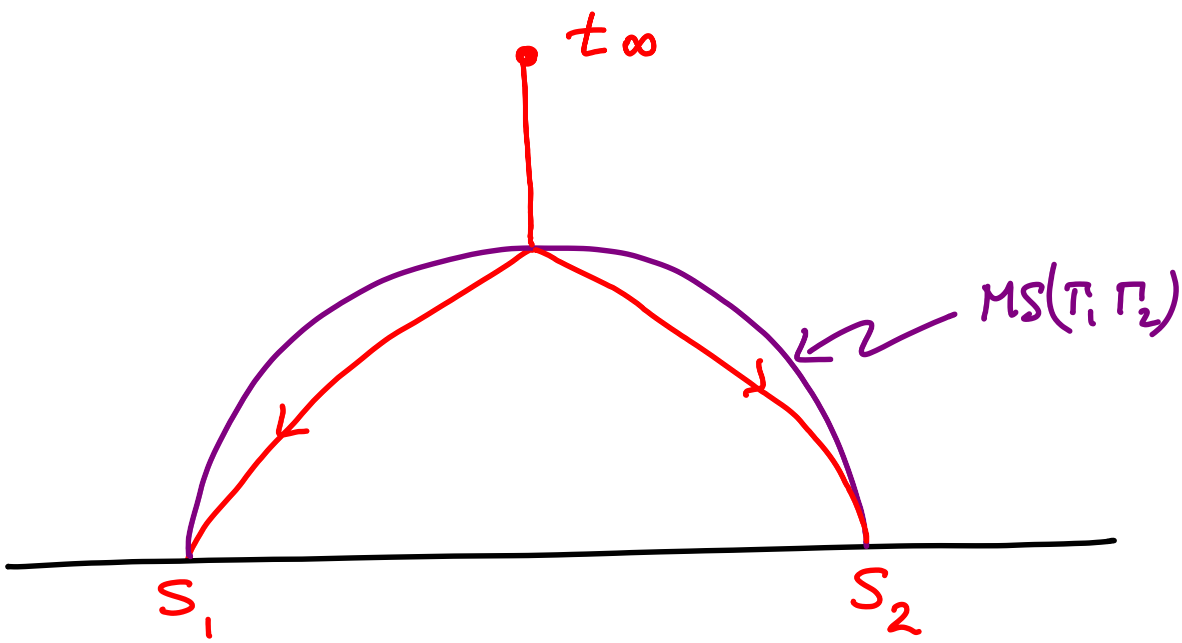
$$\Gamma = P + q_0 dV = \Gamma_1 + \Gamma_2$$

$$= \underbrace{e^{S_1}\left(1 + \frac{c_2(X)}{24}\right)}_{\text{1 D6 WITH FLUX} = S_1} - \underbrace{e^{S_2}\left(1 + \frac{c_2(X)}{24}\right)}_{1\ \overline{\text{D6}}\ \text{w/ FLX}\ S_2}$$

$$S_1 - S_2 = P$$



MOREOVER – YOU CAN COMPUTE THE POLAR DEGENERACY:



$$\Omega(\Gamma, t_\infty) = (-1)^{I_{12}-1} |I_{12}| \, \Omega(\Gamma_1)\Omega(\Gamma_2) = (-1)^{I_{12}-1} |I_{12}|$$

$$I_{12} = \langle \Gamma_1, \Gamma_2 \rangle = \frac{P^3}{6} + \frac{c_2(X)\cdot P}{12}$$

INDEED = THE CORRECT ANSWER FOR

χ (MODULI OF PURE D4) $= \chi(|P|)$

DESCRIBING THE SPLIT ATTRACTOR FLOWS FOR $0 < \hat{q}_0 < \frac{\chi(P)}{24}$ IS MUCH MORE COMPLICATED...

IN GENERAL, POLAR STATES CAN BE VERY COMPLICATED SPLIT ATTRACTORS, REALIZED IN MANY DIFFERENT WAYS....

