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**Joint ICTP-IAEA Workshop on Nuclear Structure and Decay Data:
Theory and Evaluation**

28 April - 9 May, 2008

**Theory Nuclear Structure
(II) Collective Models**

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Nuclear Structure (II) Collective models

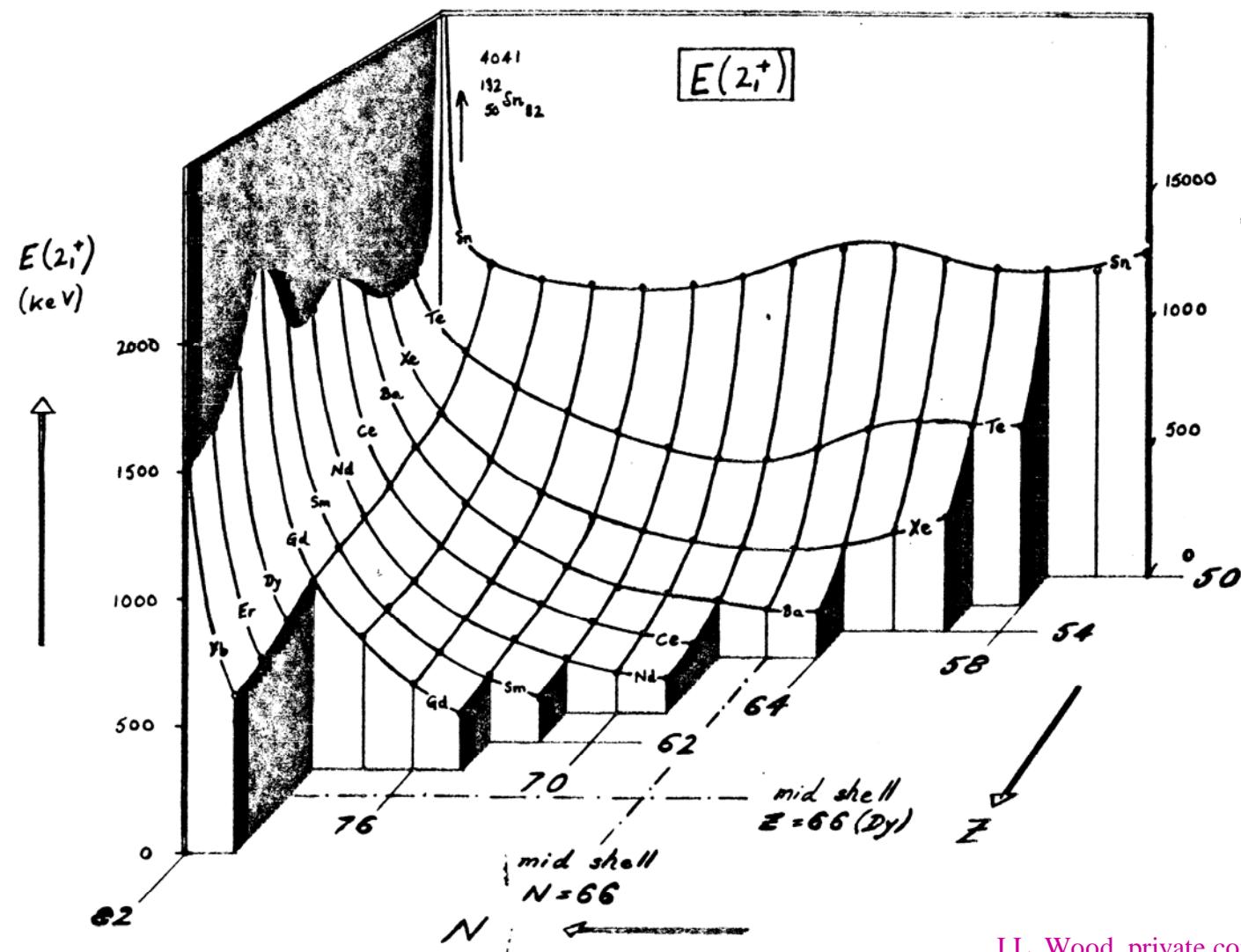
P. Van Isacker, GANIL, France

NSDD Workshop, Trieste, April-May 2008

Overview of collective models

- (Rigid) rotor model
- (Harmonic quadrupole) vibrator model
- Liquid-drop model of vibrations and rotations
- Interacting boson model
- Particle-core coupling model
- Nilsson model

Evolution of $E_x(2^+)$



J.L. Wood, private communication

Quantum-mechanical symmetric top

- Energy spectrum:

$$E_{\text{rot}}(I) = \frac{\hbar^2}{2\mathfrak{I}} I(I+1)$$

$$\equiv A I(I+1), \quad I = 0, 2, 4, \dots$$

$$\begin{array}{c} E(I) - E(I-2) \\ \hline 6^+ \qquad \qquad \qquad 42A \end{array}$$

- Large deformation \Rightarrow

large $\mathfrak{I} \Rightarrow$ low $E_x(2^+)$.

$$\begin{array}{c} 4^+ \qquad \qquad \qquad 22A \end{array}$$

- R_{42} energy ratio:

$$E_{\text{rot}}(4^+)/E_{\text{rot}}(2^+) = 3.333\dots$$

$$\begin{array}{c} 2^+ \qquad \qquad \qquad 14A \\ 0^+ \qquad \qquad \qquad 6A \end{array}$$

Rigid rotor model

- Hamiltonian of quantum-mechanical rotor in terms of ‘rotational’ angular momentum \mathbf{R} :

$$\hat{H}_{\text{rot}} = \frac{\hbar^2}{2} \left[\frac{R_1^2}{\mathfrak{I}_1} + \frac{R_2^2}{\mathfrak{I}_2} + \frac{R_3^2}{\mathfrak{I}_3} \right] = \frac{\hbar^2}{2} \sum_{i=1}^3 \frac{R_i^2}{\mathfrak{I}_i}$$

- Nuclei have an additional intrinsic part H_{intr} with ‘intrinsic’ angular momentum \mathbf{J} .
- The total angular momentum is $\mathbf{I}=\mathbf{R}+\mathbf{J}$.

Rigid axially symmetric rotor

- For $\mathfrak{J}_1 = \mathfrak{J}_2 = \mathfrak{J} \neq \mathfrak{J}_3$ the rotor hamiltonian is

$$\hat{H}_{\text{rot}} = \sum_{i=1}^3 \frac{\hbar^2}{2\mathfrak{J}_i} (I_i - J_i)^2 = \underbrace{\sum_{i=1}^3 \frac{\hbar^2}{2\mathfrak{J}_i} I_i^2}_{\hat{H}'_{\text{rot}}} - \underbrace{\sum_{i=1}^3 \frac{\hbar^2}{\mathfrak{J}_i} I_i J_i}_{\text{Coriolis}} + \underbrace{\sum_{i=1}^3 \frac{\hbar^2}{2\mathfrak{J}_i} J_i^2}_{\text{intrinsic}}$$

- Eigenvalues of H'_{rot} :

$$E'_{KI} = \frac{\hbar^2}{2\mathfrak{J}} I(I+1) + \frac{\hbar^2}{2} \left(\frac{1}{\mathfrak{J}_3} - \frac{1}{\mathfrak{J}} \right) K^2$$

- Eigenvectors $|KIM\rangle$ of H'_{rot} satisfy:

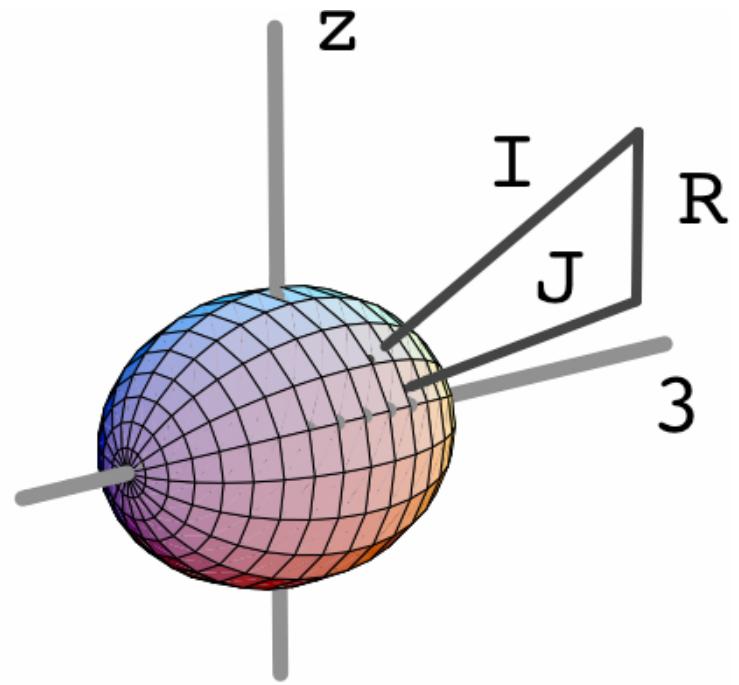
$$I^2 |KIM\rangle = I(I+1) |KIM\rangle,$$

$$I_z |KIM\rangle = M |KIM\rangle, \quad I_3 |KIM\rangle = K |KIM\rangle$$

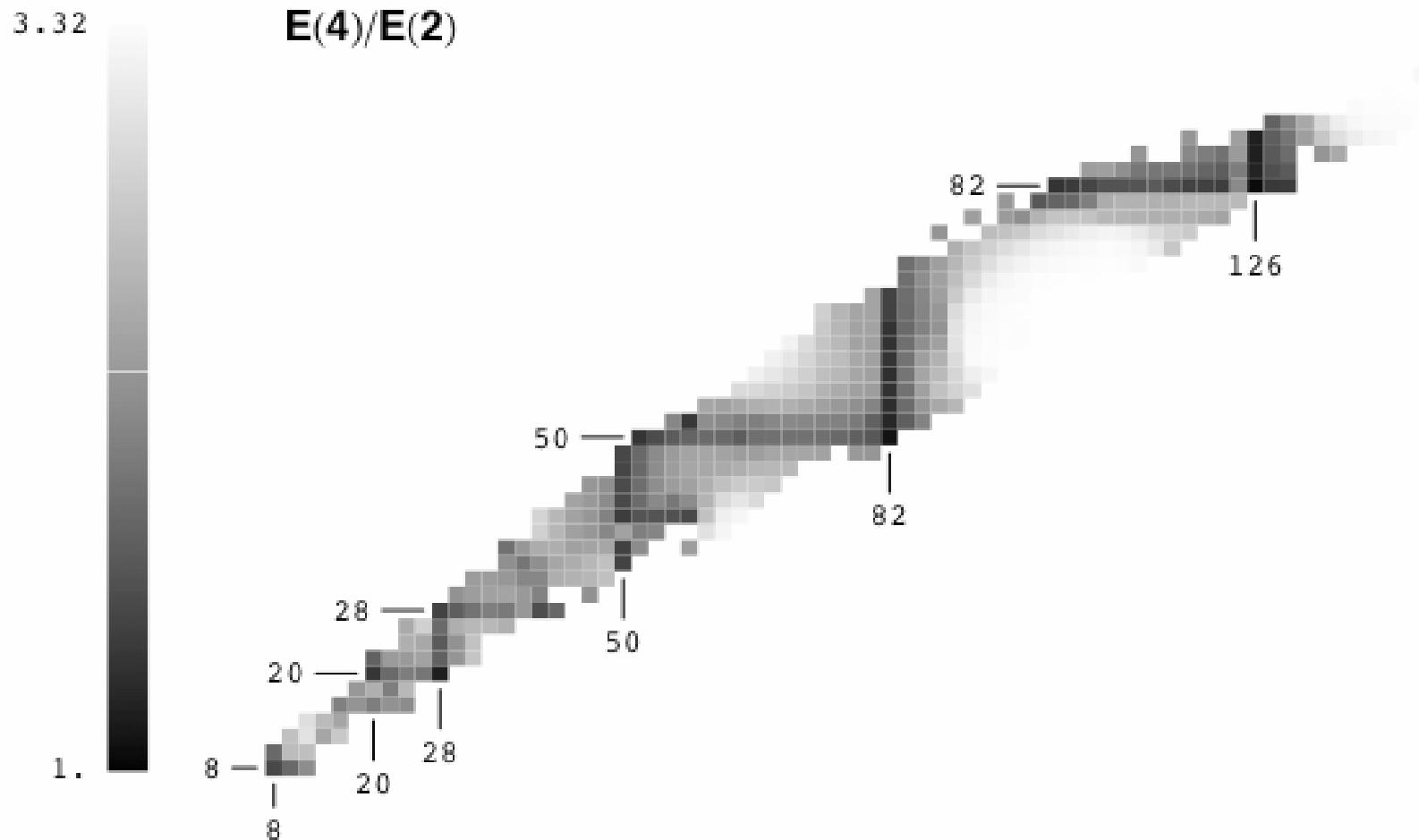
Ground-state band of an axial rotor

- The ground-state spin of even-even nuclei is $I=0$. Hence $K=0$ for ground-state band:

$$E_I = \frac{\hbar^2}{2\mathfrak{I}} I(I+1)$$



The ratio R_{42}



Electric (quadrupole) properties

- Partial γ -ray half-life:

$$T_{1/2}^\gamma(E\lambda) = \ln 2 \left\{ \frac{8\pi}{\hbar} \frac{\lambda+1}{\lambda[(2\lambda+1)!!]} \left(\frac{E_\gamma}{\hbar c} \right)^{2\lambda+1} B(E\lambda) \right\}^{-1}$$

