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Introductory School on Gauge Theory/Gravity Correspondence

19 - 30 May 2008

A very very very basic introduction to the AdS/CFT correspondence

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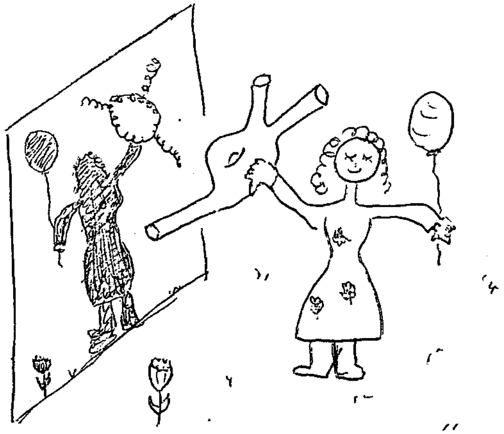
A VERY VERY VERY BASIC INTRODUCTION TO THE AdS/CFT CORRESPONDENCE.



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A.S. ICTP

MAY 2008.

THE REFERENCE:

JUAN MARTIN MALDACENA:

"The large N Limit of superconformal field theories and supergravity".

hep-th/9711200.

ALSO:

* Gubser, Klebanov, Polyakov
  "Gauge theory correlators from noncritical string theory"
  hep-th/9802109

* Witten
  "Anti-de Sitter space and holography
  hep-th/9802150

REVIEWS:

* Aharony, Gubser, Maldacena, Ooguri, Oz
  hep-th/9905111
* D'Hoker, Freedman hep-th/0201253
* Maldacena hep-th/0309246.

Just to give away the punchline, in 1997 Maldacena proposed that a certain very special type of gauge theory, the $\mathcal{N}=4$ SYM, is "dual" (read "the same") to a very special type of string theory: IIB on <u>Anti</u> deSitter space in FIVE dimensions (times a sphere). AdS is particularly symmetric solution of Einsteins equations with a <u>NEGATIVE</u> cosmological constant whereas the real world is well described by an (effective?) POSITIVE cosmological constant in FOUR dimensions. Also, the $\mathcal{N}=4$ SYM is a far cry from any phenomenologically relevant gauge theory...

Still, at the time I am writing this sentence, Maldacena's paper has 5188 citations in the SPIRES database (If you check now it will be more) To set the scale: the papers containing <u>experimental evidence</u> that $\Lambda > 0$ (or something very close to that...) have 4776, 3813, 3641, ... citations

Have we all gone astray?

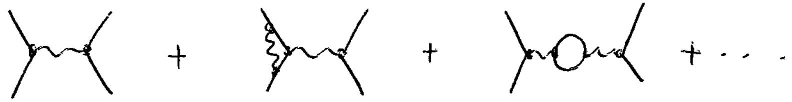
I will argue that the answer is a resounding NO!

Maldacena's idea (and its generalizatio is the closest we have ever come to a theoretical understandi of deep issues in strongly couple gauge theories and quantum gravity. It is of monumental importanc

Everything we know (know ≡ experimental test!) about particle physics today can be described by gauge theories.

Everything we know about gravity is described by general relativity.

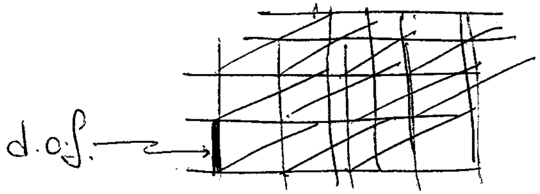
We can systematically compute quantum corrections in a gauge theory by doing "perturbation theory".



but this works only at energy scales where the renormalized coupling constants are small.

On the other hand, most of the interesting problems occour when the coupling gets large (This happens at LOW energies in QCD!): chiral symmetry breaking, confinment...

There are methods to deal with strongly coupled gauge theories e.g. lattice gauge theory



but it is cleary important to develop more.

On the other hand, we are not even entirely sure (entirely sure ≡ corroborat by experiments) on how to compute quantum corrections to gravity.

Many of us think string theory is BY FAR our best bet but we need more wor

Now here is the beauty: the AdS/CFT provides a bridge between these two worlds, albeit in a higly controlled enviroment (not directly phenomenological. (See other lecturers for more realistic model

Some of the questions that are difficult in one world become easy in the other. For instance, gauge perturbation theory (easy) corresponds to string dynamics in a highly curved enviroment (hard). Alternatively, classical string propagation (easy) corresponds to gauge dynamics at strong coupling (hard).

Notice how different the two worlds are (and yet they are conjectured to be the same...). For instance, one contains gravity, one does not. Perhaps even "real" gravity is an "illusion"? (Paraphrasing Maldacena's article on Scientific American)

PLAN

THE TWO WORLDS:

•) $N=4$ SYM in 3+1 dim Minkowski

••) IIB strings in $AdS_5 \times S^5$.

MOTIVATING THE CONJECTURE: D3 BRANES.

THE CONJECTURE

LARGE N LIMIT and SUPERGRAVITY.

TESTING / APPLYING THE CONJECTURE

- GLOBAL SYMMETRIES.
- OPERATORS ANOMALOUS DIMS., 2pt funct
- WILSON LOOP.

More advanced stuff still related to the CONFORMAL CASE that we probably will not have time for include:

• 3 & 4 pt funct's, • conf. anomalies,

• p-p waves, • integrability ...

Maldacena

1) $\mathcal{N}=4$ SYM on 3+1 dim Minkowski space

2) IIB string theory on $AdS_5 \times S_5$ space.

Before we start discussing how they are related we must remind ourselves what they are!

$\mathcal{N}=4$ SYM

Gauge group: $SU(N)$ (can be generalized)

Coupling constant: g_{YM} (and possibly a θ-angle).

Field content:

$$A_\mu^A \,,\quad \lambda_{\alpha a}^A \,,\quad X^{Ai}$$

Gauge f., Weyl sp., real scalars.

$A = 1, \dots N^2-1 = \dim(SU(N))$

$\mu = 0,1,2,3$ space time $\eta = \begin{pmatrix} - & & & \\ & + & & \\ & & + & \\ & & & + \end{pmatrix}$

$\alpha = 1,2$ Weyl index

$\left.\begin{array}{l} i = 1,2,3,4,5,6 \\ a = 1,2,3,4 \end{array}\right\}$ R symmetry ($SU(4)$) indices.

We can use the generators of $SU(N)$:

$$[T^A, T^B] = i f^{ABC} T^C \qquad N\times N \text{ hermitian matrices}$$

to turn all fields into $N\times N$ matrices:

$$A_\mu \equiv A_\mu^A T^A \,,\quad \lambda_{\alpha a} = \lambda_{\alpha a}^A T^A \,,\quad X^i = X^{Ai} T^A$$

(sum over $A = 1 \dots N^2$).

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

$$D_\mu X^i = \partial_\mu X^i + i[A_\mu, X^i] \qquad (\text{same for } \lambda_{\alpha a})$$

$$\mathcal{L} = \frac{1}{g_{YM}^2} \operatorname{tr}\Big(-\frac{1}{2} F_{\mu\nu}^2 - i \bar{\lambda}^a \not{D} \lambda_a - (D_\mu X^i)^2 +$$

$$+ C_i^{ab} \lambda_a [X^i, \lambda_b] + \text{c.c.}$$

$$+ \frac{1}{2}[X^i, X^j]^2 \Big)$$

Plus (possibly) a θ-term: $\dfrac{\theta}{16\pi^2} \operatorname{tr} \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}$

And, if one wants to do quantum computatio a gauge fixing + ghost.

Probably the only thing that needs some explanation is C_i^{ab}. They are the "Clebsh – Gordan" coefficients needed to make a singlet out of λ_a, λ_b and X^i transforming under the $\underline{4}$, $\underline{4}$ and $\underline{6}$ of the global SU(4) symmetry known as R-symmetry:

A_μ is a singlet

$\lambda \in \square \equiv \underline{4}$ $\qquad$ $\underline{4}\otimes\underline{4}\otimes\underline{6}$ = singlet $\oplus \dots$

$X \in \begin{smallmatrix}\square\\ \square\end{smallmatrix} \equiv \underline{6}$.

The C_i^{ab}, (and the whole lagrangian for that matter) can be easily derived by dimensional reduction of $\mathcal{N}=1$ SYM in 9+1 dimensions, containing only

a gauge field $\qquad A_M \rightsquigarrow A_\mu, X^i$

and a Majorana-Weyl spinor $\qquad \Psi_\Sigma \rightsquigarrow \lambda_{\alpha a}$.

$$\mathcal{L}_{10} = \frac{1}{g^2_{YM}} \cdot \mathrm{tr}\left(-\frac{1}{2} F_{MN}^2 + i\bar{\Psi}\Gamma^M D_M \Psi\right)$$

$\Gamma^M \equiv (\Gamma^\mu, \Gamma^i)$ $\qquad \Gamma^\mu = \gamma^\mu \otimes \mathbb{1}$

$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}$ $\qquad \Gamma^i = \gamma^{(5)} \otimes \tilde{\gamma}^i$

where $\gamma^{\mu=0,1,2,3}$ are the usual 4×4 Dirac

$\gamma^{(5)} = i\gamma^0\gamma^1\gamma^2\gamma^3$.

$\tilde{\gamma}^i$ are 8×8 Dirac m. of SO(6)

and in the "Weyl representation"

$$\gamma^\mu = \left(\begin{array}{c|c} 0 & \sigma^\mu \\ \hline \bar{\sigma}^\mu & 0 \end{array}\right) \qquad \tilde{\gamma}^i = \left(\begin{array}{c|c} 0 & C^i \\ \hline \bar{C}^i & 0 \end{array}\right).$$

<u>EXERCISE</u>: Show where all the terms in the D=4 Lagrangian come from the D=10 Lagr

Since C^i_{ab} are constructed to be invar under SU(4), $\mathcal{L}$ is manifestly inv under R-symmetry. The other manife symmetries are, of course, gauge SU(and Poincaré (Transl + Lorentz).

But there is MUCH MORE!

First of all, it is called $\mathcal{N}=4$ the because there are 4 SUPERSYMMETRIES:

$$Q^{a}_{\alpha} : \begin{cases} X^i \longrightarrow C^{iab} \lambda_{\alpha b} \\ \lambda_{\beta b} \longrightarrow f_{\alpha\beta}\delta^a_b + [X^i, X^j]\epsilon_{\alpha\beta} C_{ij}{}^{a}{}_{b} \\ \bar{\lambda}^b_{\dot\beta} \longrightarrow C_i^{ab} \bar{\sigma}^{\mu}_{\alpha\dot\beta} D_\mu X^i \\ A_\mu \longrightarrow \sigma_{\mu\alpha}{}^{\dot\beta} \bar{\lambda}^a_{\dot\beta} \end{cases}$$

(they can also be easily derived from 10 d)

$f_{\alpha\beta}$ is the "self dual" component of $F_{\mu\nu}$ written ~~as~~ : $f_{\alpha\beta} = F_{\mu\nu}\sigma^{\mu\nu}{}_{\alpha\beta}$

Note that Q^a_α has the R-symmetry index a $(a=1,..4)$. This is the general definition of an R-symmetry: "a symmetry that acts on the supercharges,,..

Fundamental relations:

$$\{Q^a_\alpha, \bar{Q}_{\dot\beta b}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^a_b$$

$$\{Q^a_\alpha, Q^b_\beta\} = [P_\mu, Q^a_\alpha] = 0$$ but there is MORE!...

It can be shown that the β-function of $\mathcal{N}=4$ SYM VANISHES to all loops

Let's do. it to ONE loop:

$$\beta(g)_{YM} = -\frac{1}{16\pi^2}\left(\frac{11}{3}C(A) - \frac{2}{3}\sum_{\lambda} C(\lambda) - \frac{1}{6}\sum_{X} C(X)\right) g_{Y}$$

All Weyl fermion

All real scalar.

$C(A) = C(\lambda) = C(X) = C(Adj) = N$ factors out we are left with

$$\frac{11}{3} - \frac{2}{3}\times 4 - \frac{1}{6}\times 6 = 0 \quad \checkmark$$

In a theory with $\beta=0$ No dynamical scale is generated, no "particles,, and, strictly speaking no S-matrix, altho one can talk of perturbative S-matrix for scattering of (gauge variant) gluons, gluinos etc..

For a theory w/ $\beta=0$ the Poincaré group has a larger bosonic extensio known as the CONFORMAL GROUP.

consider, for example a single field $\phi(x)$ in 4 dim with a free action:

$$S = \frac{1}{2}\int d^4x\, \partial_\mu\phi\,\partial^\mu\phi \equiv \frac{1}{2}\int d^4x\, \partial_\mu\phi\,\partial_\nu\phi\,\eta^{\mu\nu}.$$

The form of the action (including the explicit form of the metric $\eta = \begin{pmatrix}-1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1\end{pmatrix}$) is invariant under the following changes of variable:

- * spacetime translations $x^\mu = x'^\mu + a^\mu$
  * lorentz transform: $x^\mu = \Lambda^\mu{}_\nu x'^\mu$
- as long as ϕ transforms as a scalar: $\phi(x) \equiv \phi'(x')$.