BUT IN THE LIMIT $P \to \infty$ WE CAN SAY SOMETHING

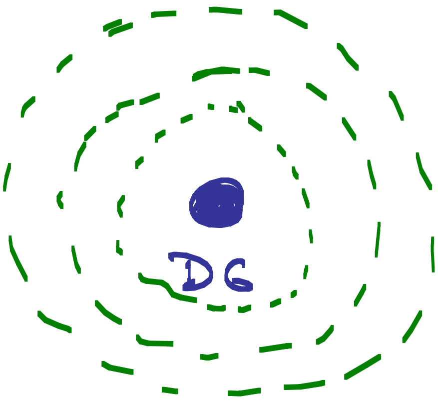
EXTREME POLAR STATES

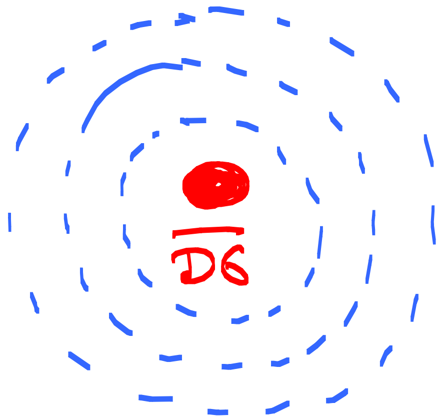
$$H^{\text{POLAR}}(\tau) = \underbrace{|I_p|\, e^{-2\pi i \tau \frac{\chi(p)}{24}} + \cdots}_{\text{“EXTREME POLAR”}} \quad \underbrace{+\mathcal{O}\left(e^{\frac{-2\pi i \tau}{|p|}}\right)}_{\text{“BARELY POLAR”}}$$

E.P.S. CONJECTURE: $\exists\ \epsilon < 1$ SO THAT

$$\frac{\hat{q}_0^{\max} - \hat{q}_0}{\hat{q}_0^{\max}} < \epsilon \quad \Longrightarrow$$

POLAR STATES SPLIT AS $D6\overline{D6}$ + HALOS :





$$\Gamma_1 = e^{S_1}(1 - \beta_1 + n_1 dV) \qquad \Gamma_2 = -e^{S_2}(1 - \beta_2 + n_2 dV)$$

6. ROUTE TO THE OSV CONJECTURE

A. BY THE W.C.F. THE (EXTREME) POLAR DEGENERACIES GO LIKE

$$\Omega(D6\text{-}D2\text{-}D0) \times \Omega(\overline{D6\text{-}D2\text{-}D0})$$

B. BUT BPS INVARIANTS OF THE D6-D2-D0 SYSTEM ARE RELATED TO GROMOV-WITTEN INVARIANTS COUNTING WORLDSHEET INSTANTONS IN X

SO, BY THE W.C.F. TOGETHER WITH RESULTS ON Z_{D6D2D0} THE EXTREME POLAR DEGENERACIES ARE RELATED

$$|Z_{TOP}|^2$$

SUGGESTING A RELATION LIKE THE OSV CONJECTURE

$$\Omega(\Gamma)_\infty = \int d\phi \; |Z_{top}(g_{top}, t)|^2 e^{-2\pi \hat{q}_0 \phi}$$

- $\exists$ STRONG ARGUMENTS FOR $|\hat{q}_0| \gg P^3$
- $\exists$ POTENTIAL COUNTEREXAMPLES FOR $|\hat{q}_0| \lesssim P^3$: "ENTROPY ENIGMA"

IN THE CHARGE REGIME

$$g_{top} \sim \sqrt{\frac{-\hat{q}_0}{p^3}} \lesssim \mathcal{O}(1)$$

THE DERIVATION IN DENEF-MOORE BREAKS DOWN.

- BARELY POLAR DEGENERACIES BECOME LARGE
- CORRECTIONS TO THE CARDY FORMULA BECOME LARGE.

THERE IS A GOOD PHYSICAL REASON THE DERIVATION BREAKS DOWN...