- Electric quadrupole transitions:

$$B(E2; I_i \rightarrow I_f) = \frac{1}{2I_i + 1} \sum_{M_i} \sum_{M_f \mu} \left| \left\langle I_f M_f \left| \sum_{k=1}^A e_k r_k^2 Y_{2\mu}(\theta_k, \varphi_k) \right| I_i M_i \right\rangle \right|^2$$

- Electric quadrupole moments:

$$eQ(I) = \left\langle IM = I \left| \sqrt{\frac{16\pi}{5}} \sum_{k=1}^A e_k r_k^2 Y_{20}(\theta_k, \varphi_k) \right| IM = I \right\rangle$$

Magnetic (dipole) properties

- Partial γ -ray half-life:

$$T_{1/2}^\gamma(M\lambda) = \ln 2 \left\{ \frac{8\pi}{\hbar} \frac{\lambda + 1}{\lambda [(2\lambda + 1)!!]} \left(\frac{E_\gamma}{\hbar c} \right)^{2\lambda+1} B(M\lambda) \right\}^{-1}$$

- Magnetic dipole transitions:

$$B(M1; I_i \rightarrow I_f) = \frac{1}{2I_i + 1} \sum_{M_i} \sum_{M_f \mu} \left| \left\langle I_f M_f \left| \sum_{k=1}^A (g_k^l l_{k,\mu} + g_k^s s_{k,\mu}) \right| I_i M_i \right\rangle \right|^2$$

- Magnetic dipole moments:

$$\mu(I) = \left\langle IM = I \left| \sum_{k=1}^A (g_k^l l_{k,z} + g_k^s s_{k,z}) \right| IM = I \right\rangle$$

E2 properties of rotational nuclei

- *Intra-band E2 transitions:*

$$B(\text{E2}; KI_i \rightarrow KI_f) = \frac{5}{16\pi} \langle I_i K | I_f K \rangle^2 e^2 Q_0(K)^2$$

- *E2 moments:*

$$Q(KI) = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)} Q_0(K)$$

- $Q_0(K)$ is the ‘intrinsic’ quadrupole moment:

$$e\hat{Q}_0 \equiv \int \rho(r') r'^2 (3\cos^2 \theta' - 1) dr', \quad Q_0(K) = \langle K | \hat{Q}_0 | K \rangle$$

E2 properties of ground-state bands

- For the ground state (usually $K=I$):

$$Q(K=I) = \frac{I(2I-1)}{(I+1)(2I+3)} Q_0(K)$$

- For the gsb in even-even nuclei ($K=0$):

$$B(\text{E}2; I \rightarrow I-2) = \frac{15}{32\pi} \frac{I(I-1)}{(2I-1)(2I+1)} e^2 Q_0^2$$

$$Q(I) = -\frac{I}{2I+3} Q_0$$

$$\Rightarrow |eQ(2_1^+)| = \frac{2}{7} \sqrt{16\pi \cdot B(\text{E}2; 2_1^+ \rightarrow 0_1^+)}$$

Generalized intensity relations

- Mixing of K arises from
 - Dependence of Q_0 on I (stretching)
 - Coriolis interaction
 - Triaxiality
- Generalized *intra-* and *inter*-band matrix elements (*e.g.* E2):

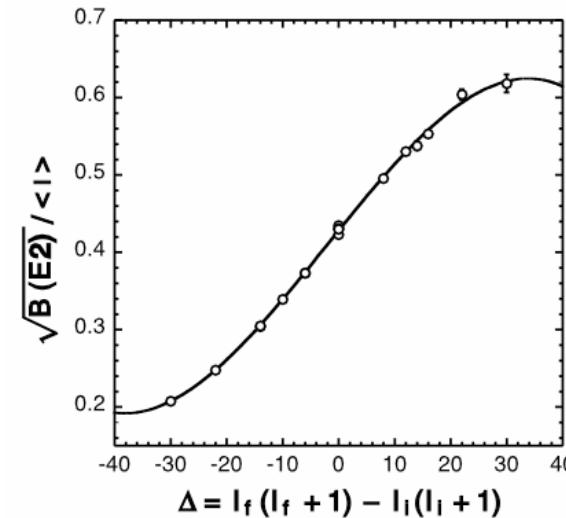
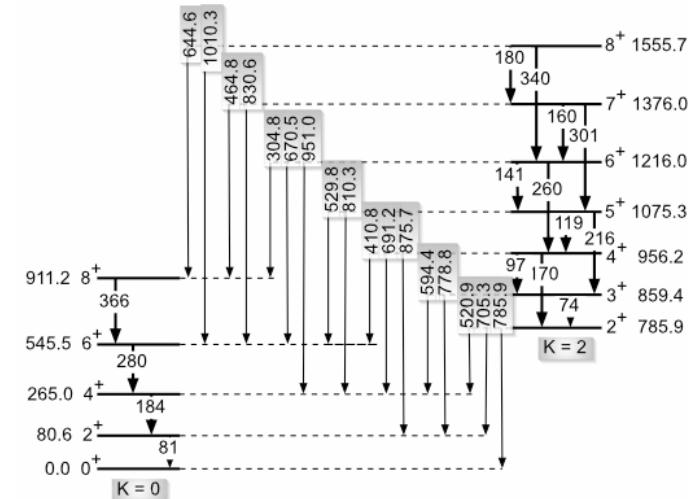
$$\frac{\sqrt{B(E2; K_i I_i \rightarrow K_f I_f)}}{\langle I_i K_i | 2K_f - K_i | I_f K_f \rangle} = M_0 + M_1 \Delta + M_2 \Delta^2 + \dots$$

with $\Delta = I_f(I_f + 1) - I_i(I_i + 1)$

Inter-band E2 transitions

- Example of $\gamma \rightarrow g$ transitions in ^{166}Er :

$$\begin{aligned} & \sqrt{B(\text{E2}; I_\gamma \rightarrow I_g)} \\ & \frac{\langle I_\gamma 2\ 2-2 | I_g 0 \rangle}{=} \\ & = M_0 + M_1 \Delta + M_2 \Delta^2 + \dots \\ & \Delta = I_g(I_g + 1) - I_\gamma(I_\gamma + 1) \end{aligned}$$



W.D. Kulp *et al.*, Phys. Rev. C **73** (2006) 014308

NSDD Workshop, Trieste, April-May 2008

Modes of nuclear vibration

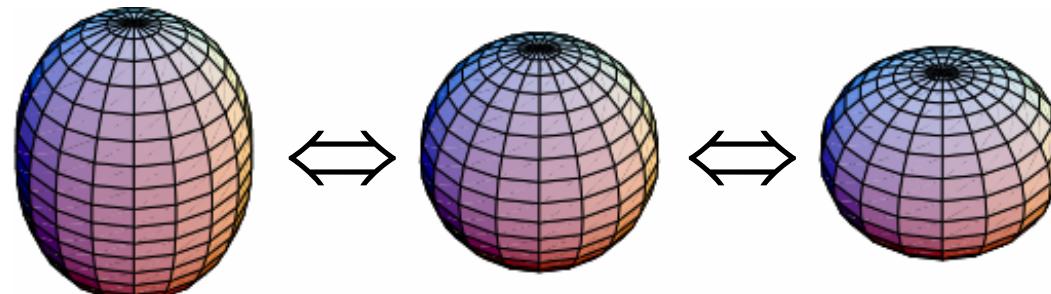
- Nucleus is considered as a droplet of nuclear matter with an equilibrium shape. Vibrations are modes of excitation around that shape.
- Character of vibrations depends on symmetry of equilibrium shape. Two important cases in nuclei:
 - Spherical equilibrium shape
 - Spheroidal equilibrium shape

Vibrations about a spherical shape

- Vibrations are characterized by a multipole quantum number λ in surface parametrization:

$$R(\theta, \varphi) = R_0 \left(1 + \sum_{\lambda} \sum_{\mu=-\lambda}^{+\lambda} \alpha_{\lambda\mu} Y_{\lambda\mu}^*(\theta, \varphi) \right)$$

- $\lambda=0$: compression (high energy)
- $\lambda=1$: translation (not an intrinsic excitation)
- $\lambda=2$: quadrupole vibration



Properties of spherical vibrations

- Energy spectrum:

$$E_{\text{vib}}(n) = \left(n + \frac{5}{2}\right)\hbar\omega, n = 0, 1, \dots$$

3 6⁺ 4⁺ 3⁺ 2⁺ 0⁺

- R_{42} energy ratio:

$$E_{\text{vib}}(4^+)/E_{\text{vib}}(2^+) = 2$$

2 4⁺ 2⁺ 0⁺

- E2 transitions:

$$B(\text{E2}; 2_1^+ \rightarrow 0_1^+) = \alpha^2$$

1 2⁺

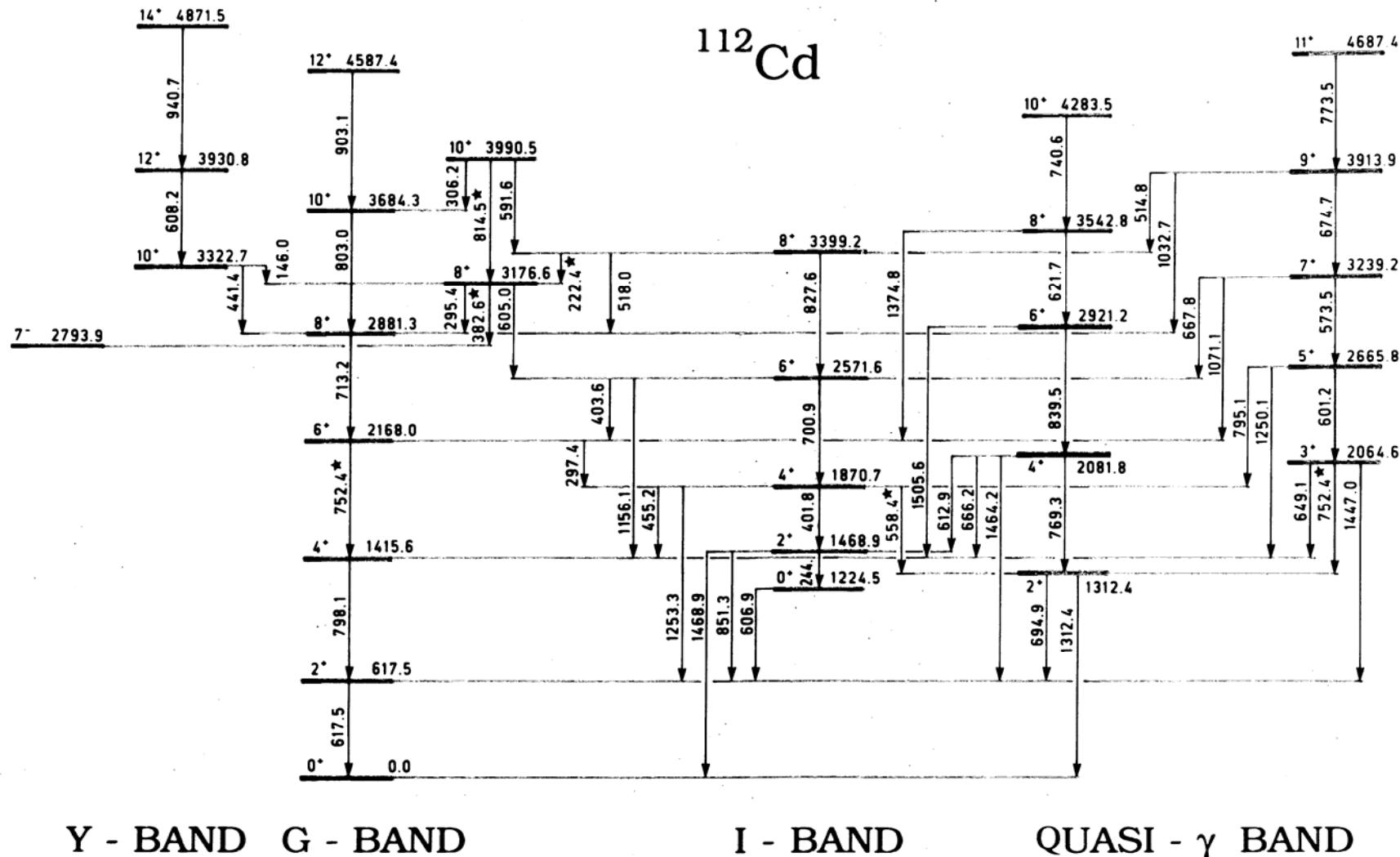
$$B(\text{E2}; 2_2^+ \rightarrow 0_1^+) = 0$$