What happens if we let $x^\mu = A x'^\mu$ instead?
This is clearly NOT a lorentz transform.
We could say that the metric changes:

- $\eta'_{\mu\nu} = A^2 \eta_{\mu\nu}$

OR, since the change is proportional to the metric itself, we could try to compensate by rescaling the fields:

$$\frac{1}{2}\int d^4x \frac{\partial\phi(x)}{\partial x^\mu}\frac{\partial\phi(x)}{\partial x^\nu}\eta^{\mu\nu} = \frac{1}{2}\int d^4x'\, A^4 \cdot \frac{1}{A}\frac{\partial}{\partial x'^\mu}\phi(Ax')\,\frac{1}{A}\frac{\partial}{\partial x'^\nu}\phi(Ax')\,\eta^{\mu\nu}$$

Can be brought to the same form as before (including η) by letting

$$A\phi(Ax') = \phi'(x')$$

Note that this is NOT General Relativity. In G.R. we accept that the metric changes (covariantly) and that $\phi'(x') \equiv \phi(x)$ for ALL coord. changes, includi dilation. Here we want to keep the action (better, the eq. of motion) INVARIANT (use η all the time) and we can only hope to do that by scaling the fields if the change in η is prop. to η itsel

- Such transformations: $\eta_{\mu\nu} \to \omega(x)\,\eta_{\mu\nu}$ are called CONFORMAL:
  - Dilations: $x^\mu \to x'^\mu = A x^\mu$
  - Special Conf: $x^\mu \to x'^\mu = \dfrac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2}$

- EXERCISE Show this!

- Obviously, not all Poincaré inv. actions can be made conformally invariant.

Eg: $S = \int d^4x \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - g\phi^n$

is invariant ONLY for $n=4$.

At this level, this is just DIMENSIONAL ANALYSIS

Now we

$\beta=0$ is. In theories where $\beta\neq 0$ a scale parameter will be generated in the quantum theory spoiling conformal invariance even if it was present in the classical action.

(Eg in QCD with massless quarks)

In $\mathcal{N}=4$ conformal invariance is exact!

Let's look at the generators of the conformal group. Let $A \cong 1+\epsilon$ $(\epsilon \ll 1)$

$$\delta\phi(x) = \phi'(x) - \phi(x) = (1+\epsilon)\phi(x+\epsilon x) - \phi(x) \simeq$$
$$\simeq \epsilon(1 + x^\mu\partial_\mu)\phi(x).$$

$\Rightarrow$ Generator : $D = i(1 + x^\mu\partial_\mu)$.

(will be generalized to $D = i(\Delta + x^\mu\partial_\mu)$)

Similarly: (w/o spin).

$$\left.\begin{array}{l} \left.\begin{array}{l} P_\mu = i\partial_\mu \\ M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \end{array}\right\} \text{Poincaré} \\ D = i(\Delta + x^\mu\partial_\mu) \\ K_\mu = i(x^2\partial_\mu - 2x_\mu x^\rho\partial_\rho - 2\Delta x_\mu) \end{array}\right\} \begin{array}{l}\text{Conformal group.} \\ \text{Take } [\,,\,] \text{ and show} \\ \simeq SO(4,2) \\ \simeq SU(2,2).\end{array}$$

We are almost done...

The last thing to notice is that K_μ and Q^a_α do not commute (See exercise below...). Thus we need to introduce 4 extra fermionic generator $\bar{S}^a_{\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}[K_\mu, Q^{a\alpha}]$: to close the (super) algebra :

$$PSU(2,2|\underset{\overset{\uparrow}{\mathcal{N}}}{4}) : \left(\begin{array}{c|c} \begin{array}{c} SU(2,2) \simeq \\ SO(4,2) \\ P_\mu, M_{\mu\nu}, K_\mu, D \end{array} & Q, \bar{S} \\ \hline \bar{Q}, S & \begin{array}{c} SU(4) \\ \text{R-SYMMETRY} \end{array} \end{array}\right)$$

THIS IS THE GLOBAL SYMMETRY OF $\mathcal{N}=4$ SYM

EXERCISE Show that $[K_\mu, Q^{a\alpha}] \neq 0$.

Hint: assume it is zero, take a further (anti)-commutat use Jacobi's identity...

Some loose ends...

1) The classical potential $V \propto \mathrm{tr}[X^i, X^j]^2$ vanishes for X^i = commuting set. However, if some $\langle X^i \rangle \neq 0$ then the conformal invariance is broken by the vev. We will only consider the unbroken (superconformal) phase: $\langle X^i \rangle = 0$

2) Setting $\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2_{YM}}$ one can argue that the theory is invariant under the (non-perturbative) discrete transformation: $\tau \to \frac{a\tau + b}{c\tau + d}$

$a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$. $(SL(2, \mathbb{Z}))$

Know as S-duality.

This is an amazing fact but we will not use it.

3) It is sometimes usefull to write the $\mathcal{N}=4$ theory in a language where at least one supersymmetry is manifest. This can be obtained at the expenses of ruining manifest $SU(4)_R$ and making ONLY $SU(3)$ manifest. (NB the symmetries are always the same, only some are "manifest" others require work!)

3 chiral Superfields:
$$\begin{cases} \Phi^1 = X^1 + i X^2 + \text{fermi} + \dots \\ \Phi^2 = X^3 + i X^4 + \quad // \\ \Phi^3 = X^5 + i X^6 + \quad // \end{cases}$$

+

1 Vector superfield W_α

SUPERPOTENTIAL: $W = \mathrm{tr}(\Phi^1 [\Phi^2, \Phi^3])$

EXERCISE: Show that the D-term and the F-terms reconstruct the full bosonic superpotential

We

$\mathcal{N}=4$ SYM after presenting the AdS/CFT duality. Now it's time to look at the other "world":

IIB STRING THEORY on $AdS_5 \times S_5$

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- In FLAT MINKOWSKI 9+1 dim space, type II (A or B) string theory, after fixing the world sheet metric, has an action:

$$\underset{\substack{\uparrow \\ \text{WORLD-SHEET}}}{S_{WS}} = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \left(\partial_\alpha X_M \partial^\alpha X^M - i\overline{\psi}^M \gamma^\alpha \partial_\alpha \psi_M\right)$$

- Where the symbols now mean something completely different from before!

- $$\left.\begin{array}{l} X^M : (\sigma,\tau) \longrightarrow \text{Minkowski } 9+1. \\ \psi^M : (\sigma,\tau) \longrightarrow \text{ten 2 components Majorana fermions} \end{array}\right\} \begin{array}{l}\text{WORLD sheet} \\ \text{fields.}\end{array}$$

$\alpha = 0,1$ is now the LORENTZ index on the world sheet, raise/lower with $\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ 2 dim Dirac matrices

$(\sigma^0 = \tau, \sigma^1 = \sigma)$.

Eq of motion: $\begin{cases} \partial_\alpha \partial X = 0 \\ \gamma^\alpha \partial_\alpha \psi^M = 0 \end{cases}$

Solution: $X^M = X^M_+(\tau+\sigma) + X^M_-(\tau-\sigma)$

$$\psi^M = \begin{pmatrix} \psi^M_-(\tau-\sigma) \\ \psi^M_+(\tau+\sigma) \end{pmatrix}$$

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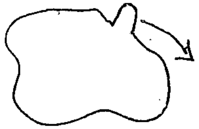
- If we only look at closed strings the left/right movers $(\tau \pm \sigma)$ are completely decoupled



For the string to be closed we need

- $X^M_\pm$ to be periodic (period $= \pi$, conventio

- But $\psi^M_\pm$ can be periodic or Antiperiodi
(RAMOND) (NEVEU-SCHWARZ

$$X_-^M = \frac{1}{2}x^M + \alpha' p^M(\tau-\sigma) + i\sqrt{\frac{\alpha}{2}}\sum_{n\neq 0}\alpha_n e$$

$$X_+^M = \frac{1}{2}x^M + \alpha' p^M(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\tilde{\alpha}_n^M e^{-2in(\tau+\sigma)}.$$

$$\psi_-^M = \sum_{n\in\mathbb{Z}} d_n^M e^{-2in(\tau-\sigma)} \quad \underline{\text{OR}} \quad \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^M e^{-2ir(\tau-\sigma)}$$

(R) (NS)

$$\psi_+^M = \sum_{n\in\mathbb{Z}} \tilde{d}_n^M e^{-2in(\tau+\sigma)} \quad \underline{\text{OR}} \quad \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{b}_r^M e^{-2ir(\tau+\sigma)}$$

We have four sectors:

$$\underbrace{\text{NS-NS}\ ,\ \text{R-R}}_{\text{space time bosons.}}\ ,\ \underbrace{\text{R-NS}\ ,\ \text{NS-R}}_{\text{space time fermions}}$$

Upon quantization, $\alpha, \tilde{\alpha}, d, \tilde{d}, b, \tilde{b}$ become creation/annihilation operators generating the quanta of space-time fields

Let us work only w

(X_-^M and ψ_-^M).

NS: $|0\rangle_{NS}$, $b_{-\frac{1}{2}}^M|0\rangle_{NS}$, massive.

$m^2 = -\frac{1}{2\alpha'}$ $m^2 = 0$

tachyon removed by GSO proj. Corresponds to massless vector field with $10-2 = 8_v$ d.o.f.

R: $|0\rangle_R$, massive

$m^2 = 0$

The g.s. is massless already. Since here we have $\{d_0^M, d_0^N\} = 2\eta^{MN}$ acting as Dirac-matrices, the g.s. carries a representation, i.e. it transforms like a spinor: $\overset{\text{positive chirality}}{8_s} \oplus \overset{\text{negative chirality}}{8_c}$

The GSO projection removes ONE of the two chiralities.

IIA : remove opposite chiralities in LEFT/RIGHT mov.

IIB " SAME " " "

Since we are interested in IIB I leave you the

EXERCISE Repeat the analysis below for IIA string theory –

For type IIB:

BOSONS

NS-NS : $8_v \otimes 8_v = 1 \oplus 28 \oplus 35$

$b^i_{-\frac{1}{2}} \tilde{b}^j_{-\frac{1}{2}} |0\rangle \otimes |\tilde{0}\rangle$ = trace $\oplus$ antisym $\oplus$ symm.

ϕ DILATON $B_{ij} = -B_{ji}$ B-field $G_{ij} = G_{ji}$ METRIC

RR : $8_c \otimes 8_c = 1' \oplus 28' \oplus 35'$

$|0\rangle_\alpha \otimes |0\rangle_\beta$ = scalar $\oplus$ 2 form $\oplus$ 4^+ form

C C_{ij} C^+_{ijkl}

In particular, C^+_{ijkl} are the physical components of a 4-form $C_{\mu\nu\rho\lambda}$ which has a self dual field strength:

$$dC^+_4 = * dC^+_4 , \quad \text{i.e.} \quad \partial C^+_{[\mu_1 \mu_2 \cdots \mu_5]} = \frac{1}{5!} \epsilon_{\mu_1 \cdots \mu_{10}} \partial C^{+[\mu_6 \mu_7 \cdots \mu_{10}]}$$

(not quite if other fields are non zero, see later).

The fermionic d.o.f. come from two gravitinos ψ^i_M of positive chirality and two dilatinos λ^i of negative chirality $(i=1,2)$. The theory is chiral but all anomalies cancel.

String theory contains also an ∞ # of massive fields, but in the low energy limit we can restrict to the massless fields with only leading interaction. This is IIB SUPERGRAVITY. (Schwarz, Nucl. Phys B

Bosonic part: (NB: $\kappa \sim \alpha'^2 g_s$, $g_s = e^{\phi_\infty}$,

$$S = \frac{1}{2\kappa^2} \int \Big\{ e^{-2\phi} \big(*R + 4 d\phi \wedge * d\phi - \tfrac{1}{2} H_3 \wedge * H_3 \big) - \frac{1}{2} F_1 \wedge * F_1 - \frac{1}{2} \tilde{F}_3 \wedge * \tilde{F}_3 - \frac{1}{4} \tilde{F}_5 \wedge * \tilde{F}_5 - \frac{1}{2} C^+_4 \wedge H_3 \wedge F_3 \Big\}$$

where: $F_1 = dC_0$, $H_3 = dB$, $F_3 = dC_2$

$F_5 = dC^+_4$, $\tilde{F}_3 = F_3 - C_0 \wedge H_3$

$\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B \wedge F_3$

and, strictly speaking it is $\tilde{F}_5$ which is self dual : $\tilde{F}_5 = * \tilde{F}_5$.

Note that this makes the kinetic term $\tilde{F}_5 \wedge * \tilde{F}_5$ vanish! Strictly speaking we cannot write an action but we will do it anyway and impose $\tilde{F}_5 = *\tilde{F}_5$ at the level of the eq. of motion.

The action above is in the so called "String frame". One can go to the "Einstein frame" by setting $\hat{G}_{MN} = e^{-\phi/2} G_{MN}$

$$S = \frac{1}{2k^2}\int \sqrt{\hat{G}}\cdot\hat{R} + \dots$$

For all our purposes this will not make a difference since the solutions we are interested in have $\phi = 0$ (ϕ is only the deviation from the asymptotic value ϕ_∞ that we have already used to define g_s and absorbed into k).

Certainly, flat space-time: $G_{MN} = \eta_{MN}$ with all other fields vanishing is a solution that preserves all SUSY (32 supercharges $= \mathcal{N}=2$ in $d=9+1$).

To see if there are others, study $\delta_\eta \lambda^i = \delta_\eta \Psi^i_M = 0$.

We will only be interested in solutions only with $\tilde{F}_5 \neq 0$ ($\Rightarrow F_5 \equiv \tilde{F}_5$).

In this case $\delta\lambda^i = 0$ is trivially satisfied since it does not depend on F_5 whereas $\delta\Psi^i_M = 0$ reduces to: ($\eta = \mathbb{C}$ Weyl spinor in 9+1 DIM)

$$D_M\eta + \frac{i}{480}\Gamma^{NPQRS}\Gamma_M F_{NPQRS}\,\eta = 0.$$

(With, remember: $D_M\eta = \left(\partial_M + \frac{1}{4}\omega_M^{\ AB}\Gamma_{AB}\right)\eta$

$$\omega_M^{AB} = e^{NA}\partial_{[M}e_{N]}^{\ B} - e^{NB}\partial_{[M,N]}e^{A} - e^{RA}e^{SB}e^{C}_{\ M}\partial_{[R}e_{S]C}$$).

spin connection   zehnbein

Obviously flat 9+1 Minkowski and $F_5 \equiv 0$ gives $\partial_M\eta = 0 \Rightarrow 32_{\mathbb{R}}$ supersymmetries.

