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Data Analyses
(Evaluation of Discrepant Data I)

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IAEA Training Workshop

Nuclear Structure and Decay Data

Evaluation of Discrepant Data I

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Evaluation of Discrepant Data

- What is the half-life of ^{137}Cs?
- What is its uncertainty?

- Look at the published data from experimental measurements:

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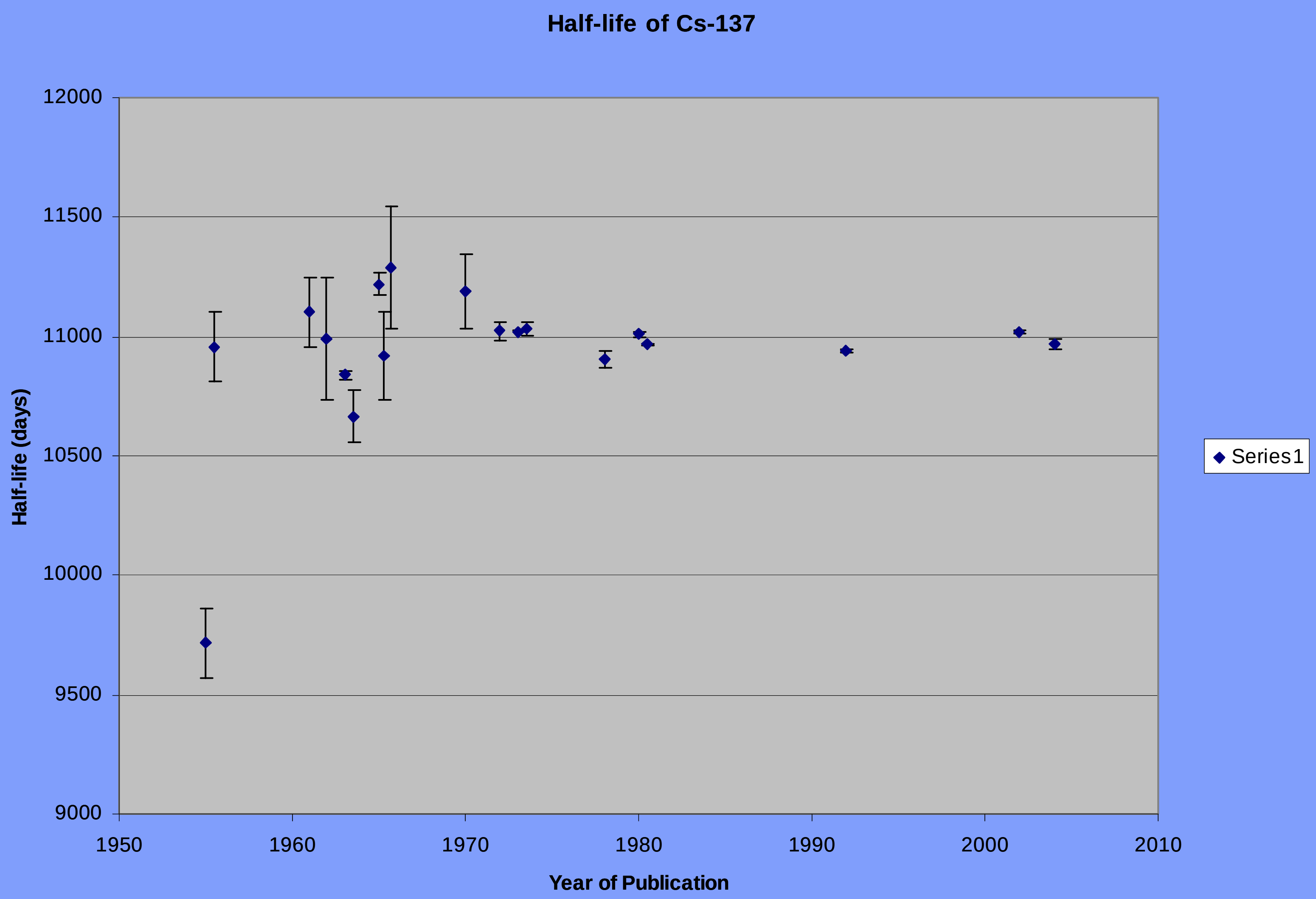
Measured Half-lives of Cs-137