1 2⁺

$$B(\text{E2}; n=2 \rightarrow n=1) = 2\alpha^2$$

0 0⁺

Example of ^{112}Cd

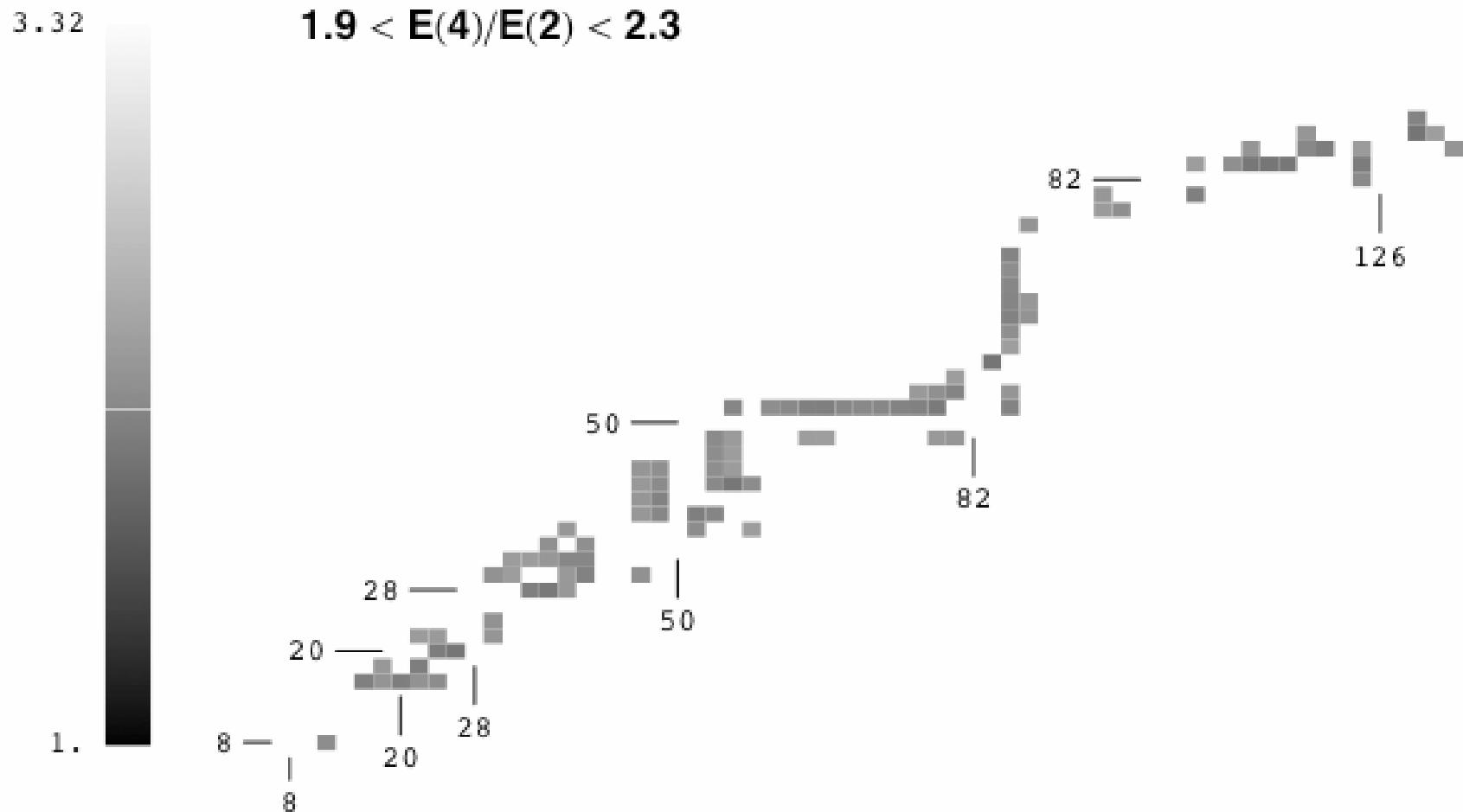


Y - BAND G - BAND

I - BAND

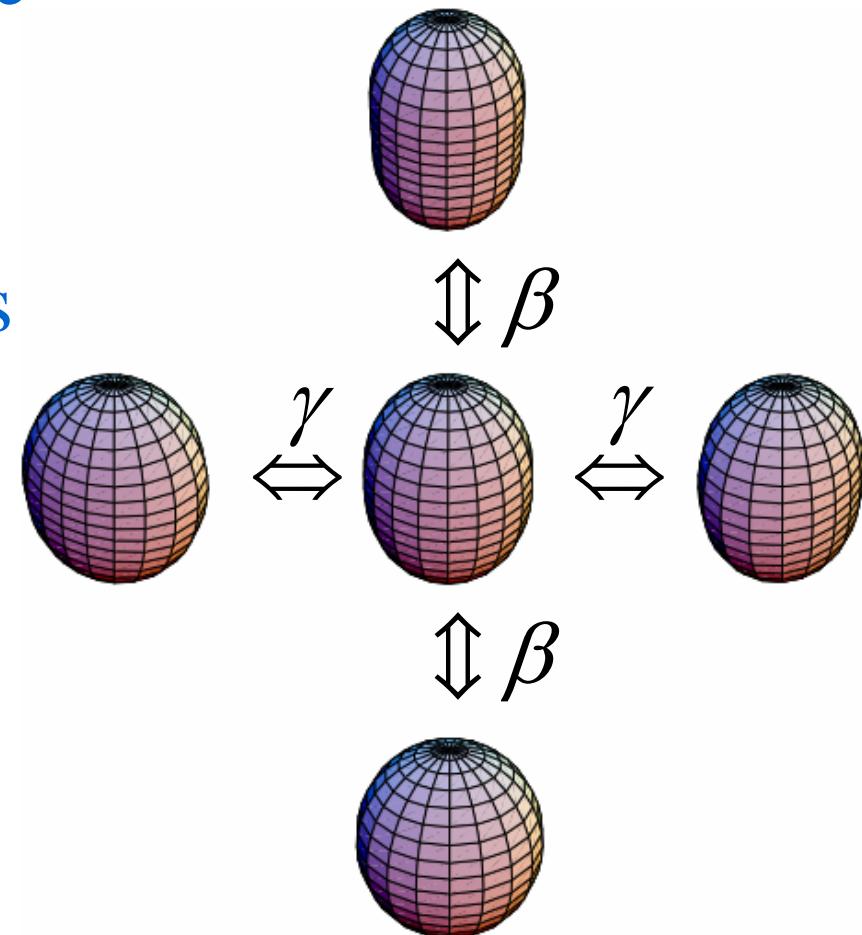
QUASI - γ BAND

Possible vibrational nuclei from R_{42}

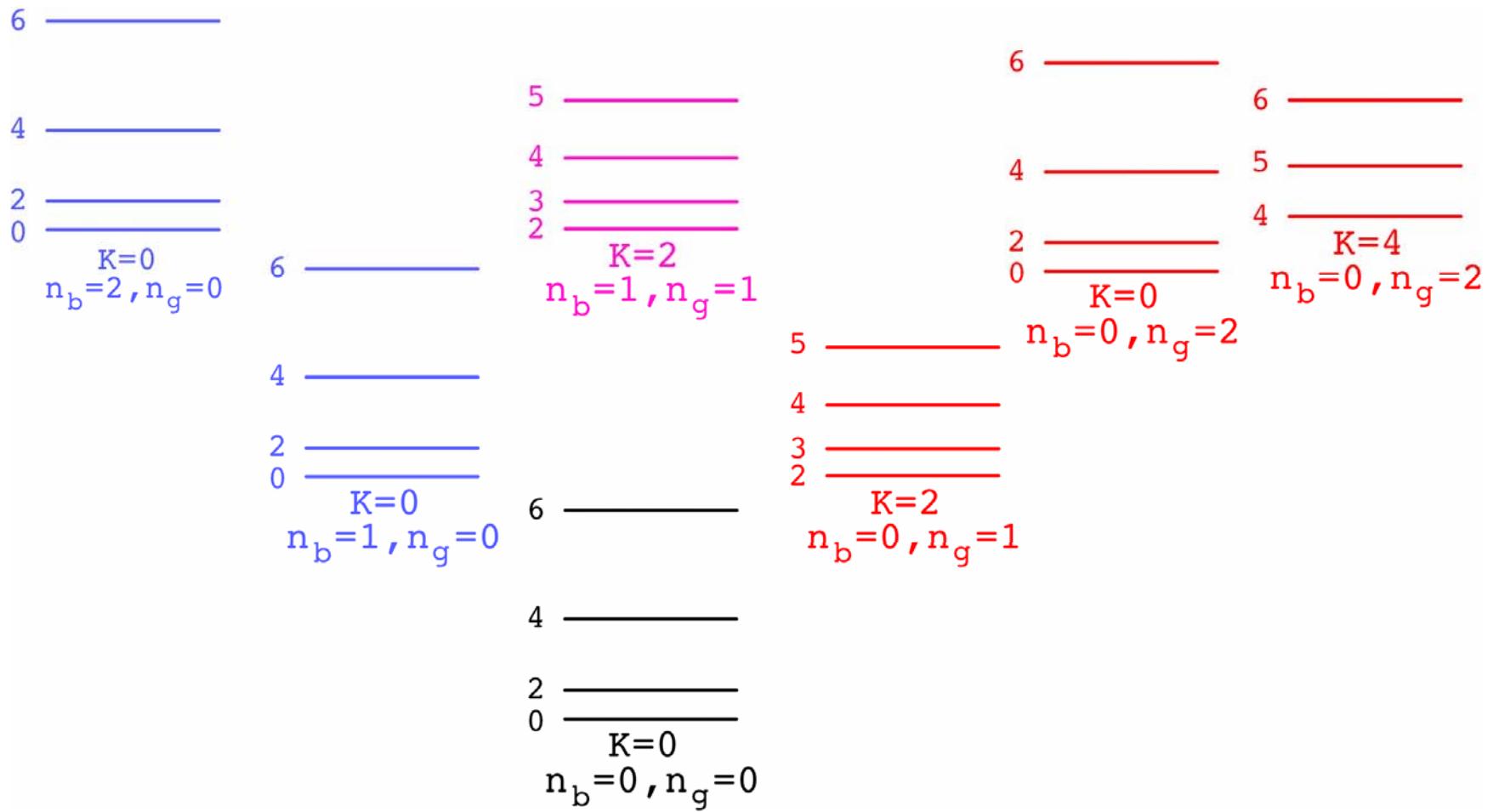


Vibrations about a spheroidal shape

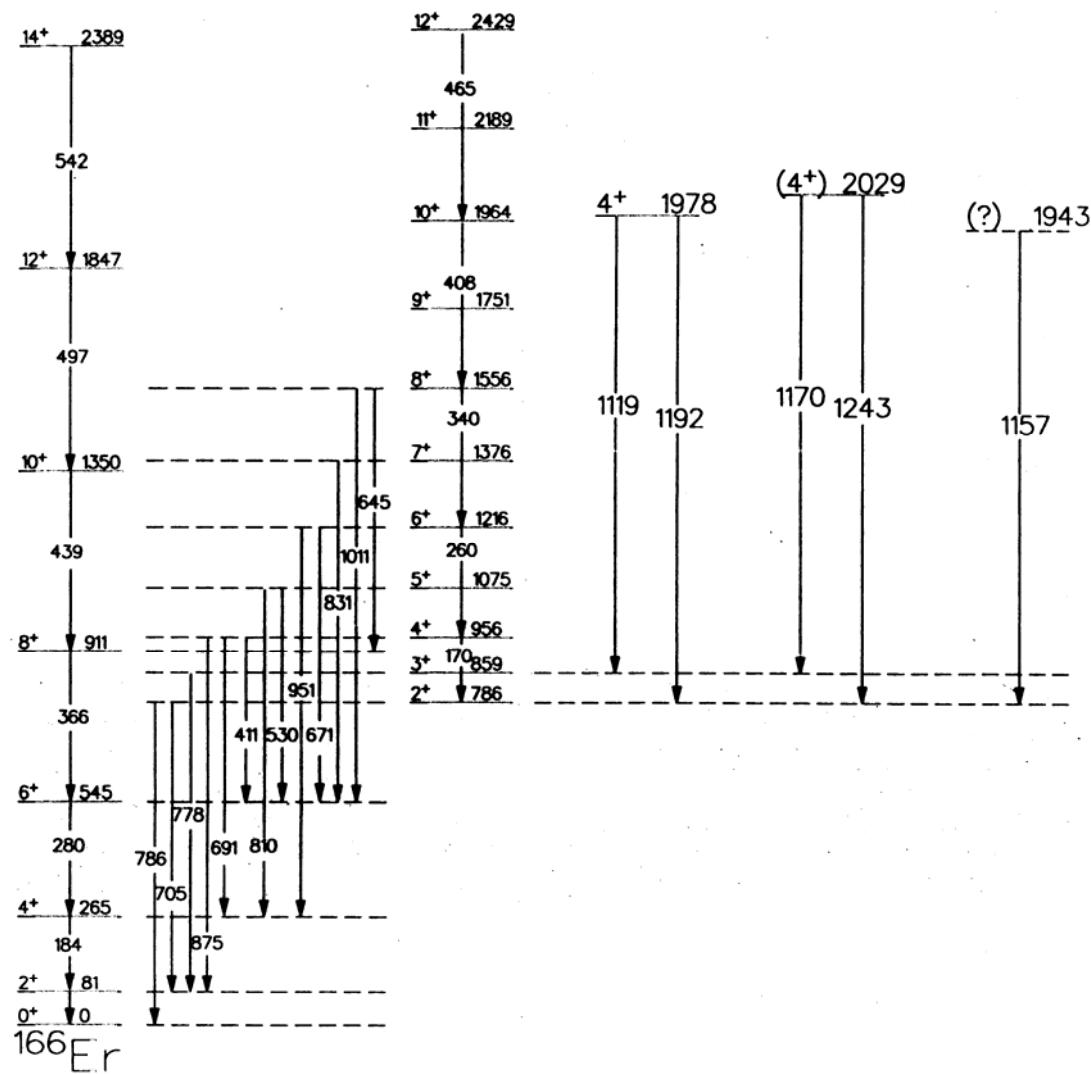
- The vibration of a shape with axial symmetry is characterized by $a_{\lambda\nu}$
- Quadrupole oscillations
 - $\nu=0$: along the axis of symmetry (β)
 - $\nu=\pm 1$: spurious rotation
 - $\nu=\pm 2$: perpendicular to axis of symmetry (γ)