But there are (two) other solutions.

The one of interest here is

$$ds^2 = ds^2(AdS_5(L)) + ds^2(S^5(L)).$$

$$F_5 = \frac{1}{L}Vol(AdS_5(L)) + \frac{1}{L}Vol(S^5(L)).$$

where $ds^2(AdS_5(L))$ is the metric of

AdS_5 w/ "radius L", $Vol(Ads_5(L))$ its volume 5-form and similarly for S^5.

More explicitly, we can write in "Poincaré coordinates":

$$ds^2(AdS_5(L)) = \frac{r^2}{L^2}\big(\underbrace{-(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2}_{\equiv \eta_{\mu\nu}dx^\mu dx^\nu}\big) + \frac{L^2}{r^2}dr^2$$

$$ds^2(S^5(L)) = L^2 \underbrace{d\Omega_5}_{\text{unit radius}} = \hat{g}_{\alpha\beta}d\theta^\alpha d\theta^\beta .$$

$$\underbrace{F_{0123r}}_{\text{and. antisymetric}} = \frac{r^3}{L^4} \quad , \quad \underbrace{F_{\alpha_1\ldots\alpha_5}}_{S^5(L)} = L^4 \underbrace{\sqrt{\hat{g}}\,\epsilon_{\alpha_1\cdots\alpha_5}}_{\substack{\text{volume}\\\text{form of}\\\text{unit } S^5.}}$$

Notice that $F_5 = *_{10}F_5$

as it should $\left(\sqrt{\det|G_{10}|} = L^2 r^3 . \sqrt{\hat{g}}\right)$.

Now, with this explicit form, it is reasonably easy to investigate the # of sol. of the eq. $\tilde{D}_M \eta = 0$.

Here we cannot just hope to reduce it to $\partial_\mu \eta = 0$ (which is a coord. dependent statement anyways).

What we should do instead is to compute the integrability condition:

$$[\tilde{D}_M, \tilde{D}_N]\eta \overset{\text{def}}{\equiv} \Xi_{MN}\,\eta$$

where $\Xi_{MN} \equiv \Xi_{NM}$ is a matrix acting on η

(Recall that $[D_M, D_N]\eta \equiv \frac{1}{2}R_{MNPQ}\Gamma^{PQ}\eta$

but here $\tilde{D}_M = D_M + O(F)$ and you pick up extra terms).

The point is that the # of independent sol's to $\tilde{D}_M\eta = 0$ is the same as the # of zero eigenvalues of $\Xi_{MN}\eta = 0 \quad \forall M,N$

In this case the claim is that the solution is MAXIMALLY SUPERSYMM, that is $\Xi_{MN} \equiv 0$ and all $32_{\mathbb{R}}$ η's are allo

EXERCISE Show that $[\tilde{D}_M, \tilde{D}_N] = 0$, for the $AdS_5 \times S^5$ solution.

NB: there is a third solution preserving all SUSY, known as p-p-wave (p-p ≡ plane-parallel). It plays an important role in testing the conject but we will not get there.

At this point you should complain: we quantized the string in flat space and now we are using the $AdS_5 \times S^5$ solution. The answer of course is that strictly speaking we are not allowed to do it and we should quantize in $AdS_5 \times S^5$. However, this is still an OPEN PROBLEM. Nevertheless, when the radius of curvature R is large (in units of $\sqrt{\alpha'}$) (that is the curvature itself is small) the SUGRA action is a good approx and this is the region where most tests have been done.

The parameters of the compactification are thus $g_s = e^{\Phi_\infty}$ and $N = \int_{S^5} F_5$ flux.
(the flux is quantized in the full string theory)

Since AdS is perhaps unfamiliar to some of you, here is a little more details in a simpler setting:

AdS space (in any dim.) is an example of MAXIMALLY SYMMETRIC SPACE.

A max. symm. space in D dimensions (of arbitrary signature, Euclidian, Minkowski... is a space with $\frac{1}{2}D(D+1)$ killing vectors.

Our "working definition" will be that its metric obeys:

$$R_{MNPQ} = K\left(g_{MP}\,g_{NQ} - g_{MQ}\,g_{NP}\right)$$

(Recal: $\Gamma^P_{MN} = \frac{1}{2} g^{PQ}\left(\partial_M g_{QN} + \partial_N g_{QM} - \partial_Q g_{MN}\right)$

$$R_{MNP}{}^{Q} = \partial_M \Gamma^Q_{NP} - \partial_N \Gamma^Q_{MP} + \Gamma^Q_{MR}\Gamma^R_{NP} - \Gamma^Q_{NR}\Gamma^R_{MP}.$$

K is a CONSTANT known as "curvature const

Any two spaces with same K and same signature are isomorphic.

Be careful with the many different sign conventions in the literature, INCLUDING MINE!

Euclidean signature $\begin{cases} K>0 & S^D & \text{sphere} \\ K=0 & \mathbb{R}^D & \text{Euclidian sp.} \\ K<0 & H^D & \text{Hyperbolic sp.} \end{cases}$

Minkowski signature $\begin{cases} K>0 & dS_D & \text{de Sitter} \\ K=0 & \mathbb{R}^{D,1} & \text{Minkowski} \\ K<0 & AdS_D & \text{ANTI de Sitter} \end{cases}$

⋮

A max. symm. space obeys Einstein's eqs with a cosmological constant (usually done in Minkowski signature but it makes sense in general).

$$S = \frac{1}{2k^2}\int d^Dx\sqrt{g}\,(R-\Lambda)$$

$$\delta S = 0 \Rightarrow R_{MN} - \frac{1}{2}g_{MN}R + \frac{\Lambda}{2}g_{MN} = 0$$

$$R_{MNPQ} = K(g_{MP}g_{NQ} - g_{MQ}g_{NP}) \Rightarrow R_{MP} \equiv g^{NQ}R_{MNPQ} = K(D-1)g_{MP}$$

$$\Rightarrow R \equiv g^{MP}R_{MP} = KD(D-1) \quad \text{constant.}$$

$$\Rightarrow K = \frac{\Lambda}{(D-2)(D-1)} \qquad (D>2 \text{ only}).$$

EXERCISE What went wrong for $D=1,2$ in the last steps?

We see that $\Lambda > 0 \Rightarrow$ de Sitter
$\Lambda < 0 \Rightarrow$ Anti de Sitter

(But please don't confuse this with real life cosmology. There one uses the FRW metric (NOT maximally symm. in general), $K=0,\pm 1$ denotes the curvat of space (not space-time) and $T_{\mu\nu} \neq$

I can embed all of these spaces into a flat space one dim higher as hyperboloids: $\mathbb{R}^{p,q}$ $\eta = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & +1 & \\ & & & & +1 \ddots \end{pmatrix}$ $\begin{matrix} \}p \\ \}q \end{matrix}$

$$\eta_{AB}Y^AY^B = L^2$$

(That is: $-(\vec{Y}_p)^2 + (\vec{Y}_q)^2 = L^2$)

is a max symm. space of signature $(p, q-1)$ (since) can always solve $(\vec{Y}_q)^2 = L^2 + (\vec{Y}_p)^2$.) and manifest isometry $SO(p,q)$.

$+(Y_1)^2 + (Y_2)^2 + \cdots \quad +(Y_{D+1})^2 = L^2$ $\quad S^D$

$-(Y_1)^2 + (Y_2)^2 + \cdots \quad +(Y_{D+1})^2 = L^2$ $\quad dS_D$

$\vdots$ $\Big\}$ weird stuff

$-(Y_1)^2 - \cdots -(Y_{D-1})^2 + (Y_D)^2 + (Y_{D+1})^2 = L^2$ $\quad AdS_D$

$-(Y_1)^2 - \cdots \quad -(Y_D)^2 + (Y_{D+1})^2 = L^2$ $\quad H^D$

$-(Y_1)^2 - \cdots \quad -(Y_{D+1})^2 = L^2$ $\quad \emptyset$

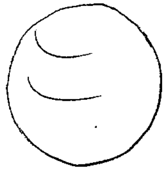
EXERCISE Find the relation between K and L. (You should at least show $K \propto \frac{1}{L^2}$).

ISOMETRY

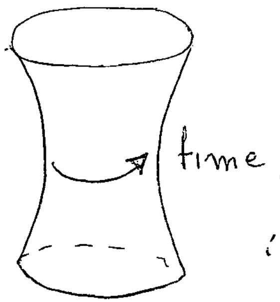
Thus: S^5: $+Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 + Y_6^2 = L^2 \quad SO(6)$

AdS_5: $-Y_1^2 - Y_2^2 - Y_3^2 - Y_4^2 + Y_5^2 + Y_6^2 = L^2 \quad SO(4,2)$

S^5



AdS_5



To avoid clo
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EXERCISE Check that the metric

$$ds^2 = \frac{r^2}{L^2}(dx^\mu)^2 + \frac{L^2}{r^2}dr^2$$

is a metric of (a patch of) AdS

That is: $R_{MNPQ} \propto g_{MP}g_{NQ} - g_{MQ}g_{NP}$

It is usually more practical to change variable $r \longrightarrow z = L^2/r$ for which:

$$ds^2 = L^2 \, \frac{(dx^\mu)^2 + dz^2}{z^2}$$

Which makes it clear that the "boundary" ($z=0$) has Minkowski signature.

One can add the point "$u=\infty$" if needed.

One can also Wick rotate to H_5 whose "boundary" is $\mathbb{R}^4$ (or S^4 if compactified

The previous metric does not cover – the whole hyperboloid:

$$Y_1 = L\frac{X_1}{Z},\ Y_2 = L\frac{X_2}{Z},\ Y_3 = L\frac{X_3}{Z},\ Y_4 = \frac{Z}{2}\left(1-\frac{L^2-X^{\mu 2}}{Z^2}\right)$$

$$Y_5 = L\frac{X^0}{Z},\ Y_6 = \frac{Z}{2}\left(1+\frac{L^2+X^{\mu 2}}{Z^2}\right).$$

It is also possible to find a global metric covering the whole hyperboloid

$$Y_1 = L\sinh\rho\,\hat{n}_1 \quad \dots \quad Y_4 = L\sin\rho\,\hat{n}_4$$

$$Y_5 = L\cosh\rho\cos\tau\ ,\quad Y_6 = L\cosh\rho\sin\tau\,.$$

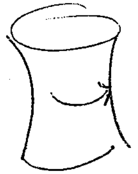
with $\hat{n}^2 = 1$ parameterizing a 3sphere.

$$\Rightarrow ds^2 = L^2\left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3\right)$$

Notice that to have a mostly plus metric one must define ds^2 as the pull-back of $-\eta_{AB}dY^A dY^B$ for Ads (and + for dS)

<u>EXERCISE</u> Work out the above metrics in detail from the pull-back of the embedding.

We will always let time ∈ ℝ thus effectivelly going to the coverin space of

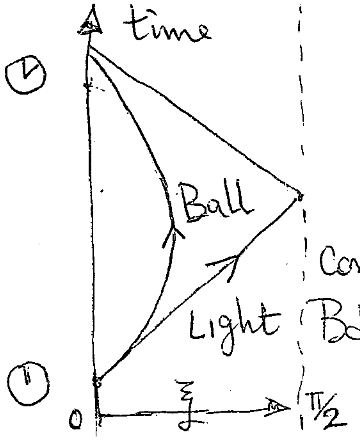


. The covering has trivial topology and its "conformal boundary" is $\mathbb{R}^{3,1}$. (more on this later.

Living in Ads space would be very stra Every thing would behave like a boomera and, if thrown, would came back at you with the same time delay, including light itself which would "reach the boundary" and come back. The boundary of AdS is a very concrete thing!

$$d\xi = \frac{d\rho}{\cosh 2\rho}$$

$$\Rightarrow \xi = \arctan(\tanh($$



At this point we have all the basic ingredients and we could just go ahead and write the conjecture (which, in case you have not guessed relates $\mathcal{N}=4$ SYM to IIB strings on $AdS_5 \times S^5 \ldots$).

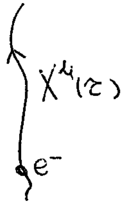
● However, it is more pedagogical ● to follow the road taken by Maldacena to arrive at the proposal. This was done by studying the

$$\boxed{\text{D3 - BRANE}}.$$

● The existence of RR potentials ● (C_0, C_2, C_4^+ in IIB and C_1, C_3 in IIA) has been known since the beginning of superstring theory. But untie 1995 (Polchinski) it was not clear how to describe the states charged under such fields. (the fundamental string is neutral...).

Just like an electron (point-like object) couples to a 1 form (the e.m. potential) via

$$e \int_{\mathbb{R}} d\tau \, A_\mu(x(\tau)) \, \overset{\circ}{x}^\mu(\tau)$$



● So a n-form potential $C_{\mu_1 \cdots \mu_n}$ couples to a $p = n-1$ dimensional extended ● object spanning a $p+1 \equiv n$ dim world-volume:

$$T_p \int_{\Sigma_p \times \mathbb{R}} d\tau \, d^p\sigma \; C_{\mu_1 \cdots \mu_{p+1}}(X(\tau,\sigma)) \underbrace{\partial_\tau X^{\mu_1} \cdots \partial_{\sigma^p} X^{\mu_{p+1}}}_{\text{antisymmetrize}}.$$

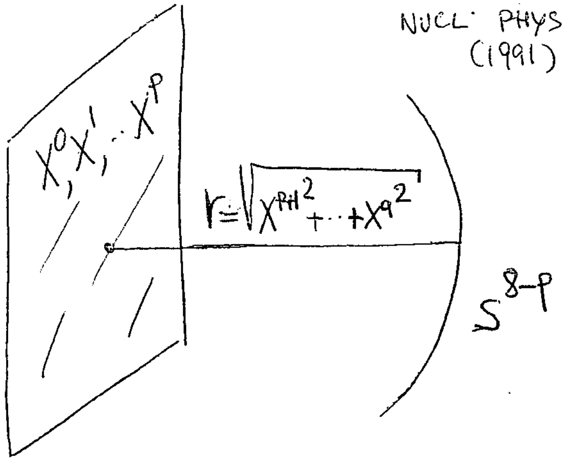
● The objects coupling to the RR fields ● are known as D(p)-branes and they have two types of description:

•) As solution to the SUGRA eqs.