ENTROPY ENIGMA

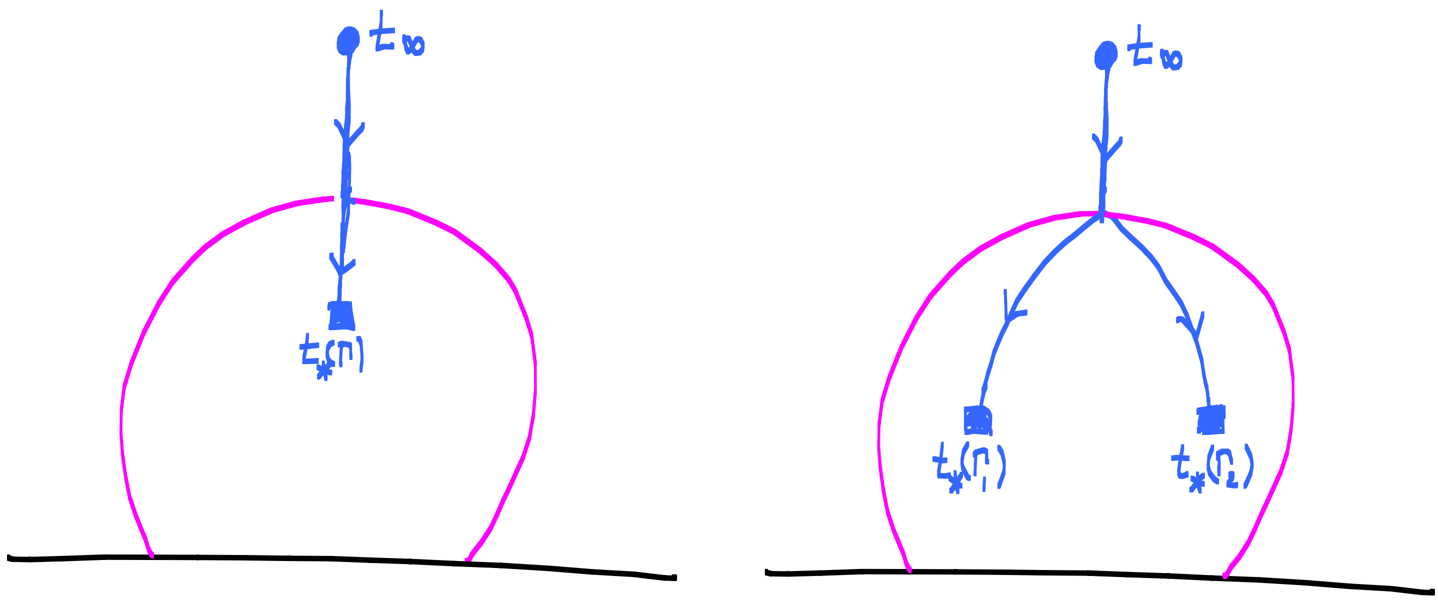
NOW CHOOSE $q_0 < 0$, P AMPLE SO

$$\Gamma = (0, P, 0, q_0)$$

HAS A REGULAR ATTRACTOR POINT

NEVERTHELESS! WE CAN CHOOSE q_0, Q_A SO THAT $\exists$ A TWO-CENTERED SOLUTION WITH $\Gamma = \Gamma_1 + \Gamma_2$

$$\Gamma_1 = \left(r, \tfrac{1}{2}P, Q, \tfrac{1}{2}q_0\right) \quad \Gamma_2 = \left(-r, \tfrac{1}{2}P, -Q, \tfrac{1}{2}q_0\right)$$



BOTH SOLUTIONS EXIST

SO... COMPARE ENTROPIES

$$S(\Gamma) \quad \text{vs.} \quad S(\Gamma_1) + S(\Gamma_2)$$

IN FACT,

$\exists$ FAMILY OF CHARGES

$$\lambda\Gamma = \lambda(0, P, 0, q_0) = \Gamma_1^\lambda + \Gamma_2^\lambda$$

$$\Gamma_1^\lambda = \left(r, \tfrac{\lambda}{2}P, \lambda^2 Q, \tfrac{\lambda}{2}q_0\right) \quad \Gamma_2^\lambda = \left(-r, \tfrac{\lambda}{2}P, -\lambda^2 Q, \tfrac{\lambda}{2}q_0\right)$$

SCALING OF ENTROPIES:

$$S(\lambda\Gamma) = \lambda^2 S(\Gamma)$$

BUT!

$$\boxed{S(\Gamma_1^\lambda) = S(\Gamma_2^\lambda) \sim \frac{(\lambda P)^3}{r} \sim \lambda^3}$$