Spectrum of spheroidal vibrations



Example of ^{166}Er



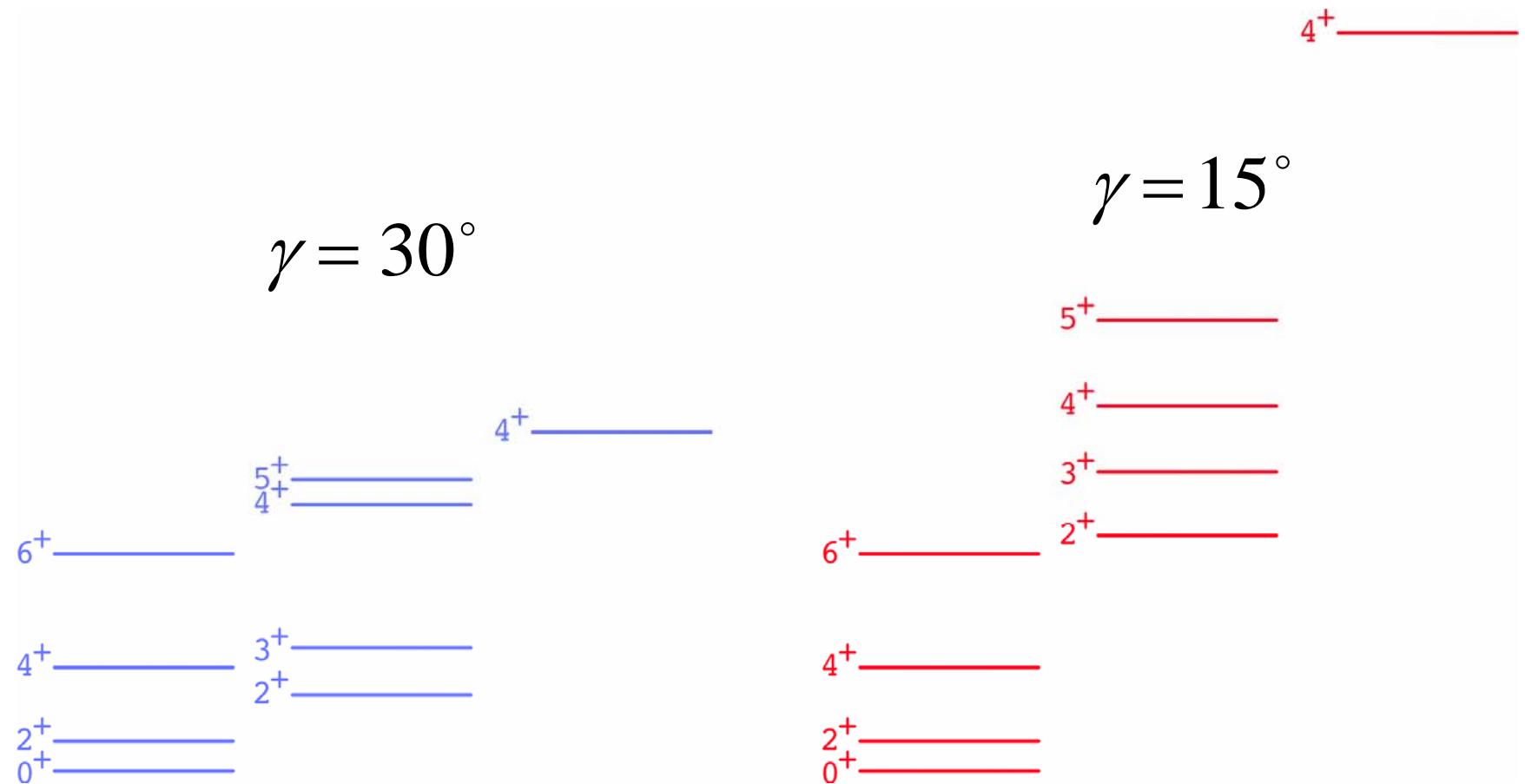
Rigid triaxial rotor

- Triaxial rotor hamiltonian $\mathfrak{I}_1 \neq \mathfrak{I}_2 \neq \mathfrak{I}_3$:

$$\hat{H}'_{\text{rot}} = \sum_{i=1}^3 \frac{\hbar^2}{2\mathfrak{I}_i} I_i^2 = \underbrace{\frac{\hbar^2}{2\mathfrak{I}} I^2}_{\hat{H}'_{\text{axial}}} + \underbrace{\frac{\hbar^2}{2\mathfrak{I}_f} I_3^2}_{\hat{H}'_{\text{mix}}} + \underbrace{\frac{\hbar^2}{2\mathfrak{I}_g} (I_+^2 + I_-^2)}$$
$$\frac{1}{\mathfrak{I}} = \frac{1}{2} \left(\frac{1}{\mathfrak{I}_1} + \frac{1}{\mathfrak{I}_2} \right), \quad \frac{1}{\mathfrak{I}_f} = \frac{1}{\mathfrak{I}_3} - \frac{1}{\mathfrak{I}}, \quad \frac{1}{\mathfrak{I}_g} = \frac{1}{4} \left(\frac{1}{\mathfrak{I}_1} - \frac{1}{\mathfrak{I}_2} \right)$$

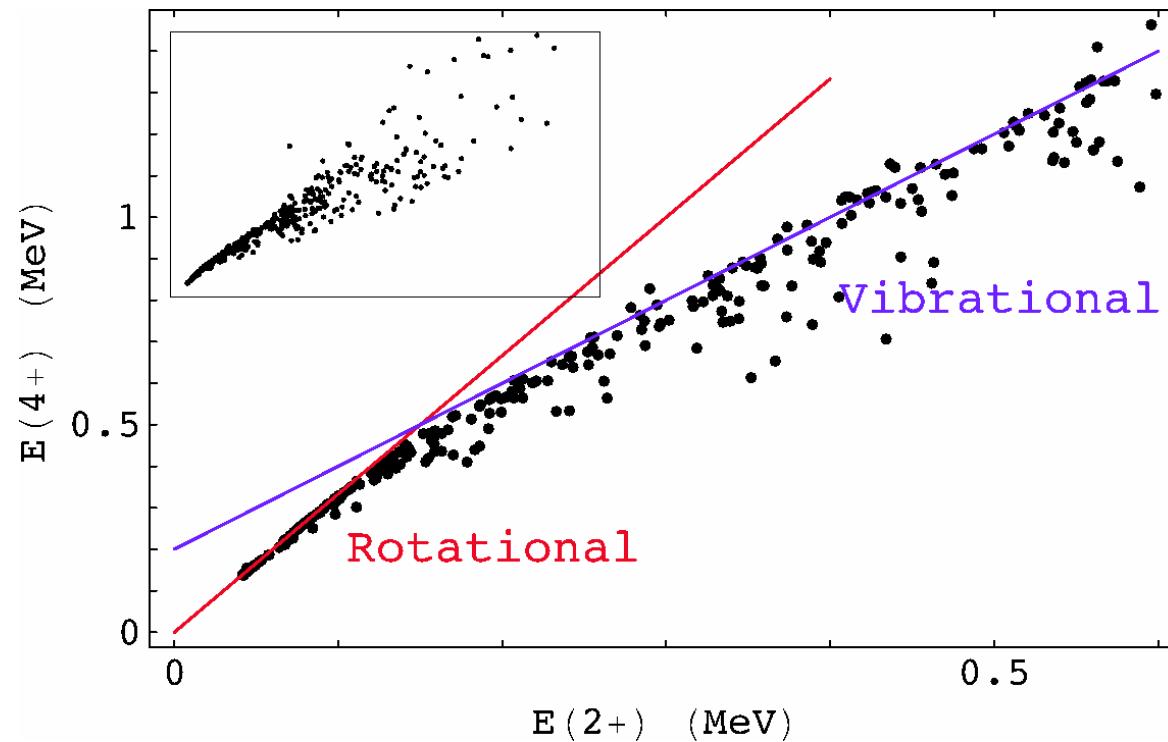
- H'_{mix} non-diagonal in axial basis $|KIM\rangle \Rightarrow K$ is *not* a conserved quantum number

Rigid triaxial rotor spectra



Tri-partite classification of nuclei

- Empirical evidence for seniority-type, vibrational- and rotational-like nuclei:



- Need for model of *vibrational* nuclei.

N.V. Zamfir *et al.*, Phys. Rev. Lett. **72** (1994) 3480

NSDD Workshop, Trieste, April-May 2008

Interacting boson model

- Describe the nucleus as a system of N interacting s and d bosons. Hamiltonian:

$$\hat{H}_{\text{IBM}} = \sum_{i=1}^6 \epsilon_i \hat{b}_i^\dagger \hat{b}_i + \sum_{i_1 i_2 i_3 i_4=1}^6 v_{i_1 i_2 i_3 i_4} \hat{b}_{i_1}^\dagger \hat{b}_{i_2}^\dagger \hat{b}_{i_3} \hat{b}_{i_4}$$

- Justification from
 - Shell model: s and d bosons are associated with S and D fermion (*Cooper*) pairs.
 - Geometric model: for large boson number the IBM reduces to a liquid-drop hamiltonian.

Dimensions

- Assume Ω available 1-fermion states. Number of n -fermion states is
$$\binom{\Omega}{n} = \frac{\Omega!}{n!(\Omega-n)!}$$
- Assume Ω available 1-boson states. Number of n -boson states is
$$\binom{\Omega+n-1}{n} = \frac{(\Omega+n-1)!}{n!(\Omega-1)!}$$
- Example: $^{162}\text{Dy}_{96}$ with 14 neutrons ($\Omega=44$) and 16 protons ($\Omega=32$) ($^{132}\text{Sn}_{82}$ inert core).
 - SM dimension: $\sim 7 \cdot 10^{19}$
 - IBM dimension: 15504

Dynamical symmetries

- Boson hamiltonian is of the form

$$\hat{H}_{\text{IBM}} = \sum_{i=1}^6 \varepsilon_i \hat{b}_i^\dagger \hat{b}_i + \sum_{i_1 i_2 i_3 i_4 = 1}^6 \nu_{i_1 i_2 i_3 i_4} \hat{b}_{i_1}^\dagger \hat{b}_{i_2}^\dagger \hat{b}_{i_3} \hat{b}_{i_4}$$