••) As hypersurfaces on which open strings are allowed to end.

NB. p = even/odd in IIA/IIB.

•) As SUGRA solutions : (Strominger NUCL PHYS B 360 (1991) 197)



$$\begin{cases} ds^2 = H_p(r)^{-1/2}\left(-dx^{0\,2} + dx^{1\,2} + \dots dx^{p\,2}\right) + H_p(r)^{\frac{1}{2}}\left(dr^2 + r^2 d\Omega^2_{8-p}\right) \\ F_{p+2} = dx^0 \wedge dx^1 \wedge \dots dx^p \wedge dH_p(r) \qquad \left(\text{plus dual for } F_5 \dots\right). \\ e^{\phi} = H_p(r)^{\frac{3-p}{4}} \end{cases}$$

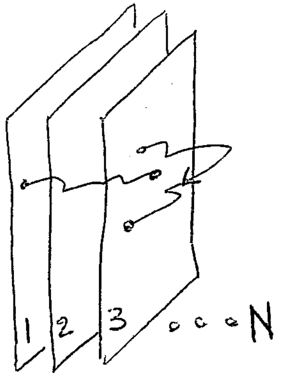
where $$H_p(r) = 1 + \underset{\uparrow}{\#}\,\frac{g_s N \alpha'^{\frac{7-p}{2}}}{r^{7-p}}$$

some numerical factor

(in writing the numerator of H_p) used some stringy information like $k \sim g_s \alpha'^2$)

<u>EXERCISE</u> Show that the above solution preserves HALF of the SUSY (16 supercharges).

••) From the stringy point of view : (Polchinski and others..)



$\underbrace{\hphantom{xxxx}}$ act as Chan-Paton factors -

$X^0, X^1 \dots X^p$ have usual Neumann-Neuman B.C.

$X^{p+1}, \dots X^9$ have Dirichlet-Dirichlet B.C.

* Now the left/right movers are couple into standing waves.
* The type of modding (integer for all X^μ and integer/half-odd for ψ^μ in the Ramond/Neveu-Schwarz sector respectively remains the same.
* However: $$X^{\mu = 0,1,\dots p} = \frac{1}{2}x^\mu + \alpha' p^\mu \tau + \text{oscill}.$$

$$X^{i=p+1,\dots 9} = \frac{1}{2}x^i - \underset{\uparrow}{d^i}\sigma + \text{oscill}$$

No p-dependence. $d^i = 0$ if all D branes are on top of each other.

* Finally GSO works as before, removing the tachyon and so on.

Hence, the WORLD-VOLUME theory is that of maximally SUSY SYM in p+1 dimensions:

$$b^{\mu}_{-\frac{1}{2}}|p^0..p^p;IJ\rangle_{NS} \rightsquigarrow A^{\mu}{}^{I}_{J}$$

$$b^{i}_{-\frac{1}{2}}|p^0..p^p;IJ\rangle_{NS} \rightsquigarrow X^{i}{}^{I}_{J}.$$

$$|p^0;..p^p;\alpha;IJ\rangle_{R} \rightsquigarrow \lambda_{\alpha}{}^{I}_{J}$$

$$g^2_{YM} \propto g_s \cdot \alpha'^{\frac{p-3}{4}}.$$

At least we see the appearence of non abelian symmetries in the type IIA/IIB context (a non trivial fact in itself!).

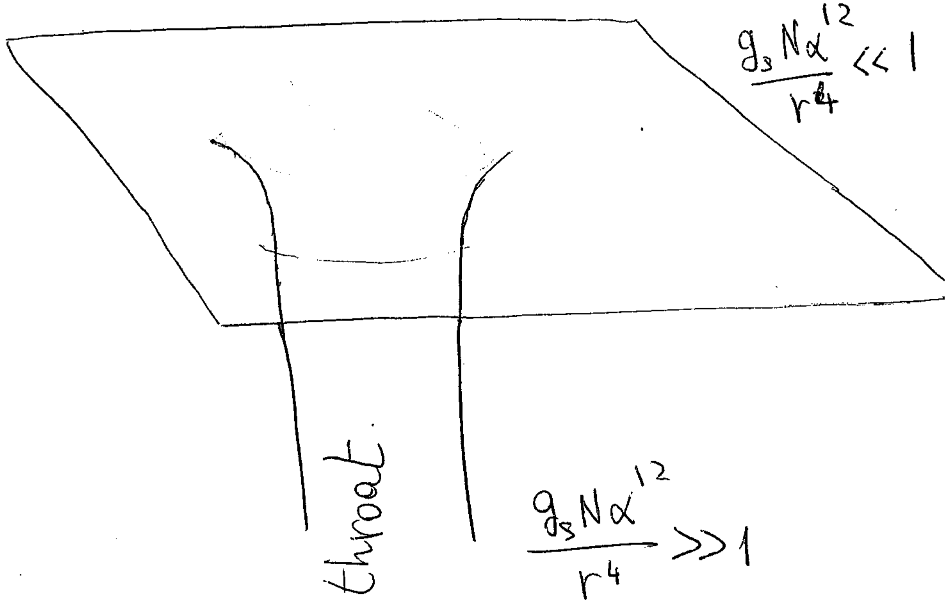
The IIB D3-brane is particularly interesting: $\boxed{p\equiv 3}$

∘∘) From the stringy point of view we have directly 3+1 dim $\mathcal{N}=4$ SYM with gauge group SU(N) and $g^2_{YM}=g_s$.

∘) From the SUGRA point of view we have:

$$e^{\phi}\equiv 1 \Rightarrow g_s=e^{\phi_\infty} \text{ constant everywhere}$$

$$ds^2=\left(1+4\pi\frac{g_sN\alpha'^2}{r^4}\right)^{-1/2}(dx^\mu)^2+\left(1+4\pi\frac{g_sN\alpha'^2}{r^4}\right)^{\frac{1}{2}}(dr^2+r^2d\Omega$$



In the region $\frac{g_sN\alpha'^2}{r^4} << 1$ the metric reduces to Minkowski 9+1 dim space:

$$ds^2 = \underbrace{(dx^\mu)^2}_{\mathbb{R}^{3,1}} + \underbrace{dr^2 + r^2 d\Omega_5}_{\mathbb{R}^6 \text{ written in polar coord's}} .$$

Much more interesting is the region $\frac{g_sN\alpha'^2}{r^4} >> 1$

$$ds^2 = \frac{r^2}{\sqrt{4\pi g_s N\alpha'^2}}(dx^\mu)^2 + \frac{\sqrt{4\pi g_s N\alpha'^2}}{r^2}\left(dr^2 + r^2 d\Omega_5\right)$$

$$= \underbrace{\frac{r^2}{\sqrt{4\pi g_s N\alpha'^2}}(dx^\mu)^2 + \frac{\sqrt{4\pi g_s N\alpha'^2}}{r^2}dr^2}_{AdS_5} + \underbrace{\sqrt{4\pi g_s N\alpha'^2}\, d\Omega_5}_{S^5}$$

Both with "radius" $L^2 = \sqrt{4\pi g_s N\alpha'^2}$

So now even $AdS_5 \times S^5$ has made its appearence!

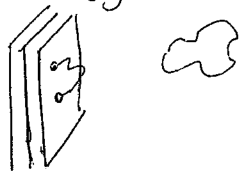
NB: Better rescale $r \to L^2 r$ so that:

$$ds^2 = L^2 d\hat{s}^2 = L^2\left(r^2(dx^\mu)^2 + \frac{dr^2}{r^2} + d\Omega_5^2\right) =$$

$$\text{"}\left(z = \frac{1}{r}\right)\text{"} \quad = L^2\left(\frac{(dx^\mu)^2 + dz^2}{z^2} + d\Omega_5^2\right).$$

There is a QUALITATIVE argument for why we should expect the tw to be the same:

In the "stringy" picture



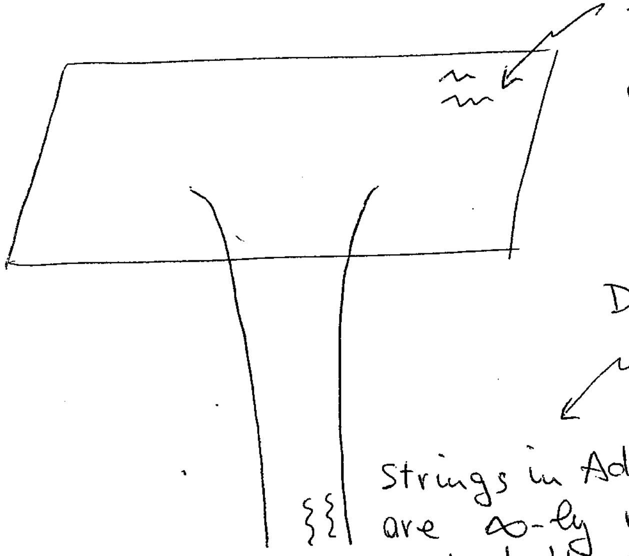
$$S_{TOT} = S_{BRANE} + S_{INT.} + S_{BULK}.$$

$$\underset{}{S_{BRANE}} = \underset{\left(\sim \int d^4x\, F^2 + ..\right)}{S_{\mathcal{N}=4}} + \underset{\left(\sim \alpha'^2 \int d^4x\, F^4 + \ldots\right)}{S_{HIGER\ ORDER}} \xrightarrow{\text{LOW ENERGY}} S_{\mathcal{N}=4}$$

$$S_{INT.} \sim \kappa \int d^4x\, h F^2 + \ldots \xrightarrow{\text{LOW ENERGY}} 0$$

$$S_{BULK} = \underset{\left(\sim \int (\partial h)^2 + \kappa h(\partial h)^2 + ..\right)}{S_{SUGRA}} + \underset{\left(\sim \kappa^6 \int (\partial h)^8 + ..\right)}{S_{HIGHER\ ORDER/MASSIVE}} \xrightarrow{\text{LOW ENERGY}} S_{FREE\ SUGRA}$$

On the other hand, in the "brany" picture at low energies:



FREE SUGRA Decoupled from the brane since $\sigma \sim \omega^3 R^8 \to 0$

DECOUPLED.

Strings in $AdS_5 \times S^5$ that are ∞-ly red shifted and don't have the energy to "come out".

(This is the usual "red shift": $g_{tt} = \frac{1}{\sqrt{1+L^4/r^4}} \to 0$ as $r \to 0$ just like $g_{tt} = (1 - \frac{r_0}{r}) \to 0$ as $r \to r_0$ in the usual Schwartshild solution)

Hence:

$$S_{\mathcal{N}=4} + \cancel{S_{\text{FREE SUGRA}}} \Longleftrightarrow S_{\substack{IIB \\ AdS_5\times S^5}} + \cancel{S_{\text{FREE SUGRA}}}$$

Don't take it too seriously, since we do not actually have full controll over the RHS (mostly).

THE CONJECTURE

(in its STRONGEST FORM)

that we will spell in details

There is a one-to-one map between the PHYSICAL QUANTITIES ("observables") of

•) $\mathcal{N}=4$ SYM with gauge group $SU(N)$ and coupling constant g_{YM}, (θ

AND

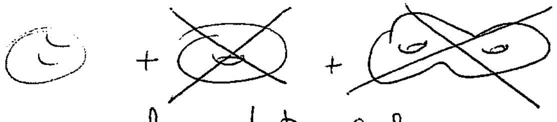
••) IIB string theory on $AdS_5 \times S^5$ with $\int_{S^5} F_5 = N$ and $g_s = g_{YM}^2$, $(C_0 \equiv \theta)$

Now you should object that I just told you that nobody knows how to formulate ••) so the whole thing seems empty.

Luckily there are various higly non trivial limits of the above conject. where many things can be tested (and works!). If nobody ever succeeds in formulating ••) (or it turns out that there are ambiguities in the formulation) we might turn the conject around and use •) to DEFINE ••)!

To see what this limit might be let us recall that, although we do not know how to quantize IIB strings in $AdS_5 \times S^5$ we do know the SUGRA limit. This will be a good approximation if:

1): $g_s << 1$ so that STRING LOOPS are supressed :



(this is sometimes referred to as "classical" but it is still higly non trivial, just remember the Veneziano amplitude in flat space! A better name would be "weakly coupled").

2) The curvature radius L is large in units of α' : $\frac{L}{\sqrt{\alpha'}} >> 1$

(this means the curvature is small. Similar to taking the zero slope limit in the Veneziano formula reducing it to a field theory amplitude).

From the SUGRA eqs. one derives:

$$L^4 = 4\pi g_s N \cdot \alpha'^2$$

So what we need in order to be able to do computations in SUGRA is:

$$\left.\begin{array}{l} g_s \equiv g_{YM}^2 \longrightarrow 0 \\ g_s N \equiv g_{YM}^2 N \equiv \lambda \longrightarrow \infty \end{array}\right\}$$

obviously this requires $N \longrightarrow \infty$

the amazing thing is that this is also a very interesting limit for the gauge theory. We need to pause and look at the gauge theory in this regime. In fact it is easier and useful to consider a MORE GENERAL regime of the gauge theory where we let

$g_{YM} \to 0$, $N \to \infty$ KEEPING $\lambda = g_{YM}^2 N$ FIN

(This is known as the 't Hooft limit in the gauge theory side)

Notice that if we succeed in defining the 't Hooft limit for ALL λ, the previous SUGRA LIMIT is obtained by letting $\lambda \to \infty$ AT THE END.