Authors		Measured half-lives	
		in days	
		$t_{1/2}$	σ
Wiles & Tomlinson (1955a)		9715	146
Brown et al. (1955)		10957	146
Farrar et al. (1961)		11103	146
Fleishman et al. (1962)		10994	256
Gorbics et al. (1963)		10840	18
Rider et al. (1963)		10665	110
Lewis et al. (1965)		11220	47
Flynn et al. (1965)		10921	183
Flynn et al. (1965)		11286	256
Harbottle (1970)		11191	157
Emery et al. (1972)		11023	37
Dietz & Pachucki (1973)		11020.8	4.1
Corbett (1973)		11034	29
Gries & Steyn (1978)		10906	33
Houtermans et al. (1980)		11009	11
Martin & Taylor (1980)		10967.8	4.5
Gostely (1992)		10940.8	6.9
Unterweger (2002)		11018.3	9.5
Schrader (2004)		10970	20


Half-life of Cs-137
12000
11500
11000
10500
10000
9500
9000
1950
1960
1970
1980
1990
2000
2010
Half-life (days)
Year of Publication
Series1

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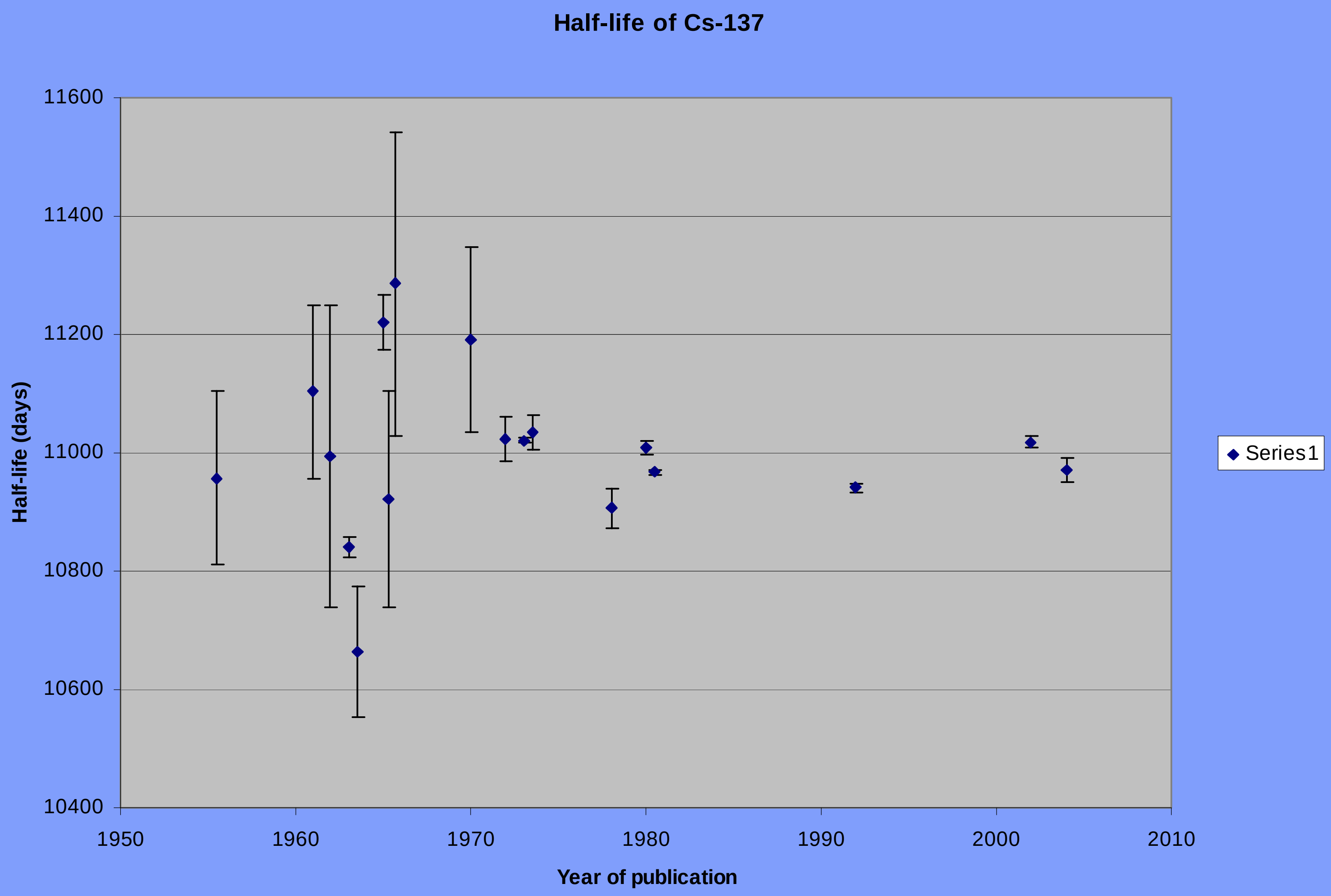