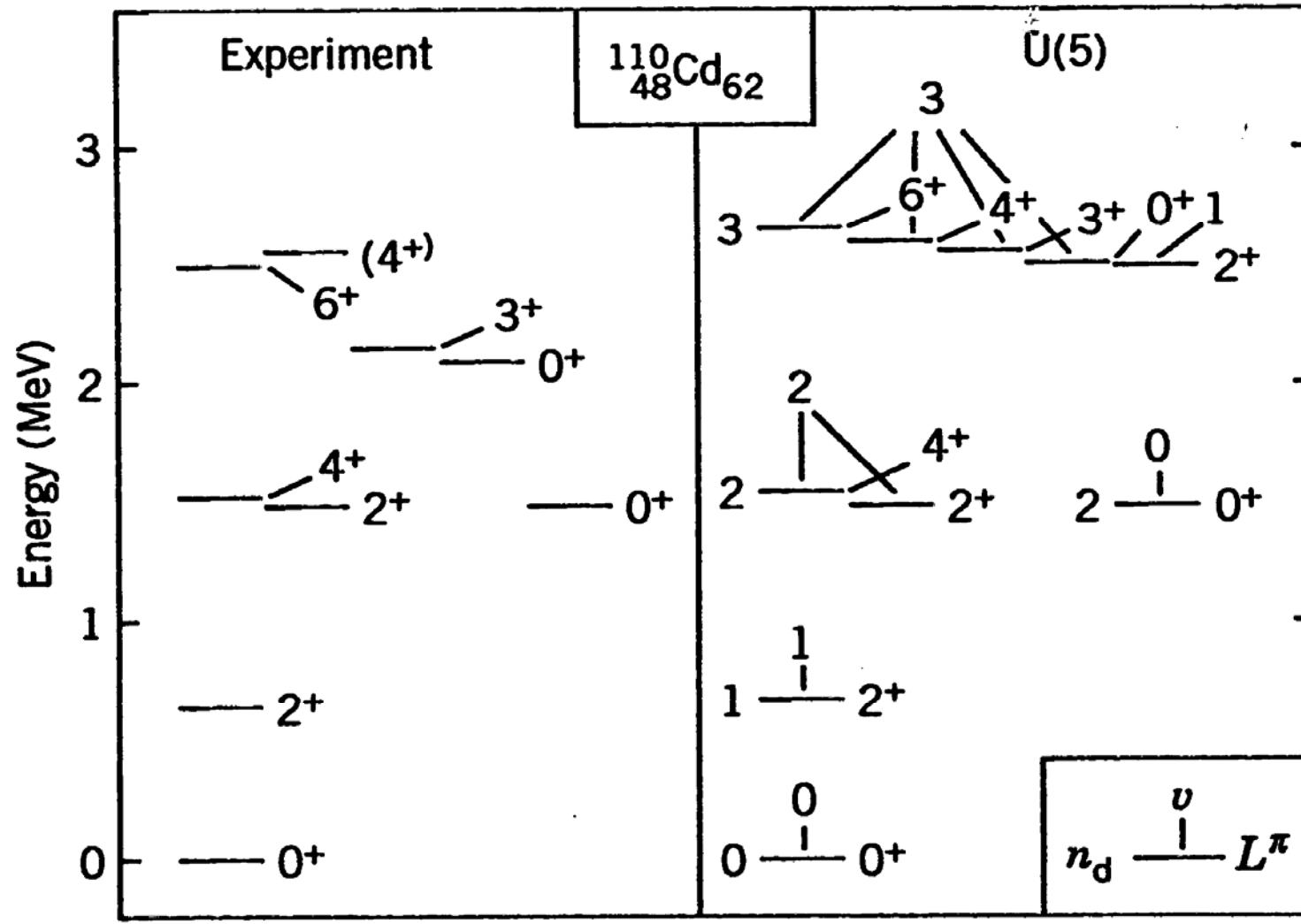
- In general not solvable analytically.
- Three solvable cases with $\text{SO}(3)$ symmetry:

$$\text{U}(6) \supset \text{U}(5) \supset \text{SO}(5) \supset \text{SO}(3)$$

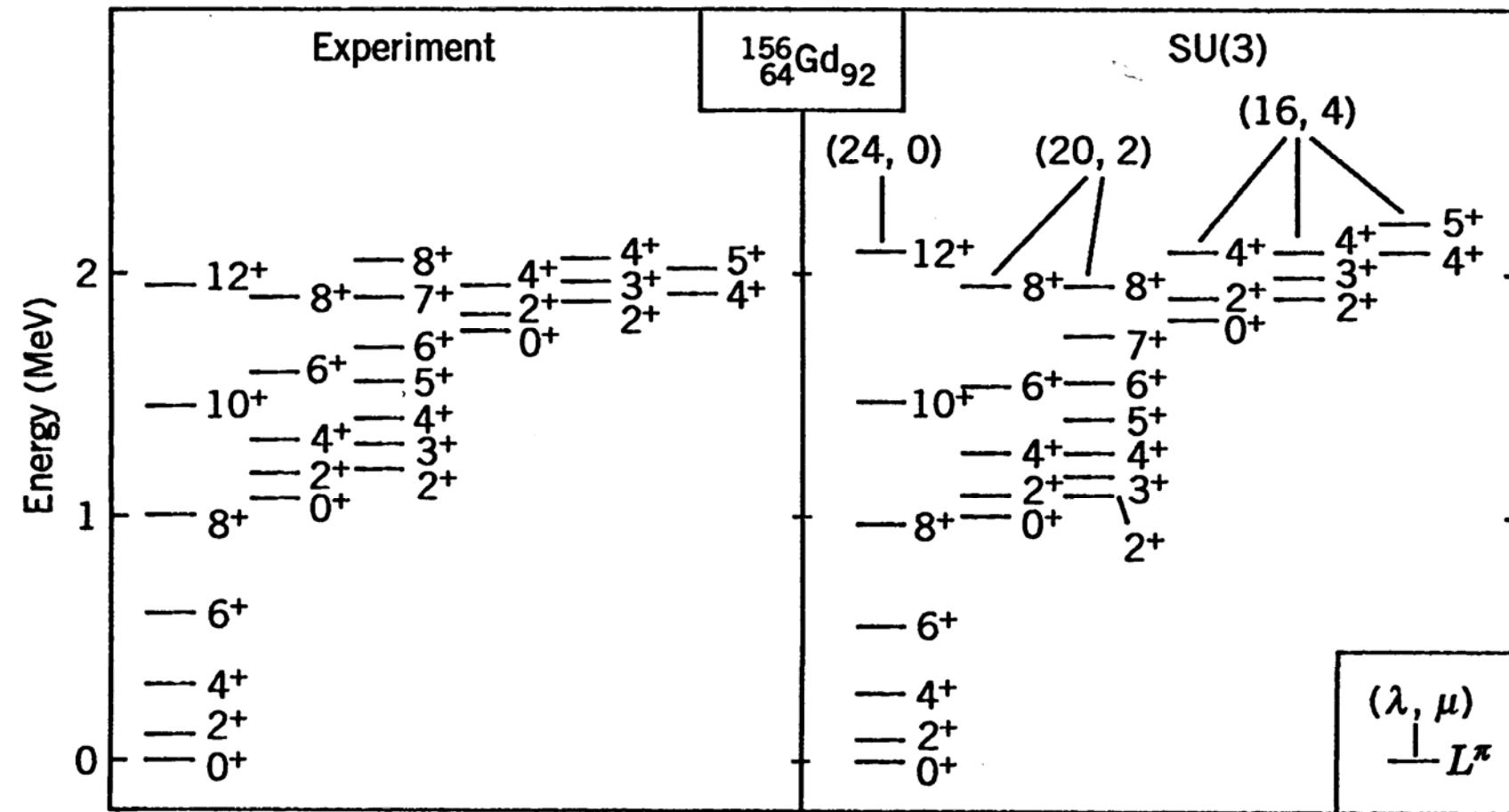
$$\text{U}(6) \supset \text{SU}(3) \supset \text{SO}(3)$$

$$\text{U}(6) \supset \text{SO}(6) \supset \text{SO}(5) \supset \text{SO}(3)$$

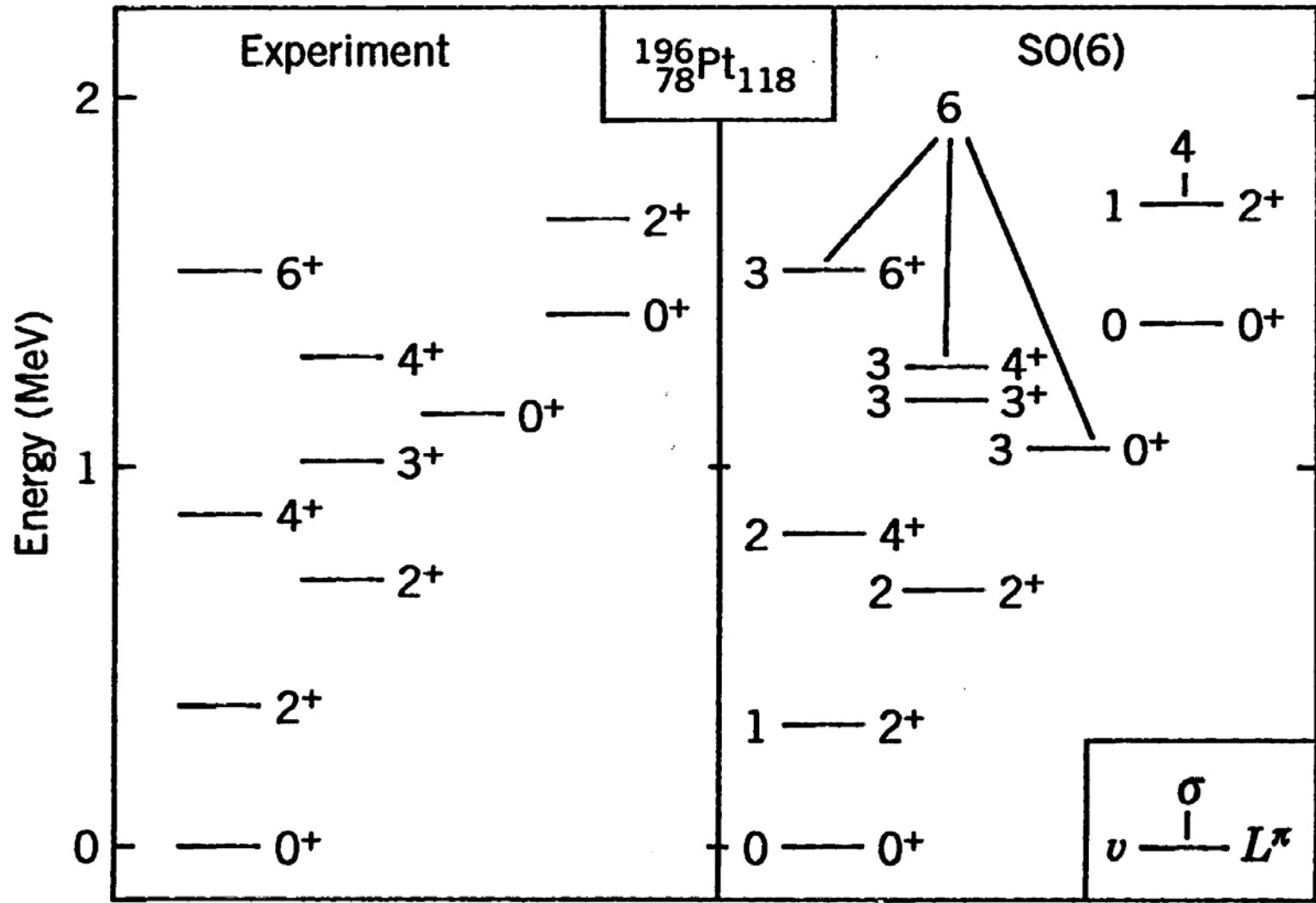
U(5) vibrational limit: $^{110}\text{Cd}_{62}$



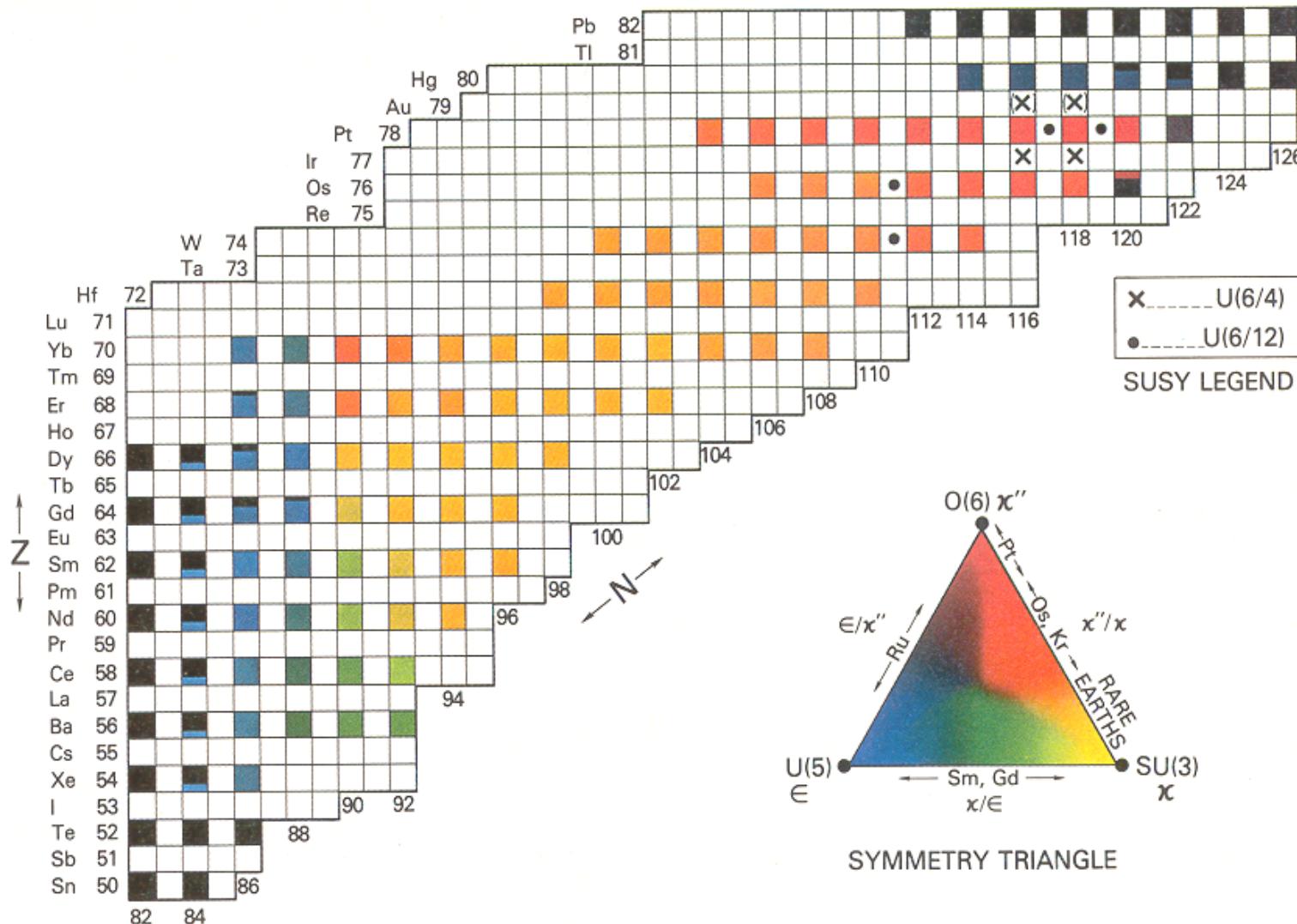
SU(3) rotational limit: $^{156}\text{Gd}_{92}$