Before we do that, we are in an intermediate regime where:

- $g_s \equiv \frac{\lambda}{N} \to 0$ (string theory is weakly coupled

)

but

$$\frac{L^4}{\alpha'^2} = 4\pi\lambda \text{ finite}$$ ($AdS_5 \times S^5$ is strongly curved : ~~SUGRA~~).

- This would require solving the full sting th. on the ⊖ S^2 (Veneziano Viresoro-Shapiro)
- an "easier" problem but also still UNSOLVED. (One can do perturbation theory).

If we write the σ-model on S^2:

$$S_{WS} = \frac{1}{4\pi\alpha'}\int_{S^2}\sqrt{\gamma}\,\gamma^{\alpha\beta}G_{MN}(X)\,\partial_\alpha X^M\partial_\beta X^N + \cdots$$

$\gamma^{\alpha\beta}$ = auxiliary metric on the world sheet.

- G_{MN} = Metric in target space ($AdS_5 \times S_5$) and rescale $G_{MN} = L^2\hat{G}_{MN}$ as before
- where $\hat{G}_{NM}$ is the metric for UNIT RADIUS:

$$S_{WS} = \frac{L^2}{4\pi\alpha'}\int_{S^2}\sqrt{\gamma}\,\gamma^{\alpha\beta}\hat{G}_{MN}(X)\,\partial_\alpha X^M\partial_\beta X^N + \dots$$

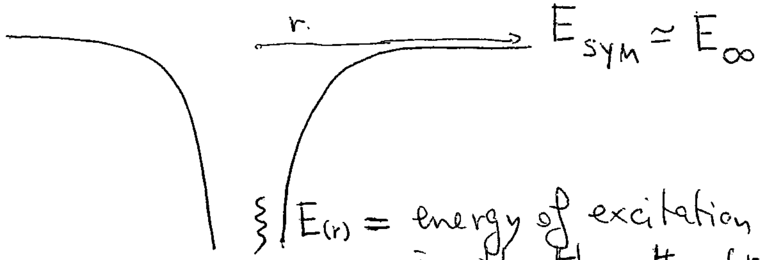
- Since $\frac{L^2}{\alpha'} \sim \sqrt{\lambda}$ we see that $\frac{1}{\sqrt{\lambda}}$ play the role of the σ-model expansion
- parameter, (like α' in flat space, th main difference being that it is dimension

This picture will be valid for $\lambda \gg 1$.

But now let us look at the gauge theor

This is the right place to comment on the exact relation between the AdS_5 radial coordinate r and the energy scale u of the SYM theory.

WE ARE IN THE CASE $N \to \infty$, $\lambda \sim$ FINITE.



We want to be able (in principle) to discuss stringy states in the throat. This means that we want $E_{(r)} \cdot \sqrt{\alpha'} \sim$ FINITE.

However, we have seen that this state observed from ∞ (SYM) would appear redshifted: $E_{SYM} \sim \frac{r}{L} E_{(r)} \sim \frac{r}{\lambda^{\frac{1}{4}}\sqrt{\alpha'}} E_{(r)} \sim \frac{r}{\sqrt{\alpha'}} E_{(r$

(we keep $\lambda \sim$ finite).

Thus: $E_{SYM} \sim \frac{r}{\sqrt{\alpha'}} E_{(r)} \sim \frac{r}{\alpha'} \underbrace{E_{(r)} \cdot \sqrt{\alpha'}}_{\text{hold finite}}$.

$\therefore$ $U \sim \frac{r}{\alpha'}$ is the ENERGY SCALE (Note that α' factors out from ds^2_{10})

The 't Hooft (Large N) Limit

NUCL PHYS B72 (1974) 461

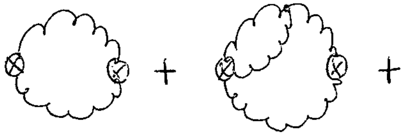
Consider ANY gauge theory w/ gauge group $SU(N)$ (and only matter fields in the Adj rep. for simplicity)

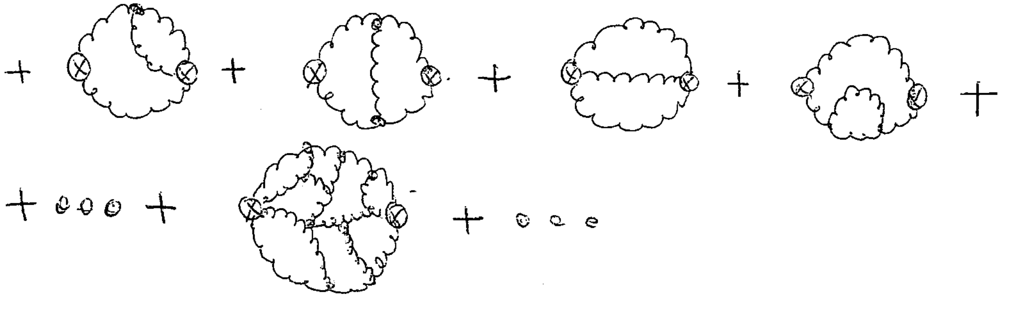
Suppose we want to compute the correlation function of some GAUGE INVARIANT OPERATOR:

e.g.: $\mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} = \mathrm{tr}(\partial A)^2 + g_{YM} A^2 \partial A + g^2_{YM} A^4$

We are familiar with the fact that perturbation theory is controlled by powers of g_{YM}.

Eg: $\langle \mathrm{tr} F^2(x)\, \mathrm{tr} F^2(y) \rangle =$



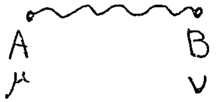


However we must be carefull because if N is large it might compensate for g_{YM} being small...

We must understand the behavior of $\langle \mathrm{tr}F^2(x)\,\mathrm{tr}F^2(y)\rangle$ in BOTH g_{YM} AND N.

- To do this is better to rewrite the
- propagators and the vertices:

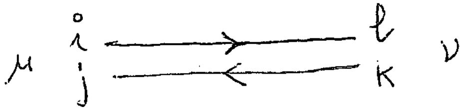
Usually $\langle 0|T(A^A_\mu(x)\,A^B_\nu(y))|0\rangle = D_{\mu\nu}(x-y)\cdot\delta^{AB}$



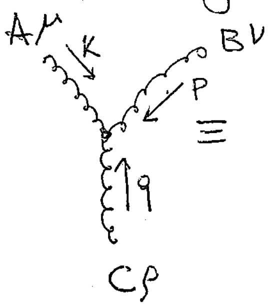
usual "photon" propagator in some gauge.

- We will think instead of A_μ as matrices:
- $A_{\mu}{}^{i}{}_{j} \equiv A^A_\mu\, T^{A\,i}{}_{j}$ $\quad A = 1\ldots N^2$, $\quad i,j = 1\ldots N$

then: $\langle 0|T(A_{\mu}{}^{i}{}_{j}(x)\,A_{\nu}{}^{k}{}_{\ell}(y))|0\rangle = D_{\mu\nu}(x-y)\,\delta^i_\ell\,\delta^k_j$

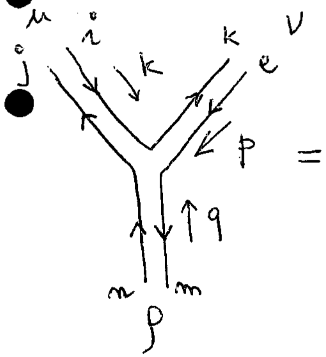


Similarly:



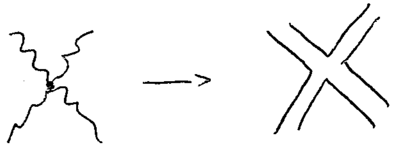
$\equiv g_{YM}\, f^{ABC}\cdot\left(\eta^{\mu\nu}(k-p)^\rho + \eta^{\nu\rho}(p-q)^\mu + \eta^{\rho\mu}(q-k)^\nu\right)$,

Becomes:



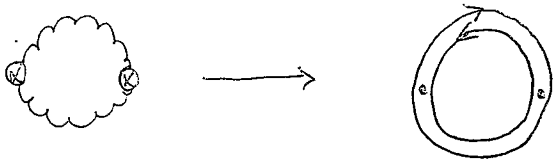
$= g_{YM}\,\delta^i_k\,\delta^\ell_m\,\delta^n_j\cdot($ same $)$

- and so on:



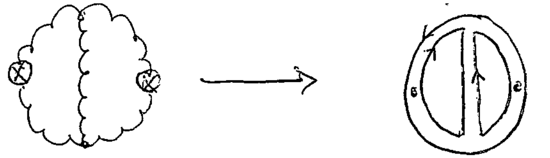
- The only thing that changes is the explicit form for the color factors. Similar treatment for the ghosts and matter (if present).

Let's play with N.



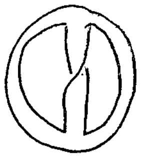
$\propto N^2$ (Since $\delta^j_i \delta^i_j = N$)

"1 loop"

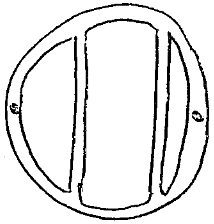


$\propto g_{YM}^2 \cdot N^3$.

"2 loops"

(Note that  is NOT allowed) (try putting the $\rightarrow$)

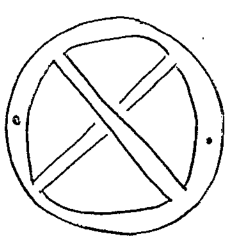
To higher loops we have various possibilities:



$\propto g_{YM}^4 \times N^4 \leftarrow$ 4 color loops

 $= \delta^j_i \delta^k_j \delta \cdot \delta^i_e = N$

"3-loops"



$\propto g_{YM}^4 \cdot N^2$ (only 2 color loops)

't Hooft observation was that each diagram scales like:

$$\lambda^{\ell-1} \cdot N^{\chi(\Sigma)}$$

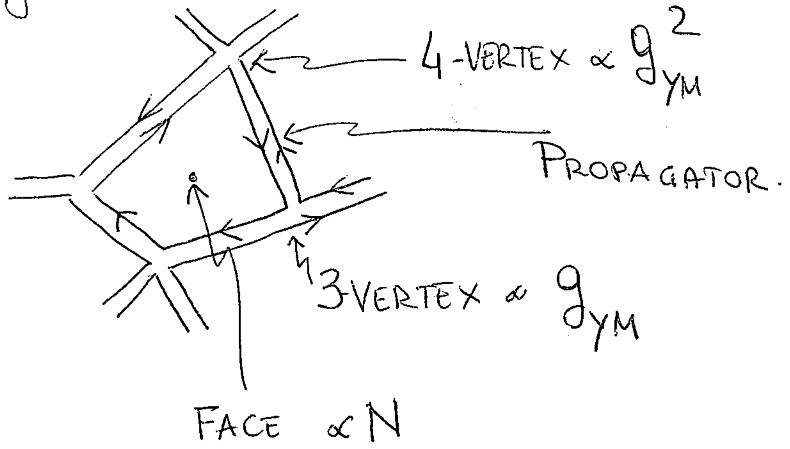
where $\lambda = g_{YM}^2 N$ as before, ℓ counts the number of momentum loops (powers of g_{YM}^2 as usual) and VERY IMPORTANTLY the extra N dependence is given by $\chi(\Sigma)$, the EULER CHARACTERISTIC of the 2 dim SURFACE Σ on which the diagram can be written without intersecting:

Thus, if I scale $N \to \infty$ keeping λ fixed ONLY those diagrams that can be written on a sphere S^2 ("PLANAR" diagrams) survive:

$\chi = 2$ $\qquad$ 0 $\qquad$ -2 $\qquad$ ...

(It would have been better to call them "spherical"...)

Proof: A generic diagram (written in double line notation a' la 't Hooft) will define an oriented, closed 2d surface.



If: F = # Faces
E = # Propagators = # Edges.
V_3 = # 3-vertices
V_4 = # 4-vertices

We can also define $V = V_3 + V_4$ = total # of vertices,

and notice that $E = \frac{1}{2}(4V_4 + 3V_3) \equiv 2V_4 + \frac{3}{2}V_3$

(each 4 vertex contributes HALF of 4 propagators:  and so on)

Then, a generic diagram will scale like: $g_{YM}^{V_3+2V_4} \times N^F \equiv$

$$\equiv \left(g_{YM}^2 N\right)^{\frac{1}{2}V_3+V_4} \cdot N^{F-\frac{1}{2}V_3-V_4}$$

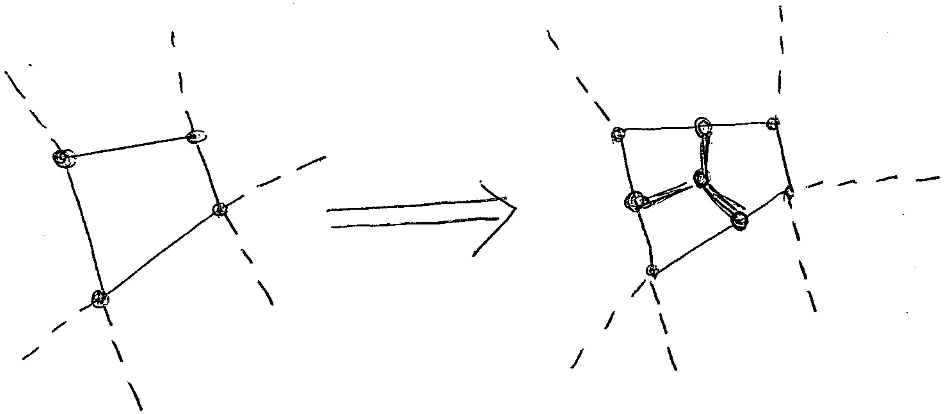
$$\equiv \left(g_{YM}^2 N\right)^{E-V} N^{F-E+V}$$

$$\equiv \left(g_{YM}^2 N\right)^{l-1} \cdot N^{\chi}$$

•) $l = E-V+1$ is the usual QFT "loop" countin it is the number of unconstrained 4-mome to be integrated over: each propagator contributes to one, each vertex has a δ-function and removes one and there is one overall δ-function that does not help because it constrains only the external momenta. For $g_{YM}^2 N \ll 1$) can still do perturbation theory.