$\Rightarrow$ MANY IMPLICATIONS FOR PHYSICS & MATHEMATICS

SOME TECHNICAL DETAILS

1. CONSTRUCT A FAMILY OF 2-CENTERED

$$\tilde{\Gamma}_1^{\lambda} = \left(r, \frac{P}{2}, Q, \lambda^{-2}\frac{q_0}{2}\right)$$

$$\tilde{\Gamma}_2^{\lambda} = \left(-r, \frac{P}{2}, -Q, \lambda^{-2}\frac{q_0}{2}\right)$$

$\tilde{\Gamma}_i^{\lambda}$ CAN BE 1-CENTERED BH'S OR CAN THEMSELVES BE POLAR

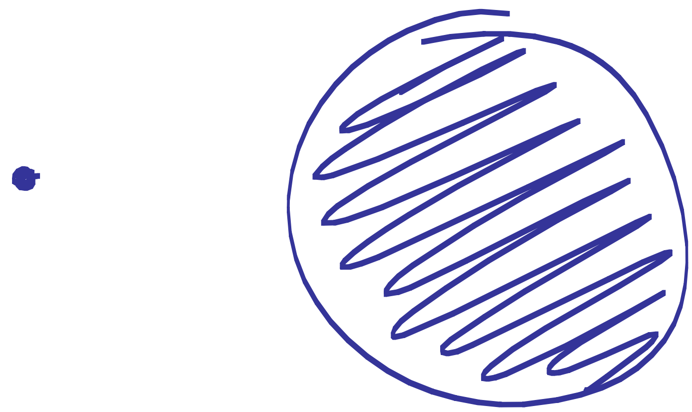
2. ATTRACTOR FORMALISM HAS A SCALING SYMMETRY UNDER

$$T_\lambda(p^0, P, Q, q_0) = (p^0, \lambda P, \lambda^2 Q, \lambda^3 q_0)$$

$$S(T_\lambda \Gamma) = \lambda^3 S(\Gamma)$$

3. APPLY TO $\quad T_\lambda \tilde{\Gamma}_1^{\lambda} + T_\lambda \tilde{\Gamma}_2^{\lambda} = \lambda \Gamma$

RECENTLY, DEBOER ET. AL.
SHOWED THAT IF WE SPLIT
THE D2-D0 CHARGE ASYMMETRICALLY
BETWEEN THE TWO CENTERS
THEN THE COEFFICIENT OF THE
λ^3 GROWTH CAN BE INCREASED:



DOMINATES

BUT BOTH CONTRIBUTIONS SCALE
LIKE λ^3.

DEGENERACY DICHOTOMY

- WE HAVE FOUND CONTRIBUTIONS TO $\Omega(\lambda\Gamma)_\infty$ GROWING LIKE e^{λ^3}

- IF INDEED $\Omega(\lambda\Gamma)_\infty \sim e^{\lambda^3}$ THEN WEAK COUPLING OSV IS WRONG, SINCE OSV $\Rightarrow$ $\Omega(\lambda\Gamma)_\infty \sim e^{\lambda^2}$.

- BUT $\Omega(\lambda\Gamma)_\infty$ IS AN INDEX. IT IS POSSIBLE THAT

$$\Omega(\lambda\Gamma)_\infty = \sum \pm e^{\lambda^3} \sim e^{\lambda^2}$$

- WE ARGUE THAT THIS IS UNLIKELY, BUT IT IS NOT EXCLUDED.

SUPPOSE THAT THERE <u>ARE</u> "MAGICAL CANCELLATIONS" AND

$$\Omega(\lambda\Gamma)_\infty \sim e^{\lambda^2}$$

- THIS RAISES THE QUESTION OF $\dim \mathcal{H}(\Gamma;t)$ vs. $\Omega(\Gamma;t)$

- PHYSICALLY: THE <u>DIMENSION</u> IS RELEVANT

- BUT ALL TESTS OF THE STROMINGER–VAFA PROGRAM USE THE <u>INDEX</u> (WITH ONE EXCEPTION).