$\text{SO}(6)$ γ -unstable limit: $^{196}\text{Pt}_{78\ 118}$



Applications of IBM

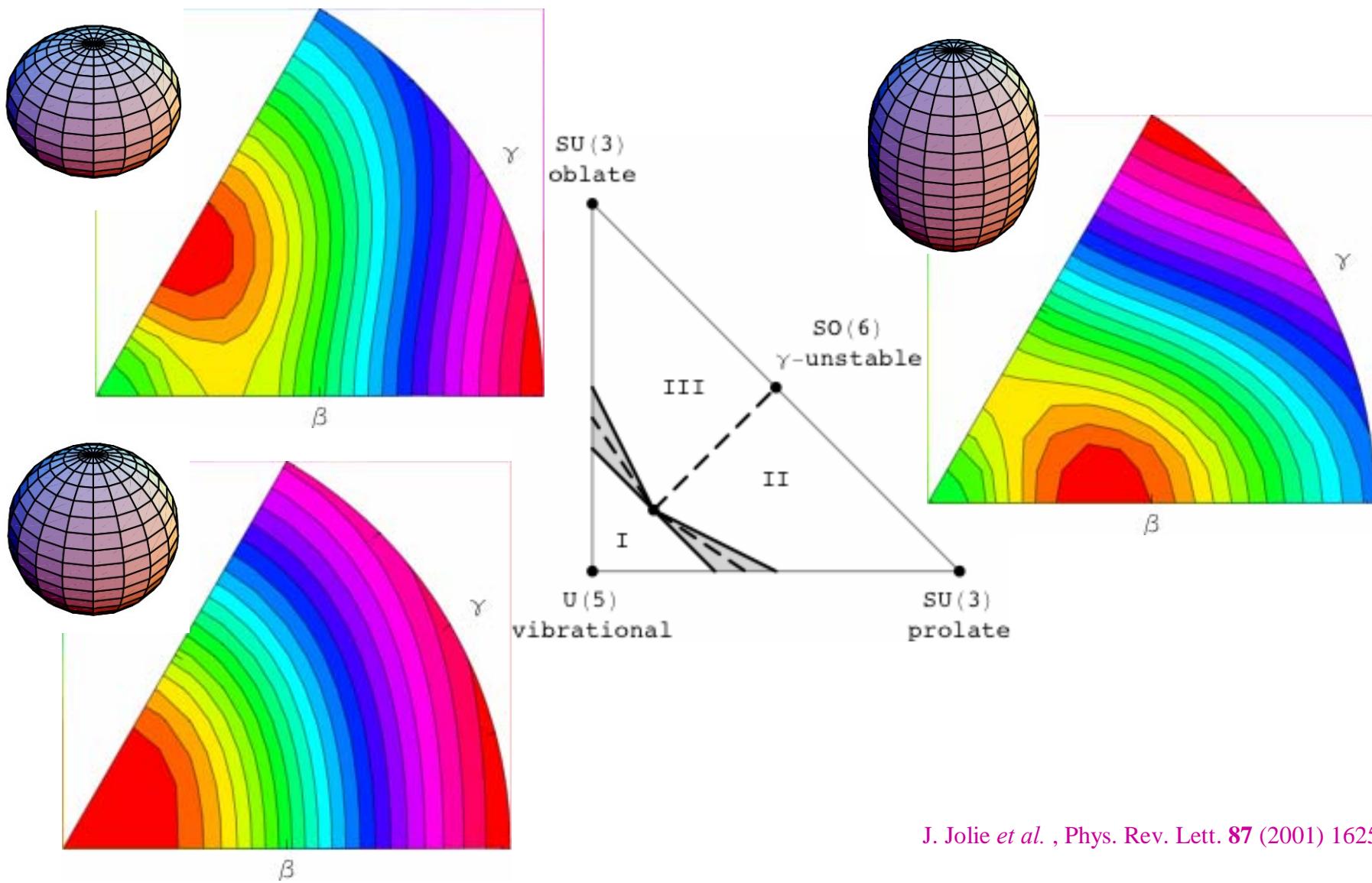


Classical limit of IBM

- For large boson number N the minimum of $V(\beta, \gamma) = \langle N; \beta\gamma | H | N; \beta\gamma \rangle$ approaches the exact ground-state energy:

$$V(\beta, \gamma) \propto \begin{cases} U(5) : & \frac{\beta^2}{1 + \beta^2} \\ SU(3) : & \frac{\beta^4 - 4\sqrt{2}\beta^3 \cos 3\gamma + 8\beta^2}{8(1 + \beta^2)^2} \\ SO(6) : & \left(\frac{1 - \beta^2}{1 + \beta^2} \right)^2 \end{cases}$$

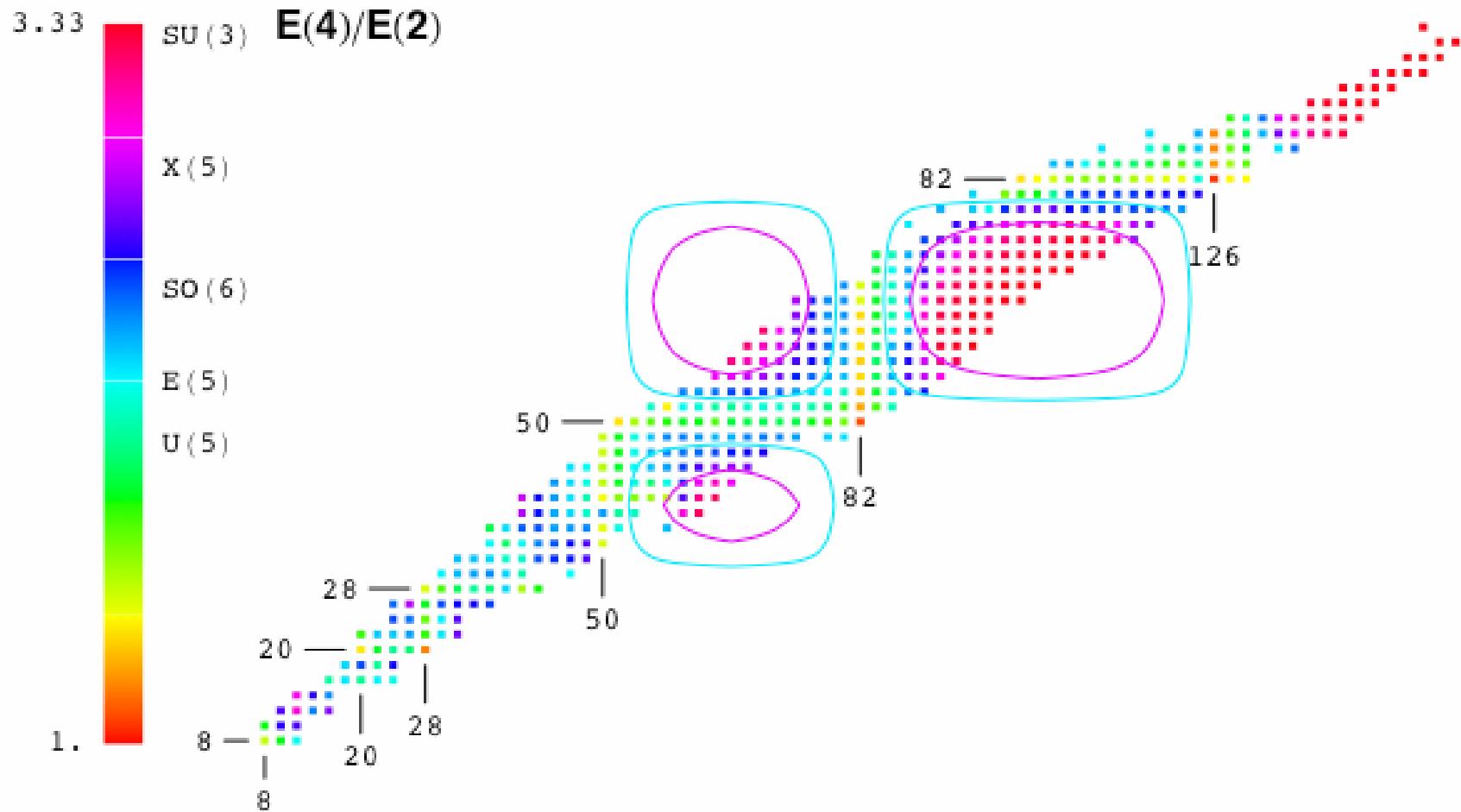
Phase diagram of IBM



J. Jolie *et al.*, Phys. Rev. Lett. **87** (2001) 162501.

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The ratio R_{42}

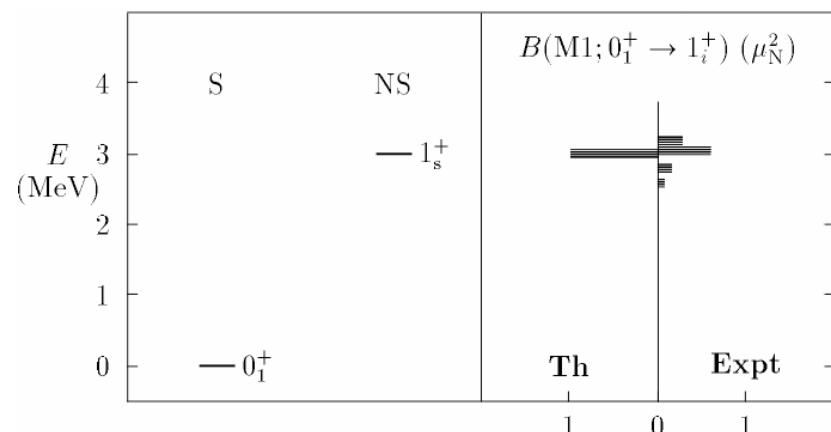
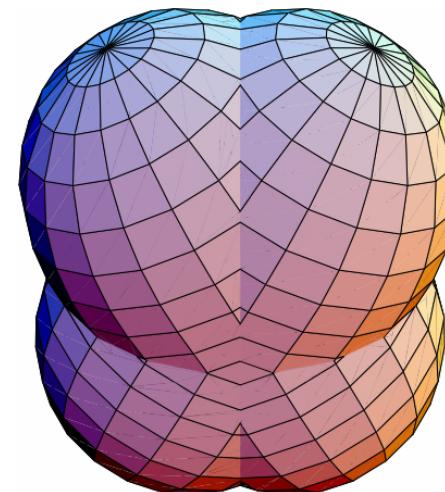
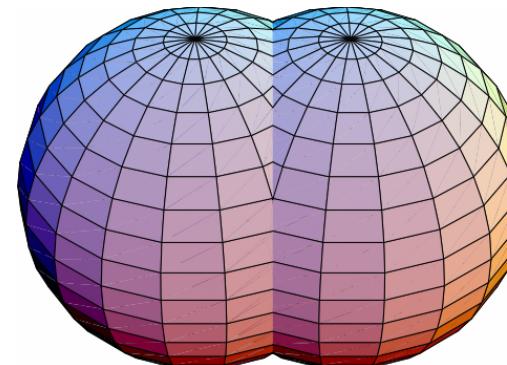