Half-life of Cs-137
Half-life (days)
11600
11400
11200
11000
10800
10600
10400
1950
1960
1970
1980
1990
2000
2010
Year of publication
Series1

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Evaluation of Discrepant Data

- The measured data range from 9715 days to 11286 days.
- What value are we going to use for practical applications?

x

- The simplest procedure is to take the unweighted mean:
- If x_i, for i = 1 to N, are the individual values of the half-life, then the unweighted mean, x_u, and its standard deviation, σ_u, are given by: -

The Unweighted Mean

$$x_u = \frac{\sum x_i}{N}$$

$$\sigma_u = \sqrt{\frac{\sum \left(x_i - x_u\right)^2}{N\left(N-1\right)}}$$

The Unweighted Mean

- This gives the result: 10936 ± 75 days
- However, the unweighted mean is influenced by outliers in the data, in particular the first, low value of 9715 days.
- Secondly, the unweighted mean takes no account of the fact that different authors made measurements of different precision, so we have lost some of the information content of the listed data.

The Weighted Mean

- We can take into account the authors' quoted uncertainties, σ_i, i = 1 to N, by weighting each value, using weights w_i, to give the weighted mean, x_w.

$$w_i = \frac{1}{\sigma_i^2}$$

$$x_w = \frac{\sum x_i w_i}{\sum w_i}$$

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The Weighted Mean

- The standard deviation of the weighted mean, σ_w, is given by:

$$\sigma_w = \sqrt{\frac{1}{\sum w_i}}$$

- And for the half-life of Cs-137 the result is 10988 ± 3 days

The Weighted Mean

- This result has a small uncertainty, but how do we know how reliable it is?
- How do we know that all the data are consistent?
- We can look at the deviations of the individual data from the mean, compared to their individual uncertainties.
- We can define a quantity 'chi-squared'

$$\chi_i^2 = \frac{(x_i - x_w)^2}{\sigma_i^2}$$

The Weighted Mean

- We can also define a ‘total chi-squared’

$$\chi^2 = \sum_i \chi_i^2$$

- In an ideal consistent data set the ‘total chi-squared’ should be equal to the number of degrees of freedom, i.e. to the number of data points minus one.

The Weighted Mean

- So, we can define a 'reduced chi-squared':

$$\chi_R^2 = \frac{\chi^2}{N-1}$$

- which should be close to unity for a consistent data set.

The Weighted Mean

- For the Cs-137 data which we have considered, the 'reduced chi-squared' is 18.6, indicating significant inconsistencies in the data.

- Let us look at the data again.

- Can we identify the more discrepant data?

Measured Half-lives of Cs-137

Authors		Measured half-lives	
		in days	
		$t_{1/2}$	σ
Wiles & Tomlinson (1955a)		9715	146
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The Weighted Mean

- The highlighted values are the more discrepant ones.
- In other words their values are far from the mean and their uncertainties are small.
- It is clear that, in cases such as the Cs-137 half-life, the uncertainty, σ_w, ascribed to the weighted mean, is much too small.
- One way of taking into account the inconsistencies is to multiply the uncertainty of the weighted mean by the Birge ratio:-

The Weighted Mean

- <u>The Birge Ratio</u>

$$\sqrt{\frac{\chi^2}{N-1}} = \sqrt{\chi_R^2}$$

- In the case of Cs-137 this would increase the uncertainty of the weighted mean from 3 days to 13 days, which would be more realistic.