••) $\chi = F-E+V$ is the EULER CHARACTERISTIC

It is easy to show that χ does not depend on the specifics of the tiling:



$$V \longrightarrow V+4$$

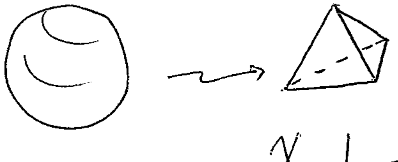
$$E \longrightarrow E-4+10 = E+6$$

$$F \longrightarrow F-1+3 = F+2$$

$$\chi = F-E+V \longrightarrow \chi \quad \checkmark .$$

So I can pick any tiling I want

e.g. S^2:



$$\chi = 4-6+4 = 2 .$$

(If it stresses you that $\langle \mathrm{tr}F^2(x)\, \mathrm{tr}F^2(y)\rangle_{S^2} \sim N^2$, just rescale $\mathrm{tr}F^2 \longrightarrow \frac{1}{N}\mathrm{tr}\,F^2$.)

EXERCISES

1) We did the scaling using usual pert. theory where $S = -\int d^4x \frac{1}{2} \mathrm{tr}\, F^2$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g_{YM} [A_\mu, A_\nu]$

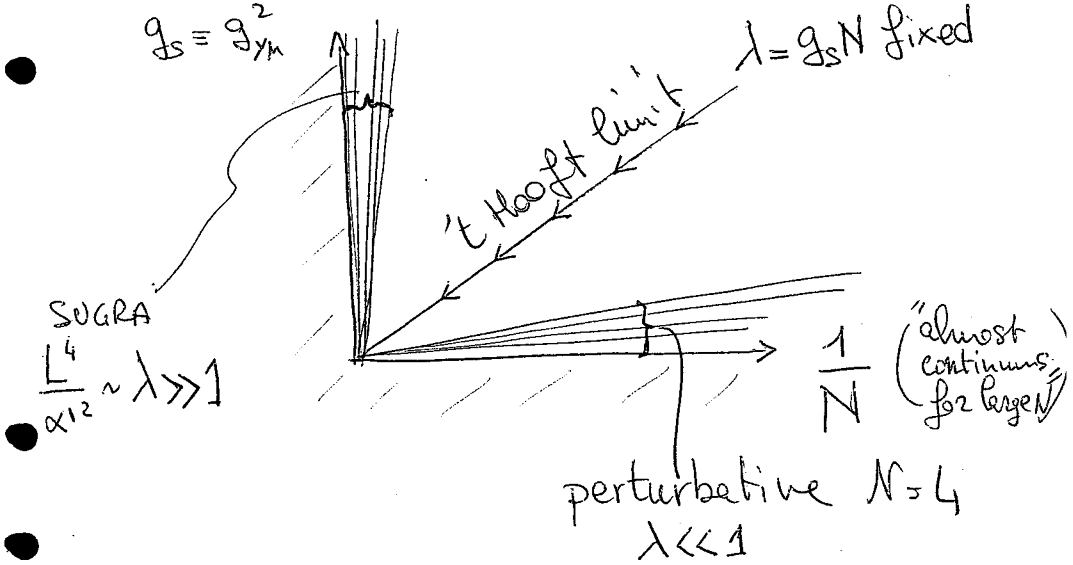
Show that you get the SAME results if you use $S = -\frac{1}{g_{YM}^2} \int d^4x\, \frac{1}{2} \mathrm{tr} F^2$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu]$ instead

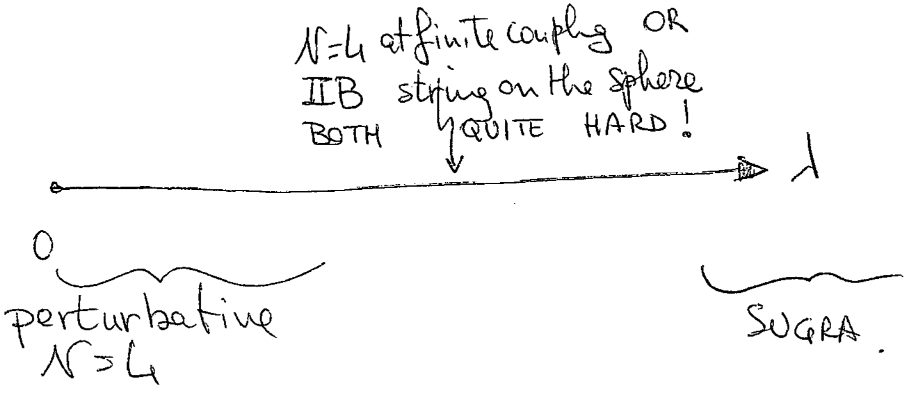
2) For $\mathcal{N}=4$ we have $\beta = 0$ but the 't Hooft limit applies to theories with $\beta \neq 0$ as well. For a theory with only adjoint fields, show that $\beta(\lambda)$ is well defined (finite as $N \to \infty$) to one loop. (It is true to ALL LOOPS!)

3) QCD has also fields in the fundamental (the quarks). How does the counting generalize? What about theories with $SO(N)$ gauge group.

Thus, the STRONG form of the conjecture relates the two theories ($\mathcal{N}=4$ and IIB) in a 2 dim parameter space ($g_s \equiv g_{YM}^2$, N) (3 dim if one considers $\theta = C_0 \neq 0$).

The 't Hooft limit $N \to \infty$ $\lambda = g_{YM}^2 N$ fixed reduces the number of parameters to one.





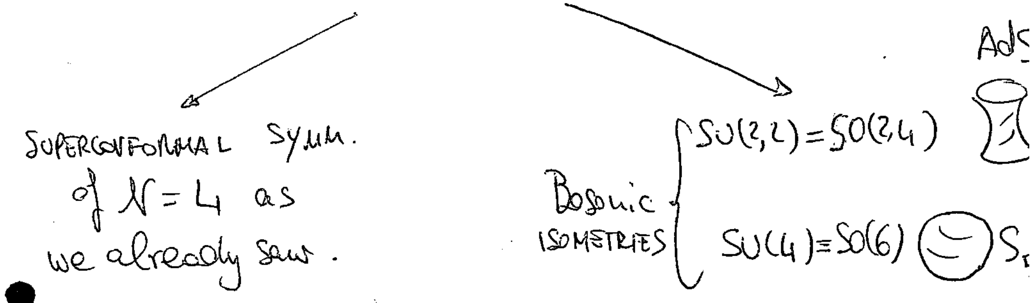
REFINING and TESTING the CONJECTURE

1) GLOBAL SYMMETRIES

If you are claiming that two theories are the same, the very first thing you should do is to check that the GLOBAL symmetri are the same. (Gauge / Diffeo.. "symmetries" need not be, think of Seiberg duality).

In this case they are:

$$PSU(2,2|4)$$



Plus 32 Supercharges..

Notice that in general, the conformal group in D ~~space-time~~ dimensions : $SO(2,D)$ is the SAME as the isometry group of AdS_{D+}. This fact was well known but it found its place in the AdS/CFT corresp. Here even R-symmetry has a geometrical explanatio

4) OPERATORS

Now it is time to specify our "map" between the two theories in more detaie.

Let us look at LOCAL quantities first (There are also interesting non-local observables like the Wilson loop, see later).

From the $\mathcal{N}=4$ SYM point of view, they are all GAUGE INVARIANT LOCAL OPERATORS (they need to be renormalized even in a "FINITE" theory)

(ex: $\mathrm{tr}(F_{\mu\nu}(x)\dots)\,\mathrm{tr}(X_i(x)\dots)$)

They are constructed in the following way: Consider the set of all "LETTERS" build out of ONE of the three gauge co-variant fields F, λ, X and an ARBITRARY # of co-variant der. D.

$$\text{Letters: } \left\{ X_i,\ D_\mu X_i,\ D_\mu D_\nu X_i \dots;\ \lambda_{\alpha a},\ D_\mu \lambda_{\alpha a}, \dots\ F_{\mu\nu},\ DF_{\mu\nu\rho}\dots \right\}$$

NB there is some REDUNDANCY in this set since for instance:

$$D_\mu(D_\nu X_i) - D_\nu(D_\mu X_i) - F_{\mu\nu} X_i + X_i F_{\mu\nu} = 0$$

More redundancy comes from imposing the equations of motion

$$\begin{cases} D^\mu F_{\mu\nu} = \frac{i}{2}\left[\bar\lambda^a_{i\dot\alpha} \lambda_{a\alpha}\right] \sigma_{\nu\alpha\dot\alpha} + i\left[X_i, D_\nu X_i\right] \\ i \not{D} \lambda_{a\alpha} = [X_i, \bar\lambda^b_\alpha]\, C^i{}_{ab} \\ D^2 X_i = [X_j, [X_j, X_i]] + C_i^{ab} \lambda^\alpha_a \lambda_{b\alpha} + \text{c.c.} \end{cases}$$

And a last redundancy comes from the fact that at finite N the concept of "how many traces" is ill defined since, for $p > N$ $\mathrm{tr}\, X^p$ can be written as product of smaller traces.

Let $W_I(x) \in \{X_i(x), D_\mu X_i(x), \dots \quad D_\mu F_{\nu\rho}(x) \dots\}$ denote an arbitrary "letter".

The most general Gauge inv. local op. can be written as:

$$\mathcal{O} = tr(W_{I_1}(x) \cdots W_{I_p}(x)) \cdot tr(W_{J_1}(x) \cdots W_{J_q}(x)) \cdots$$

NB: •) $W_{I_1}^{(x)} \cdots W_{I_p}(x)$ by itself is not gauge invariant (only co-variant).

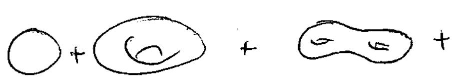
•) I cannot use a "naked" A_μ, only inside $F_{\mu\nu}$ or D_μ.

•) $tr(W_{I_1}(x) \cdots W_{I_p}(x))\, tr(W_{J_1}(y) \cdots W_{J_q}(y))$ is gauge inv. BUT NON LOCAL.

•) $tr(W_{I_1}(x) \cdots W_{I_p}(y))$ is NEITHER.

After taking the traces, [illegible] indices disappear but the Lorentz/Dirac indices μ, α and the R-symmetry indice i, a remain and help us classifying the various operators (together with Δ).

<u>From the IIB point of view</u> they are the fully quantum excitations of the IIB string in $AdS_5 \times S^5$ (whatever that means...) This would mean having control over the complete string loop expansion



+ ..., something we do not even have in flat space time. It would probably require at least letting the geometry fluctuate in the bulk and keeping $AdS_5 \times S^5$ as an asymptotic condition only

At any rate, the conjecture is therefore refined by saying that there is a ONE to ONE map between gauge inv. local operators $\mathcal{O}(x)$ of $\mathcal{N}=4$ SYM and the quanta of IIB string in $AdS_5 \times S^5$.

There should therefore be a one to one map between the CONFORMAL DIMENSIONS of such operators and the MASSES of the excitations.

$$\mathcal{N}=4 \qquad\qquad\qquad IIB.$$

$$\mathcal{O} \longleftrightarrow |\psi\rangle$$

$$\Delta \longleftrightarrow m.$$

Since we know almost nothing about this general case let us, without further ado, consider the LARGE N ('t Hooft) LIMIT:

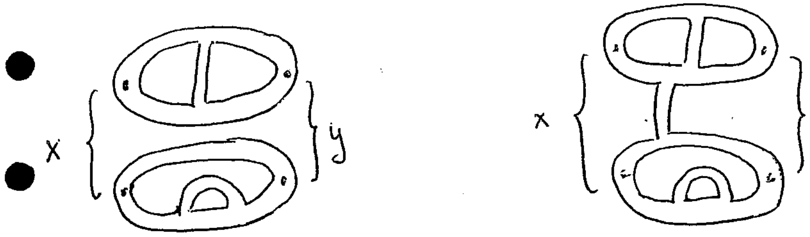
$$N \to \infty \qquad g_s N \equiv g_{YM}^2 N \equiv \lambda \quad \text{FINITE (ARBITRARY)}$$

(in a short while we will let it be large in order to use SUGRA)

We have already learned that some simplifications will occour:

*) From the $\mathcal{N}=4$ SYM ~~side~~, where perturbative computations $(\lambda \ll 1)$ will be possible, we only need consider SINGLE TRACE OPERATORS (a concept that makes sense when $N=\infty$)

The reason is that knowing the correlation functions of all single trace operators allows to construct the correlation funct. of ALL operators to leading order in $\frac{1}{N}$:



contains an extr

$$\frac{g_{YM}^2}{N} = \frac{\lambda}{N^2}$$

*) From the IIB side, as we have seen taking the opposite $(\lambda \gg 1)$ limit we supress the σ-model corrections and go to the "classical" theory

Let us see what the relation $\Delta \longleftrightarrow m$ should be.

(Usually one works in Euclidian signature $AdS_5 \to H^5$, $\mathbb{R}^{3,1} \to \mathbb{R}^4$ or even S^4. This will not matter too much for our purposes.).

The theory contains an ∞ # of fields: the SUGRA fields in AdS_5, their KALUZA-KLEIN modes, the MASSIVE string excitations, their KK modes... .

We would like to compute:

$$Z(\text{b.c.}) = \int_{\text{b.c. in } AdS_5} \mathcal{D}(\text{all fields})\, e^{-S_{TOT}}$$

Clearly an hopeless task!