Extensions of IBM

- Neutron and proton degrees freedom (IBM-2):
 - F -spin multiplets ($N_\nu + N_\pi = \text{constant}$)
 - Scissors excitations
- Fermion degrees of freedom (IBFM):
 - Odd-mass nuclei
 - Supersymmetry (doublets & quartets)
- Other boson degrees of freedom:
 - Isospin $T=0$ & $T=1$ pairs (IBM-3 & IBM-4)
 - Higher multipole (g, \dots) pairs

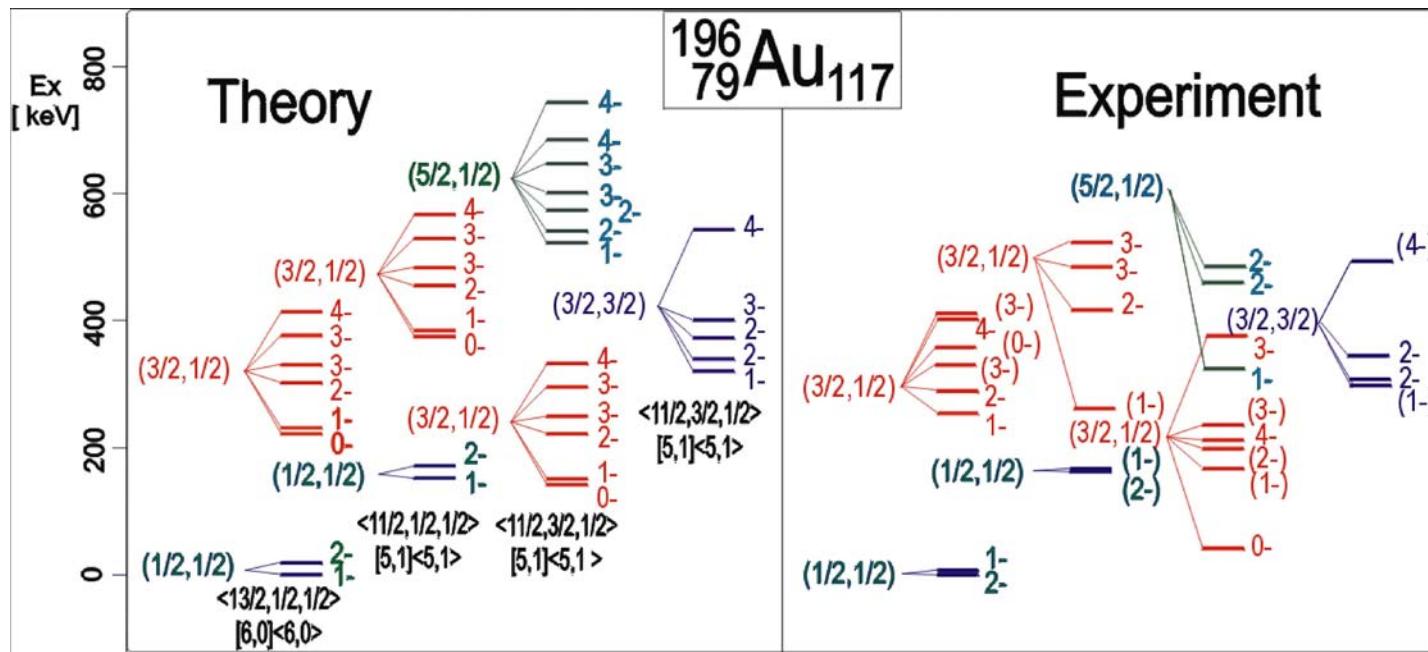
Scissors mode

- Collective displacement modes between neutrons and protons:
 - Linear displacement (giant dipole resonance):
 $R_\nu - R_\pi \Rightarrow E1$ excitation.
 - Angular displacement (scissors resonance):
 $L_\nu - L_\pi \Rightarrow M1$ excitation.



Supersymmetry

- A simultaneous description of even- and odd-mass nuclei (doublets) or of even-even, even-odd, odd-even and odd-odd nuclei (quartets).
- Example of ^{194}Pt , ^{195}Pt , ^{195}Au & ^{196}Au :



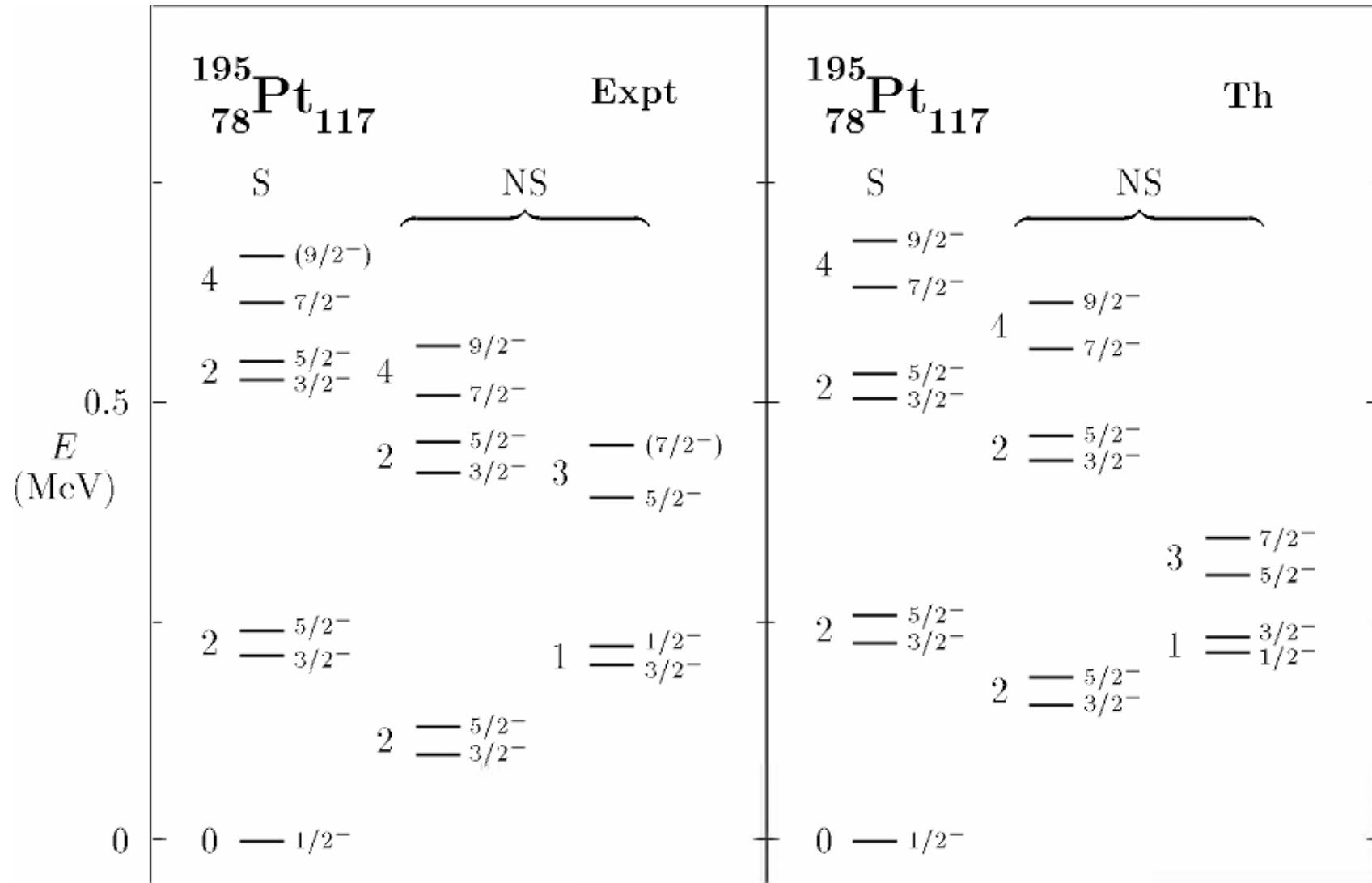
Bosons + fermions

- Odd-mass nuclei are fermions.
- Describe an odd-mass nucleus as N bosons + 1 fermion mutually interacting. Hamiltonian:

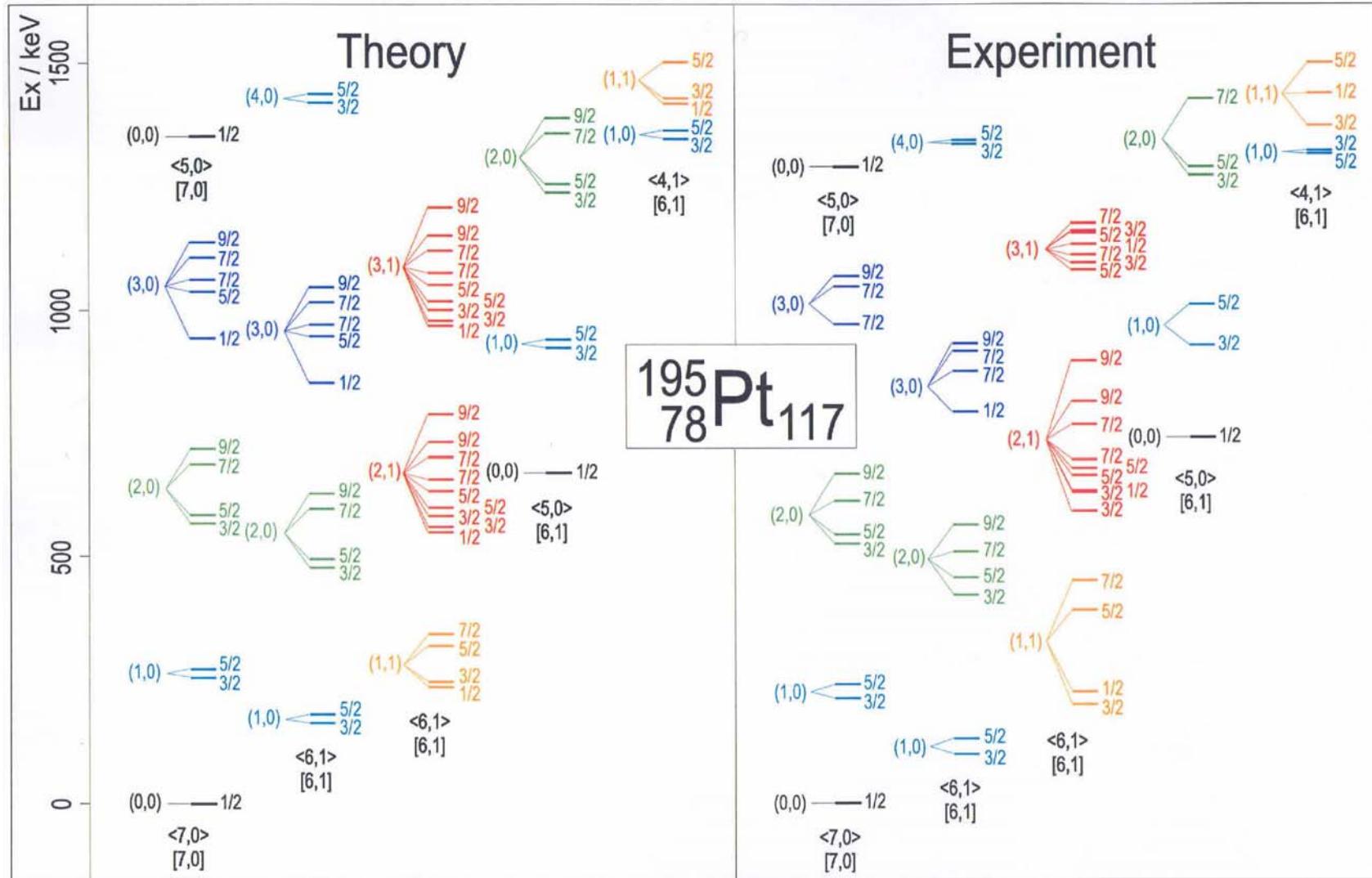
$$\hat{H}_{\text{IBFM}} = \hat{H}_{\text{IBM}} + \sum_{j=1}^{\Omega} \bar{\varepsilon}_j \hat{a}_j^\dagger \hat{a}_j + \sum_{i_1 i_2=1}^6 \sum_{j_1 j_2=1}^{\Omega} \bar{v}_{i_1 j_1 i_2 j_2} \hat{b}_{i_1}^\dagger \hat{a}_{j_1}^\dagger \hat{b}_{i_2} \hat{a}_{j_2}$$

- Algebra: $\text{U}(6) \oplus \text{U}(\Omega) = \left\{ \begin{array}{c} \hat{b}_{i_1}^\dagger \hat{b}_{i_2} \\ \hat{a}_{j_1}^\dagger \hat{a}_{j_2} \end{array} \right\}$
- Many-body problem is solved analytically for certain energies ε and interactions v .