- IT IS $\natural$ TO SUPPOSE THAT IN THE EXACT THEORY, NONPTVE STRINGY EFFECTS GIVE:

$$\dim \mathcal{H}(\Gamma;t) = \Omega(\Gamma;t)$$

IF WE GRANT THIS POINT,

AND IF, MOREOVER, THERE ARE "MAGICAL CANCELLATIONS" SO THAT

$$\log \Omega(0, \lambda P, 0, \lambda q_0) \sim \lambda^2$$

THEN THE SPECTRUM OF NEAR-BPS STATES TAKES A REMARKABLE FORM:

$$E - |Z| = 0 \qquad \sim e^{\lambda^2} \text{ states}$$

$$E - |Z| \sim e^{-1/g_s} \qquad \sim e^{\lambda^3} \text{ states}$$

7. KONTSEVICH-SOIBELMAN FORMULA

THE KS FORMULA IS A RELATION BETWEEN $\Omega(\Gamma; t_\pm)$ ACROSS MS WALLS WITH NO RESTRICTION ON PRIMITIVITY OF CONSTITUENTS.

- NO PHYSICAL DERIVATION YET

EVIDENCE THAT K & S $\Omega(\Gamma;t)$'s ARE THE SAME AS PHYSICAL $\Omega(\Gamma;t)$'s:

- CAN RECOVER PRIMITIVE WCF
- CAN RECOVER SEMI-PRIMITIVE WCF
- NONTRIVIAL CHECKS FOR SU(2) SEIBERG-WITTEN WITH $N_f = 0,1,2,3$ HYPERMULTIPLETS

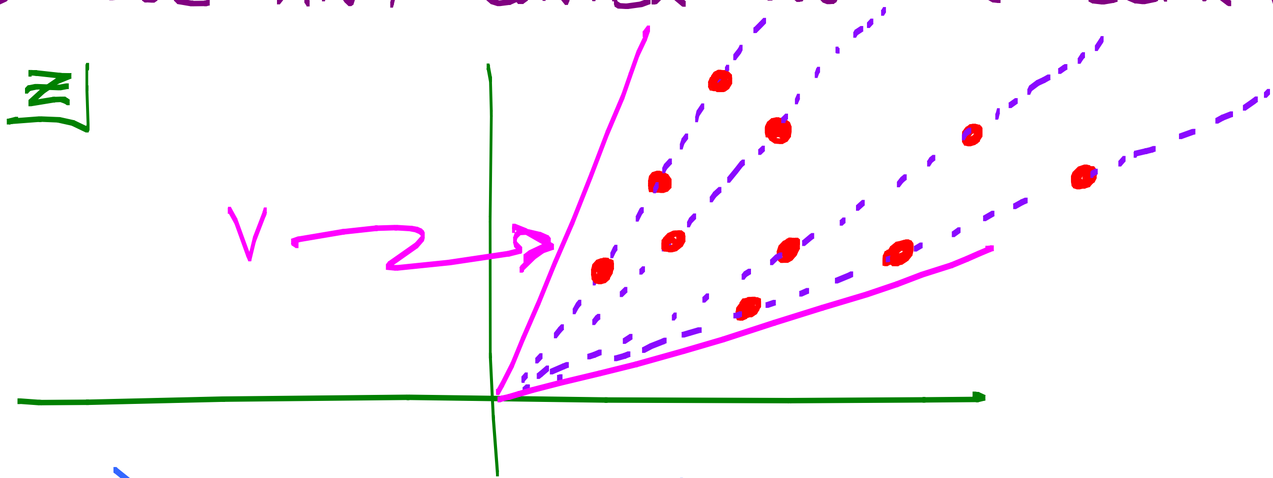
(LAST TWO ARE RESULTS w/ Wu-yen Chuang)

THE KONTSEVICH–SOIBELMAN FORMULA

FOR THE LATTICE Λ OF CHARGES INTRODUCE A LIE ALGEBRA $\mathbb{Z}[\Lambda]$ WITH ONE GENERATOR FOR EACH $\gamma \in \Lambda$:

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle \, e_{\gamma_1 + \gamma_2}$$

@ FIXED t, $Z : \Lambda \longrightarrow \mathbb{C}$, CHOOSE ANY CONVEX ANGULAR SECTOR V



$$\overrightarrow{\prod_{\gamma \in Z^{-1}(V) \cap \Lambda}} \left(\exp \sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} \right)^{\Omega^-(\gamma)} \quad \text{INCREASING SLOPE}$$

$$= \overleftarrow{\prod_{\gamma \in Z^{-1}(V) \cap \Lambda}} \left(\exp \sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} \right)^{\Omega^+(\gamma)} \quad \text{DECREASING SLOPE}$$