The Limitation of Relative Statistical Weights (LRSW)

- This procedure has been adopted by the IAEA in the Coordinated Research Program on X- and gamma-ray standards.
- A Relative Statistical Weight is defined as $$\frac{w_i}{\sum w_i}$$
- If the most precise value in a data set (the value with the smallest uncertainty) has a relative weight greater than 0.5, its uncertainty is increased until its relative weight has dropped to 0.5.

The Limitation of Relative Statistical Weights (LRSW)

- This avoids any single value having too much influence in determining the weighted mean.
- (In the case of Cs-137, there is no such value).
- The LRSW procedure then compares the unweighted mean with the new weighted mean. If they overlap, i.e.

$$\left| x_u - x_w \right| \quad \leq \quad \sigma_u + \sigma_w$$

- then the weighted mean is the adopted value.

The Limitation of Relative Statistical Weights (LRSW)

- If the weighted mean and the unweighted mean do not overlap in the above sense, it indicates inconsistency in the data, and the unweighted mean is adopted.

- Whichever mean is adopted, its uncertainty is increased, if necessary, to cover the most precise value in the data set.

The Limitation of Relative Statistical Weights (LRSW)

- In the case of Cs-137:
- Unweighted Mean: 10936 ± 75 days
- Weighted Mean: 10988 ± 3 days
- The two means do overlap so the weighted mean is adopted.
- The most precise value in the data set is that of Dietz & Pachucki (1973): 11020.8 ± 4.1 days
- The uncertainty in the weighted mean is therefore increased to 33 days: 10988 ± 33 days.

The Median

- The individual values in a data set are listed in order of magnitude.
- If there is an odd number of values, the middle value is the median.
- If there is an even number of values, the median is the average of the two middle values.
- The median has the advantage that it is very insensitive to outliers.

The Median

- We now need some way of attributing an uncertainty to the median.
- For this we first have to determine a quantity 'the median of the absolute deviations' or 'MAD'

$$MAD \quad = \quad med\,\{\left|\, x_i - \tilde{m} \,\right|\} \quad for\ i = 1, 2, 3, \ldots.. N$$

where $\tilde{m}$ *is the median value.*

The Median

- It has been shown that the uncertainty in the median can be expressed as:

$$\frac{1.9 \times MAD}{\sqrt{(N-1)}}$$

The Median

- In the case of the Cs-137 half data already presented, the median is 10970 ± 23 days.
- Note that, like the unweighted mean, the median does not use the uncertainties assigned by the authors, so again some information is lost.
- However, the median is much less influenced by outliers than is the unweighted mean.

Bootstrap Method

- A Monte Carlo procedure to estimate a best value and its uncertainty.

- A random sample (with replacement) is selected from the data set and the median of this random sample is determined, $\boxed{X_{med,j}}$

- The sampling is repeated for j = 1, 2, 3, ……..M.

Bootstrap Method

- The best estimate is then given by: -

$$\hat{x} = \frac{1}{M}\sum_{j=1}^{M} x_{med,j}$$

- With variance: -

$$\sigma_{\hat{x}}^2 = \frac{1}{M-1}\sum_{j=1}^{M}\left(x_{med,j} - \hat{x}\right)^2$$

Bootstrap Method

- Note that each sample of the data set, j, may have some values of the data set repeated and other values missing.

- As in the case of the simple median the Bootstrap Method does not make use of the uncertainties quoted with the data.

Extended Bootstrap Method

- A procedure has been devised based on the Bootstrap Method, but also making use of the quoted uncertainties.

- A Gaussian distribution is assigned to each input value taking into account its associated standard uncertainty.

Extended Bootstrap Method

- Random samples are then taken from the probability distribution for each of the input quantities

- About one million Monte Carlo trials are recommended.

- The best value and standard deviation are then calculated as shown for the Bootstrap Method.

Evaluation of Discrepant Data

- So, in summary, we have:
- Unweighted Mean: 10936 ± 75 days
- Weighted Mean: 10988 ± 3 days
- LRSW: 10988 ± 33 days
- Median: 10970 ± 23 days
- Bootstrap Method 10990 ± 26 days
- Extended Bootstrap 10992 ± 19 days

Reference

- <u>'Convergence of techniques for the evaluation of discrepant data'</u>
- Desmond MacMahon, Andy Pearce, Peter Harris
- Applied Radiation and Isotopes 60 (2004) 275-281

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National Physical Laboratory

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