Example: $^{195}\text{Pt}_{117}$



Example: $^{195}\text{Pt}_{117}$ (new data)



Nuclear supersymmetry

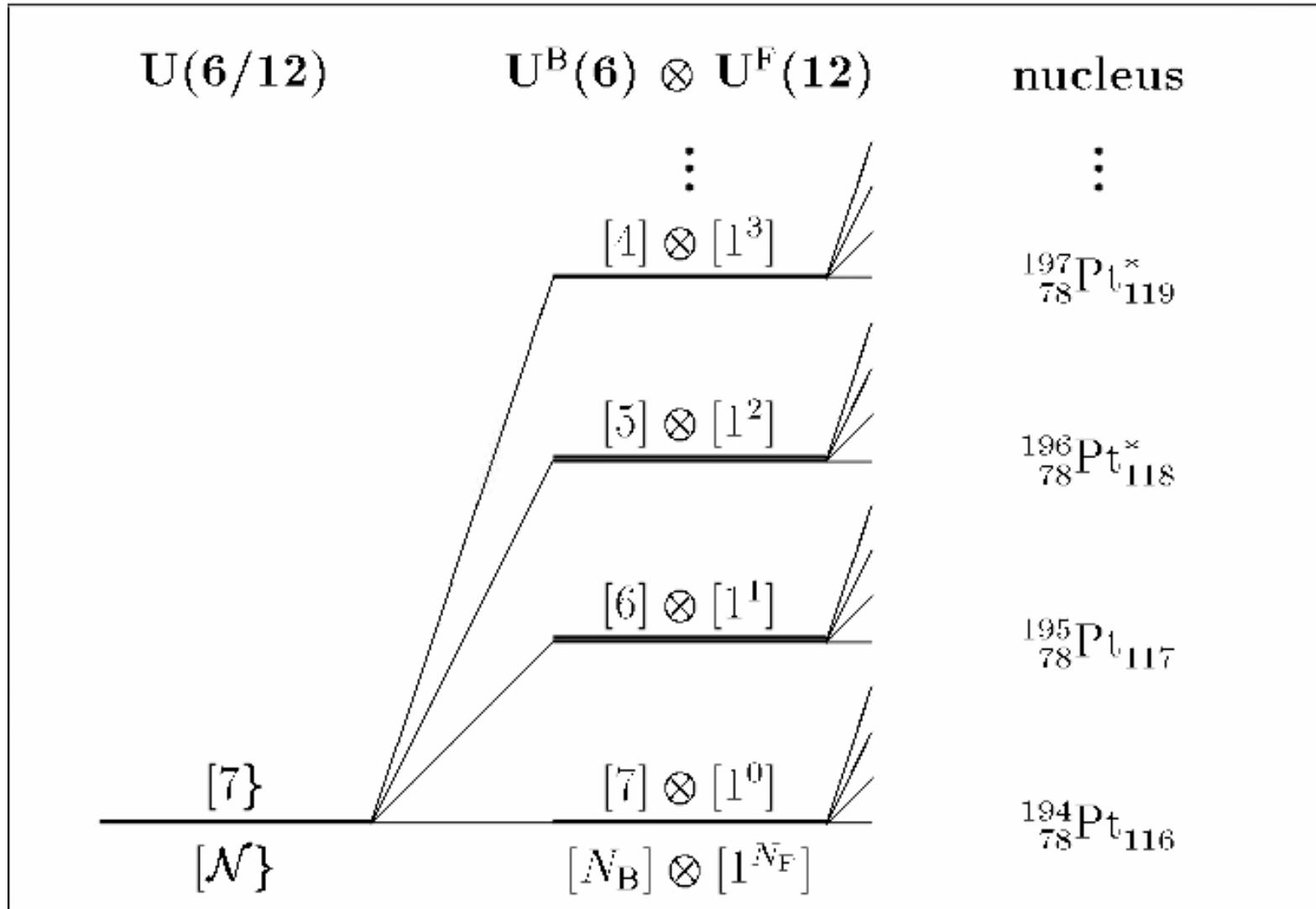
- Up to now: separate description of even-even and odd-mass nuclei with the algebra

$$U(6) \oplus U(\Omega) = \left\{ \begin{array}{c} \hat{b}_{i_1}^+ \hat{b}_{i_2} \\ \hat{a}_{j_1}^+ \hat{a}_{j_2} \end{array} \right\}$$

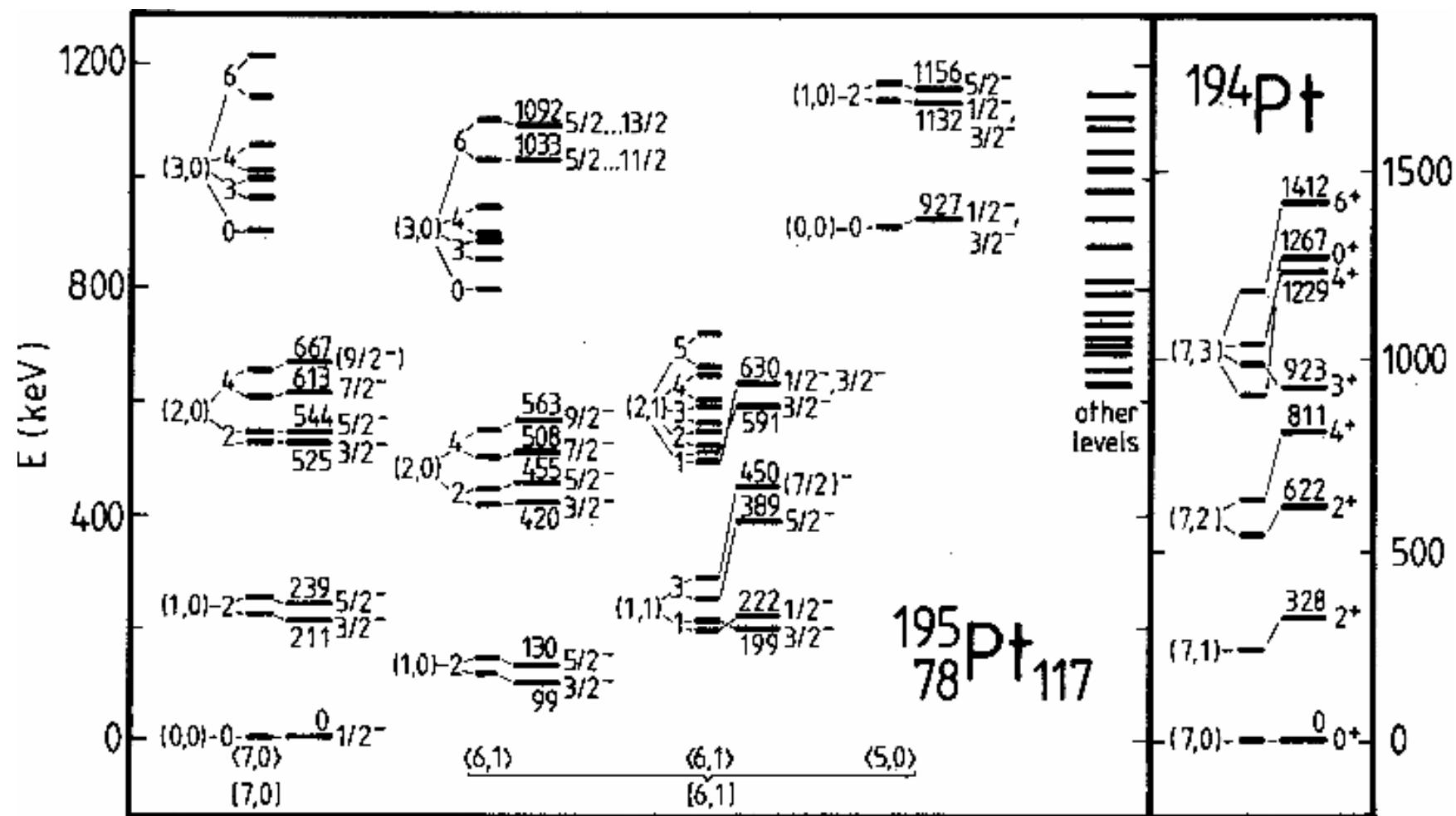
- Simultaneous description of even-even and odd-mass nuclei with the superalgebra

$$U(6/\Omega) = \left\{ \begin{array}{cc} \hat{b}_{i_1}^+ \hat{b}_{i_2} & \hat{b}_{i_1}^+ \hat{a}_{j_2} \\ \hat{a}_{j_1}^+ \hat{b}_{i_2} & \hat{a}_{j_1}^+ \hat{a}_{j_2} \end{array} \right\}$$

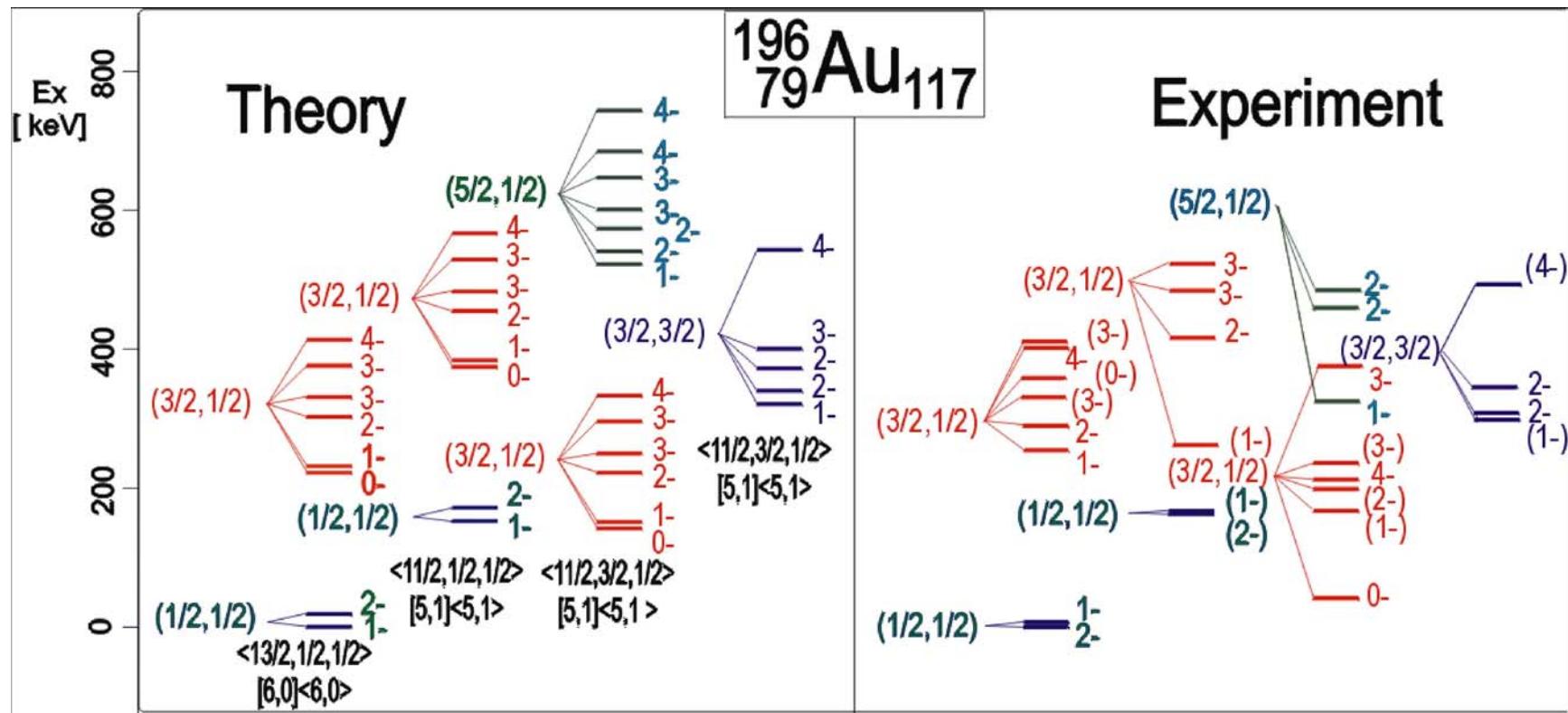
$U(6/12)$ supermultiplet



Example: $^{194}\text{Pt}_{116}$ & $^{195}\text{Pt}_{117}$



Example: $^{196}\text{Au}_{117}$



Bibliography

- A. Bohr and B.R. Mottelson, *Nuclear Structure. I Single-Particle Motion* (Benjamin, 1969).
- A. Bohr and B.R. Mottelson, *Nuclear Structure. II Nuclear Deformations* (Benjamin, 1975).
- R.D. Lawson, *Theory of the Nuclear Shell Model* (Oxford UP, 1980).
- K.L.G. Heyde, *The Nuclear Shell Model* (Springer-Verlag, 1990).
- I. Talmi, *Simple Models of Complex Nuclei* (Harwood, 1993).

Bibliography (continued)

- P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, 1980).
- D.J. Rowe, *Nuclear Collective Motion* (Methuen, 1970).
- D.J. Rowe and J.L. Wood, *Fundamentals of Nuclear Collective Models*, to appear.
- F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge UP, 1987).