AT A GENERIC POINT $t \in MS(\Gamma_1, \Gamma_2)$

$$Z(\Gamma; t) \parallel Z_1, Z_2 \implies$$

$$\Gamma = \Gamma_{a,b} = a\Gamma_1 + b\Gamma_2$$

$(\Gamma_1, \Gamma_2$ primitive $)$

FOR SMALL CONE ANGLE ONLY THE LIE SUBALGEBRA $\mathbb{Z}\Gamma_1 + \mathbb{Z}\Gamma_2$ CONTRIBUTES:

$$[e_{a,b}, e_{c,d}] = (-1)^{(ad-bc)I_{12}} (ad-bc) I_{12} e_{a+c, b+d}$$

DEFINE:

$$U_{a,b} := \exp\left(\sum_{m=1}^{\infty} \frac{e_{ma, mb}}{m^2} \right)$$

$$\prod_{\substack{a/b \uparrow \\ a \geq 0}} U_{a,b}^{\Omega^-(\Gamma_{a,b})} = \prod_{\substack{a/b \downarrow \\ a \geq 0}} U_{a,b}^{\Omega^+(\Gamma_{a,b})}$$

LIE ALGEBRA IS FILTERED $\Longrightarrow$

CAN RESTRICT TO

Heisenberg Algebra
$$\left\{\begin{array}{l} [e_{0,1}, e_{1,0}] = (-1)^{I_{12}-1} I_{12}\, e_{1,1} \\ e_{1,1} \ \underline{\text{CENTRAL}} \end{array}\right.$$

$$U_{0,1}^{\Omega^-(\Gamma_1)}\, U_{1,1}^{\Omega^-(\Gamma_1+\Gamma_2)}\, U_{1,0}^{\Omega^-(\Gamma_2)}$$

$$= U_{1,0}^{\Omega^+(\Gamma_2)}\, U_{1,1}^{\Omega^+(\Gamma_1+\Gamma_2)}\, U_{0,1}^{\Omega^+(\Gamma_1)}$$

$$\boxed{U_{01}\, U_{10} = U_{1,1}^{\pm I_{12}} \cdot U_{10}\, U_{01}} \Longrightarrow$$

$$U_{1,1}^{\Omega^+(\Gamma_1+\Gamma_2)-\Omega^-(\Gamma_1+\Gamma_2)} = U_{0,1}^{\Omega(\Gamma_1)}\, U_{1,0}^{\Omega(\Gamma_2)}\, U_{0,1}^{-\Omega(\Gamma_1)}\, U_{1,0}^{-\Omega(\Gamma_2)}$$

$$= U_{1,1}^{I_{12}\,\Omega(\Gamma_1)\,\Omega(\Gamma_2)}$$

PRIMITIVE W.C. FORMULA!

SU(2) SEIBERG–WITTEN THEORY

Γ_1 = MONOPOLE

Γ_2 = DYON

$$[e_{a,b}, e_{c,d}] = 2(bc-ad)\, e_{a+c,\, b+d}$$

STRONG : $\pm(1,0)$, $\pm(0,1)$ $\Omega = +1$ HM

WEAK: $\pm(1,1)$ $\Omega = -2$ VM

$\pm(n, n+1)$, $\pm(n+1, n)$ $\Omega = +1$ HM

STRONG $U_{1,0} \cdot U_{0,1}$

WEAK :

$$\left(U_{0,1} U_{1,2} U_{2,3} \cdots\cdots\right) U_{1,1}^{-2} \left(\cdots\cdots U_{3,2} U_{2,1} U_{1,0}\right)$$

EQUALITY APPEARS TO BE TRUE !

$\exists$ NEW IDENTITES FOR $N_f = 1, 2, 3$

8. SOME OPEN PROBLEMS

a.) PHYSICAL DERIVATION OF THE KS FORMULA

b.) HOW TO COMPUTE POLAR DEGENERACIES EFFECTIVELY ?

c.) RESOLVE THE QUESTION OF THE ENTROPY ENIGMA: ARE THERE CANCELLATIONS BRINGING $e^{\lambda^3} \to e^{\lambda^2}$?

d.) IS THERE AN OSV-LIKE RELATION FOR $\Omega(\Gamma, t_*(\Gamma))$? DO THESE ENJOY AUTOMORPHY PROPERTIES ?