Let us instead take the limits $N \to \infty$ and $\lambda \gg 1$ (after!).

(meaning that $\frac{\lambda}{N} = g_s$ is small no matter what).

Then the leading term in S_{TOT} is

$$S_{SUGRA+KK} = \frac{1}{2K^2}\int d^{10}x \sqrt{G}\big(\underbrace{R(G)+\dots}_{10 \text{ dim SUGRA}}\big) =$$

$$= \frac{Vol(S^5)}{2K^2}\int d^5x\sqrt{g}\,\big(R(g)+\dots + \text{KALUZA KLEIN}\big)$$

Where I remind you that

$Vol(S^5) \sim L^5$ with $L \sim (g_s N \alpha'^2)^{\frac{1}{4}}$

$K^2 \sim g_s^2 \alpha'^4$.

IN the SADDLE POINT APPROXIMATION:

$$Z(\text{b.c.}) \simeq e^{-S_{SUGRA+KK}\big|_{\text{b.c.}}}$$

Where $S_{\substack{SUGRA\\+KK}}\Big|_{B.C.}$ is the ON-SHELL action for the SUGRA fields obeying the specified eqs. of motion.

In other words:

① Specify the b.c. for the SUGRA fields: $h^0_{\mu\nu}(x^\mu)$, $\phi^0(x^\mu)$ ...

② Solve the eqs of motion $\delta S_{SUGRA} = 0$ for those boundary conditions.

③ Plug back into $S_{SUGRA+KK}$ which now depends only on the b.c. since the fields in the bulk are specified by the eqs of motion: $S_{\substack{SUGRA\\+KK}}\Big|_{B.C.}$

NB: $S_{SUGRA+KK}$ contains the AdS_5 sugra fields AND their KK excitations on S^5 (but no "stringy" mode).

Quick & dirty explanation of KK:

Consider a massless, free, scaler field ϕ on $\mathbb{R}^{3,1} \times S^1$ (Nothing to do with Maldacena)

x^μ , y $\phi(x^\mu, y) = \phi(x^\mu, y + 2\pi \underset{\uparrow}{L})$

RADIUS of S^1.

Eqs. of motion: $(\Box_4 + \partial_y^2)\phi(x^\mu, y) = 0$

(where $\Box_4 = \partial_\mu \partial^\mu$ is the usual D'Alambertian

Expand in Fourier modes:

$$\phi(x^\mu, y) = \sum_{m \in \mathbb{Z}} \phi_m(x^\mu)\, e^{\frac{2\pi i m y}{L}}$$

Each mode satisfy:

$$\left(\Box_4 - \left(\frac{2\pi m}{L}\right)^2\right)\phi_m(x^\mu) = 0$$

That is has a MASS: $m_n^2 = \left(\frac{2\pi m}{L}\right)^2$

Note $m_n \to 0$ at large radius.

infact, for ALL KK modes

$$m_{KK} \propto \frac{1}{L}$$

If I have fields carrying a space time index, I will get fields of different spin in $\mathbb{R}^{3,1}$:

$$\text{Eg: } A_{M=0,1,2,3,\text{"}y\text{"}}(x^\mu,y) \longrightarrow \sum \underset{\substack{\uparrow\\ \text{vectors}}}{A^{(n)}_\mu}(x)\, e^{\frac{2\pi i n y}{L}} \,;\; \sum \underset{\substack{\uparrow\\ \text{scalars}}}{A^{(n)}_{\text{"}y\text{"}}}(x)\, e^{\frac{2\pi i n y}{L}}$$

- (Remember that the gauge transformations
- must also be decomposed).

<u>**EXERCISE**</u> Carry out the reduction of the 5 dim Maxwells equations in detail.

- For more complicated manifolds (eg $AdS_5 \times S^5$) one expands in the appropriate Fourier
- modes:

$$\text{eg scalar: } \Phi(x,y) = \sum_\Gamma \phi_\Gamma(x) \cdot Y_\Gamma(y)$$

and gets a set of manes m_Γ.

This had been completely carried out for $AdS_5 \times S^5$ <u>BEFORE</u> MALDACENA started college!

Let us consider a scalar field ϕ for simplicit

Let $ds^2 = L^2 \dfrac{dx^{\mu 2} + dz^2}{z^2}$ be the AdS_5 metric

we will write $S_{\text{scalar}+\text{KE}} \longrightarrow S[\phi, g]$ only

We cannot just hope to impose

$$\phi(x^\mu, z) \longrightarrow \phi_0(x^\mu) \quad \text{as } z \to 0$$

- because the eqs of motion may not allow it!

- Let ϕ have mass m (<u>NB</u> if this mass comes from K.K compact. on S^5 which also has radius L it will go $m \sim \frac{1}{L}$ but we can be general for now).

- $$S = \frac{L^5}{2k^2}\int d^4x\,dz \sqrt{g}\left(g^{MN}\partial_M\phi\,\partial_N\phi + m^2\phi^2\right) =$$

- $$= \frac{L^5}{2k^2}\int d^4x\,dz \cdot \left(\frac{L}{z}\right)^5 \left(\frac{z^2}{L^2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{z^2}{L^2}(\partial_z\phi)^2 + m^2\phi\right.$$

$$= \frac{L^8}{2k^2}\int d^4x\,dz\, \frac{1}{z^5}\left(z^2\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + z^2(\partial_z\phi)^2 + m^2L^2\phi\right.$$

LO AND BEHOLD! $\dfrac{L^8}{k^2} \propto N^2$.

This is the right scaling for the free energ of a gas of $N^2 \sim \dim(\text{Adj}(SU(N)))$ particles

For KK modes mL is a number independent on N or g_s. (or $\lambda = g_s N$)
So we can factor out the N dependence

$$S_{SUGRA+KK} = N^2 \cdot \hat{S}$$

In general, the expansion of S_{TOT} is: making ALL DEPENDENCE on N and λ EXPLICIT:

$$S_{TOT} = N^2\left(\overset{\hat{S}}{\overset{\|}{S_{\bigcirc}}} + \frac{1}{\sqrt{\lambda}} S'_{\bigcirc} + \frac{1}{\lambda} S''_{\bigcirc} + \cdots\right) + \left(S_{\circledcirc} + \frac{1}{\sqrt{\lambda}} S'_{\circledcirc} + \cdots\right) + \frac{1}{N^2}\left(S_{\circledcirc\circledcirc} + \frac{1}{\sqrt{\lambda}} S'_{\circledcirc\circledcirc} + \cdots\right) + \cdots$$

Where $S', S''\ldots$ represent the "α'" corrections from the σ-model.
(Some of them are zero, for instance we know that the first correction starts at $\int d^{10}x\, R^4$).

The equations of motion:

$$z^2 \Box \phi + z^5 \partial_z \frac{1}{z^3} \partial_z \phi - m^2 L^2 \phi = 0$$

become a simple ODE after Fourier trans!

$$z^5 \partial_z \frac{1}{z^3} \partial_z \phi - \underbrace{z^2 p^2 \cdot \phi}_{\text{always subleading w.r.t. the mass when } z \to 0} - m^2 L^2 \phi = 0$$

Try letting $\phi \sim z^\alpha$ when $z \to 0$

$$\Longrightarrow \quad \alpha(\alpha - 4) = m^2 L^2$$

$$\Longrightarrow \quad \alpha_\pm = 2 \pm \sqrt{4 + m^2 L^2}$$

The leading solution is generically z^{α_-}
(we need ϕ to be smooth everywhere in the interior and this usually picks bothe components. We will see later how to interpret the case where the leading component is absent)

So the correct prescription for the partition function is:

$$Z[\phi_0(x^\mu), \epsilon] = e^{-N^2 \hat{S}}\Big|_{\phi(x,\epsilon)=\epsilon^{\alpha_-}\phi_0(x)}.$$

ϵ acts as a IR REGULATOR for the BULK Theory

Since ϕ_0 "disturbs" the bdry, it acts as a SOURCE for some gauge invariant local operator $O_\phi(x)$ and the punchline is that we ALSO interpret the partition function as the GENERATING FUNCTION for the correlators of O_ϕ:

$$Z[\phi_0(x^\mu), \epsilon] = \left\langle e^{\int d^4x\, \phi_0(x) O_\phi(x)} \right\rangle_\epsilon$$

where ϵ now acts as UV REGULATOR of the FIELD THEORY.

This function Z can actually be computed explicitly and the correlato can be obtained by functional derivation.

However, if we are only interested in obtaining the CONFORMAL DIM. Δ we can simply argue that under rescaling $x \to A x$, $z \to A z$ (which is an ISOMETRY of AdS_5) $\phi = \epsilon^{\alpha_-}$ does not scale and thus

$$\phi_0 \longrightarrow A^{-\alpha_-} \phi_0 .$$

We need to leave $\int d^4x\, O_\phi \phi_0$ invariant $\therefore$ $O_\phi \to A^{\alpha_- -4} O_\phi$

Meaning that its CONFORMAL DIMENSION is $\Delta = 4 - \alpha_- \equiv \alpha_+ = 2 + \sqrt{4 + (mL)^2}$

From this you

has the meaning of an (inverse) energy scale from the CFT point of view

That is $\frac{1}{Z} \sim r/\alpha' \sim E \sim u$ and you have to use the appropriate powers of α' to make the dimensionality work out as in the previous scaling argument.

Similar formulas can be obtained for other fields (spinors, vectors etc.)

In fact the conjecture can be generalized to AdS_{d+1} in some cases (for us $d \equiv 4$). The GENERAL formulas are:

scalars : $\Delta = \frac{1}{2}\left(d + \sqrt{d^2+4m^2L^2}\right)$ ($d=4$ reduces to our form).

spinors : $\Delta = \frac{1}{2}\left(d + 2|mL|\right)$

vectors : $\Delta = \frac{1}{2}\left(d + \sqrt{(d-2)^2+4m^2L^2}\right)$

p-form : $\Delta = \frac{1}{2}\left(d + \sqrt{(d-2p)^2+4m^2L^2}\right)$

(NOTE : $p=0$ scalar, $p=1$ vector).

Self dual $\frac{d}{2}$-form (if it exists) : $\Delta = \frac{1}{2}\left(d + 2|mL|\right)$

vectorspinor : $\Delta = \frac{1}{2}\left(d + 2|mL|\right)$

massless spin 2 : $\Delta = d$.

•) In some very special cases there might be ops whose anomalous dim. are given by the above formules with the sign in front of the $\sqrt{}$ reversed. Most of the time this is ruled out since it would make Δ too small to belong to a unitary repr. of the conformal group.

••) One would guess that m^2 must be positive. However, AS LONG AS the argument in the $\sqrt{}$ is positive this is actually acceptable in AdS and does not give rise to instabilities

•••) There is some ambiguity in the definition of a mass in AdS space due to the presence of terms like $R\phi^2$ (is it a mass or a coupling?).

The formulas above use the commonly accepted definition in the AdS/CFT community : "everything is a mass".

oooo) Finally, let us rivisit the α_+ vs α_- behavior. As I said, generically $\phi \sim Z^{\alpha_-}\phi_0$. Since this is not a normalizable mode, it does not correspond to a state of the bulk theory and in fact we interpreted as sources probing the theory on the bdry and coupling to an op. $\mathcal{O}$ w/ $\Delta = 4 - \alpha_- \equiv \alpha_+$.

What if, in some case (this happens mostly in theories w/ less susy as you will see from other lecturers)

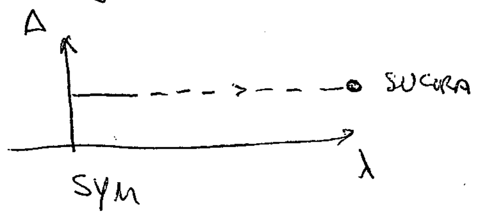
$$\phi \sim \mathcal{O} \cdot Z^{\alpha_-} + \text{Cons.}\, Z^{\alpha_+} \quad ?$$

(That is there is a conspiracy of the eq. of motions in the bulk to set the leading term (as $z \to 0$) to zero). Clearly I cannot say this corresponds to an op. w/ $\Delta = 4 - \alpha_+$ since it violates the unitary bond. But recall that we did have a $\Delta = \alpha_+$ already. Thus the most natural interpr. is to say that $\mathcal{O}$ has acquired a vev (See Balasubramanian & Kraus hep-th/9808017 Lawrence / Trivedi

Since the spectrum of

is known, the above formulas make concrete predictions for the values of anomalous dimensions AT STRONG 't HOOFT COUPLING.

To test this against field theory, our best bet is if there are operators whose anomalous dimensions are protected by susy to coincide with the engineering dimension (no renormalization). If that's the case, we should find their values among the Δ's computed from SUGRA:



It turns out that ALL of the anomalous dim's from SUGRA correspond to ALL the protected operators, known as CHIRAL OPERATORS (primaries AND descendants see later for a clarification).

The non chiral operators are mapped to the stringy excitations (and their k.k).
Here the checks are much more difficult (and stronger) but progress has been made using INTEGRABILITY (we will not discuss this).

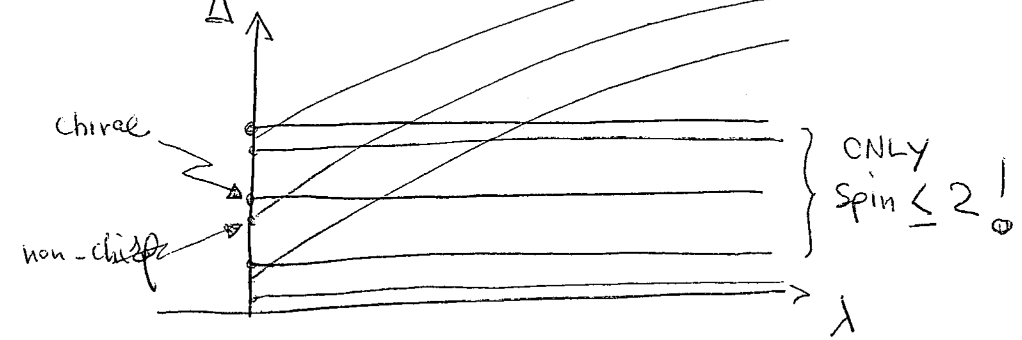
- Since $m^2_{\substack{\text{SUGRA}\\\text{and KK}}} \propto \frac{1}{L^2}$ (at large λ).

- $m^2_{\text{STRINGY}} \propto \frac{1}{\alpha'}$

And the formulas contain

- $m^2 L^2 \sim \begin{cases} \lambda^0 \\ \frac{L^2}{\alpha'} \sim \lambda^{1/2} \end{cases}$ under the $\sqrt{\;}$ sign

- Maldacena's conjecture makes the DRAMMATIC PREDICTION:



Conformal primaries, descendants, chiral ops and all that ...

Let $\mathcal{O}(x)$ be any gauge invariant local op.
(later we will look only at "single trace" ops but for now we can be even more genera

The action of some of the generators of $PSU(2,2|$
- is pretty straightforward.
- Ex., let $\mathcal{O}(x) = \text{tr}\, X^i(x) X^j(x)$

$$[P_\mu, \mathcal{O}] = i\partial_\mu \mathcal{O} \equiv i\,\text{tr}(D_\mu X^i X^j) + i\,\text{tr}(X^i D_\mu$$

$$[Q^a_\alpha, \mathcal{O}] = C^{iab}\,\text{tr}(\lambda_{\alpha b} X^j) + C^{jab}\,\text{tr}(X^i \lambda_{\alpha b}$$

- .) When taking the derivative inside the trace it can be turned into a covariant der. since the contribution from all commutators cancel (check it!)
- .) Notice that P and Q change the Lorentz + R sym. repr.
- .) $[Q\ \mathcal{O}]$ should be replaced by $\{Q, \mathcal{O}\}$ if $\mathcal{O}$ is fermionic. ~~This~~ will always be understl

DEF: operators that can be written as $[Q, \mathcal{O}'$ of another operators are called DESCENDANTS. (Super decendant)

◦) Operators that can be written as $[P, \mathcal{O}]$ are also descendants since $P \sim \{Q, Q\}$ but not viceversa (obviously).

◦) The action of other operators, such as the Lorentz generators contains x^μ explicitly:

Eg. $M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu}$

where $\Sigma_{\mu\nu}$ are the "spin" part.

The x_μ dependence of the "orbital part" is necessary to get the right commutation relations with the other generators, eg $P_\mu = i\partial_\mu$. However these pieces just "go along for the ride" and we can drop them by taking $x^\mu = 0$ AT THE END.

◦) The action of the ops K_μ and $\bar{S}^{a}_{\dot\alpha}$, $S_{\alpha a}$ is obtained in principle in the same way but in practice it's trickier,
($-[P,\mathcal{O}]$ and $[K,\mathcal{O}]$ is a bit like the difference between taking a derivative and taking an integral)

For instance, we have seen that

$$K_\mu = i(x^2 \partial_\mu - 2x_\mu x^\rho \partial_\rho - 2\Delta x_\mu)$$

and this "seems" to suggest that $[K_\mu, \mathcal{O}]_{x\to 0} = 0$

But this is NOT true if, say $\mathcal{O}$ is a total derivative $\mathcal{O}_\nu = \partial_\mu \mathcal{O}'$ because in this case the

CORRECT MEANING of $[K_\mu, \mathcal{O}_\nu]$ is:

$$[K_\mu, \partial_\nu \mathcal{O}'] = -i[K_\mu, [P_\nu, \mathcal{O}']] =$$

$$= -i[P_\nu, [K_\mu, \mathcal{O}']] + i[[P_\nu, K_\mu], \mathcal{O}'] =$$

Now, $[K_\mu, \mathcal{O}']$ should be analized in the same way, it could be zero or non zero. Suppose it is zero, (that is $\mathcal{O}'$ cannot be written as a total derivative itself) then

$$= i[[P_\mu, K_\mu], \mathcal{O}'] = [2M_{\mu\nu} - 2\eta_{\mu\nu} D, \mathcal{O}']$$

$$= 2[M_{\mu\nu}, \mathcal{O}'] - 2\eta_{\mu\nu}[D, \mathcal{O}']$$ not necessarily zero of course.

•) Since for any generator of PSU(2,2/4) the action on a gauge inv. local op is a (linear comb. of) gauge inv local ops,
The set of ALL gauge inv. local ops forms a representation of PSU(2,2/4).

This repres. is highly reducible (even after taking $x^\mu = 0$) and we would like to split it into the $\oplus$ of irreps.

•) Each irrep will still be ∞ dim since) can always take an arbitrary # of ∂_μ.

$\mathcal{O}, \ \partial_\mu \mathcal{O}, \ \partial_\mu \partial_\nu \mathcal{O} \dots$ are all in the same rep.

•) Suppose that $\mathcal{O}$ belongs to some irrep. If) hit it with enough S's at some point) will get zero.

This is so because all $\mathcal{O}$ have non negativ (mass) engineering dimension, $\mathcal{O} = \mathbb{1}$ has the lowest (zero).

Looking at $[K, Q] \sim S$, $[P, S] \sim Q$, $\{Q, Q\} \sim P$ $\{S, S\} \sim K$ since $[P] = 1$ we have $[Q] = \frac{1}{2}$, $[S] = -\frac{1}{2}$, $[K] = -1$.

So, acting with enough S (or K) will lower the dim. below zero.

<u>DEF</u> An op. st. $[S, \mathcal{O}] = 0$ is called a PRIMARY OPERATOR. (Superconformal primary

•) A primary op. also obeys $[K, \mathcal{O}] = 0$ since $K \sim \{S, S\}$ but not viceversa.

•) Ops. can be devided into PRIMARIES and DESCENDANTS Since $\{Q, S\} \sim D + R + M$ is never zero you can convince yourself that a primary canot be a descendant and vice versa.

THUS: ① Every primary gives rise to an irrep, and viceversa.

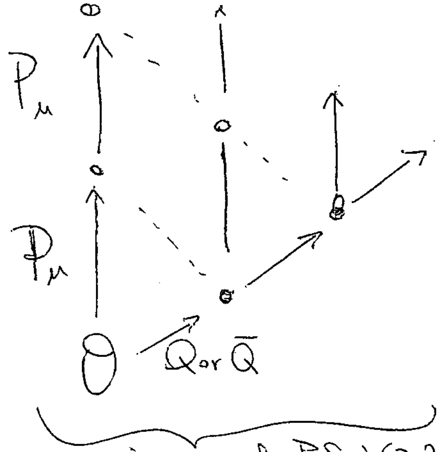
$$\mathcal{O} \xrightarrow[\substack{Q \\ (\text{or } \bar{Q})}]{} \mathcal{O}' \xrightarrow[Q]{} \mathcal{O}'' \longrightarrow \dots$$

② Every irrep can be classified by the quantum # of the primary

③ To find primaries (and thus irreps) we must classify those ops that CANNOT be written as $[Q, \cdot]$ or $[\bar{Q},$ of something (TRICKY algebraic proble

There is a SUBCLASS of primary ops that is particularly interesting since they correspond to the protected ones:

In general, since there are 8 $Q^{a=1..4}_{\alpha=1,2}$ and 8 $\bar{Q}_{a\dot{\alpha}}$, taking a (superconformal) primary) can construct 2^{16} operators that are descendant of PSU(2,2/4) but are primaries of SU(2,2) ≡ SO(2,4) only (that is anhilated by K_μ)



irrep of PSU(2,2|4) splits into irreps of SU(2,2) (vertical lines).

DEF A (superconformal) primary is called CHIRAL if it gives rise to a "smaller" irrep (less irreps of SO(2,4)).

Their importance is due to the fact that Δ can only take <u>discrete</u> values (contrary to the general case where there is only a bound Δ>...).

But if Δ is discrete it CANNOT DEPEND ON λ!

∴ CHIRAL PRIMARIES (and their descendants) obey a NON RENORMALIZATION THEOREM whereby Δ = Engineering dimension

I will not prove this fact and I refer you to the literature. I will also just state the following (crucial) result:
(see Dobrev - Petkova Lett. Math Phys 9 (1985) 287)

(In the Large N limit) the only single-trace "short" multiplets are those constructed from the primary

$$tr\left(X^{\{i_1}\dots X^{i_k\}}\right)$$

• where $\{i_1 .. i_k\}$ means the SYMMETRIZED TRACELESS combination (an irrep of SO(6)).

• Since $Q \begin{pmatrix} \lambda \\ \bar{\lambda} \\ X \\ F \end{pmatrix} \longrightarrow \begin{pmatrix} F + [X,X] \\ DX \\ \lambda \\ D\lambda \end{pmatrix}$

it should be at least reasonable that $tr(X^{\{i_1}\dots X^{i_k\}})$ is a primary. The fact

• that it is a CHIRAL PRIMARY requires some work!

• Perhaps at this point I should remind you some even more basic facts.

In ANY CFT :

•) The ONLY op. that can get a vev is the trivial identity op "$\mathbb{1}(x)$". (for which $\langle \mathbb{1}(x) \rangle = 1$). Any other vev would have a mass dim > 0 and woul

• break scale/conformal invariance

•) • It is always possible to find a set of ops for which the 2-pt func has diagonal form:

$$\langle \mathcal{O}_a(x)\, \mathcal{O}_b(y) \rangle = \frac{C_a \cdot \delta_{ab}}{|x-y|^{2\Delta_a}} \underset{\text{obviously}}{\equiv} \frac{C_b\, \delta_{ab}}{|x-y|^{2\Delta_b}}$$

• EXERCISE Show, by using the elemen

• of the conformal group mapping $x^\mu \to \frac{x^\mu}{x^2}$ (know a "inversion") that in order to get a non zero 2-pt function $\Delta_a \equiv \Delta_b$.

∘) Δ is of course the same appearing in the expression for the DILATION op:

$$[D, \mathcal{O}(0)] = \Delta\, \mathcal{O}(0)$$ (with the convention explained before)

Δ is the CONFORMAL DIMENSION

$\Delta_0 \left(\equiv \# X + \# D + \frac{3}{2} \# \lambda + 2 \# F\right)$ is the ENGINEERING DIMENSION

$\eta = \Delta - \Delta_0$ is the ANOMALOUS DIMENSION.

∘) Note that if $\eta \neq 0$ C_a must have a mass dimension in order for the engineering dims to match: normalizing:

$$\langle \mathcal{O}_a(x)\, \mathcal{O}_b(y) \rangle = \frac{\mu^{2(\Delta_{0a}-\Delta_a)} \delta_{ab}}{|x-y|^{2\Delta_a}}$$

This is the "only" place μ can appear without breaking conformal invariance

∘) In perturbation theory, for unprotected ops we would find:

$$\Delta = \Delta_0 + b\lambda + \dots$$ and:

$$\frac{\mu^{2(\Delta_0-\Delta)}}{|x-y|^{2\Delta}} = \frac{1}{|x-y|^{2\Delta_0}} (\mu|x-y|)^{2(\Delta_0-\Delta)} =$$

$$= \frac{1}{|x-y|^{2\Delta_0}} (\mu|x-y|)^{-2b\lambda} = \frac{1}{|x-y|^{2\Delta_0}} e^{-2b\lambda \log(\mu|x-y|)}$$

$$\simeq \frac{1}{|x-y|^{2\Delta_0}} \left(1 - 2b\lambda \log \mu|x-y| + \dots\right).$$

∘) Finally remember that after finding Δ for the PRIMARIES, all work is DONE. namely, any descendant $\mathcal{O}' = [Q, \mathcal{O}]$ will have $\Delta' = \Delta + \frac{1}{2}$ and so on:

$$[D, \mathcal{O}'] = [D, [Q\, \mathcal{O}]] = [Q, \underbrace{[D, \mathcal{O}]}_{\Delta \mathcal{O}}] + [\underbrace{[D, Q]}_{\frac{1}{2} Q}\, \mathcal{O}] =$$

$$= \Delta \mathcal{O}' + \frac{1}{2} \mathcal{O}' = \left(\Delta + \frac{1}{2}\right) \mathcal{O}'.$$

Quick & dirty introduction to perturbative renormalization of composite operators:

Consider a scalar theory for simplicity (Nothing to do with Maldacena, not even conformal !) -

$$S=\int d^Dx\ \frac{1}{2}\left((\partial_\mu\phi_0)^2 - m_0^2\phi_0^2\right)-\frac{\lambda_0}{4!}\phi_0^4 \equiv$$

$$\equiv \int d^Dx\ \frac{1}{2}\left(Z_\phi(\partial_\mu\phi)^2 - Z_1 m^2\phi^2\right)-Z_2\frac{\lambda\mu^\epsilon}{4!}\phi^4$$

$$\equiv \int d^Dx\ \frac{1}{2}\left((\partial_\mu\phi)^2-m^2\phi^2\right) - \frac{\lambda\mu^\epsilon}{4!}\phi^4 +$$

$$+\int d^Dx\ \frac{1}{2}\left(\delta_\phi(\partial_\mu\phi)^2-\delta_1 m^2\phi^2\right)-\frac{\lambda\mu^\epsilon}{4!}\delta_2\,\phi^4.$$

$$\left(Z_x = 1+\delta_x\right)$$

$$\phi_0=\sqrt{Z_\phi}\,\phi \implies m_0^2 Z_\phi = m^2 Z_1 \implies m_0=\sqrt{\frac{Z_1}{Z_\phi}}\,m\equiv Z_m m$$

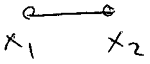
$$\lambda_0 Z_\phi^2=\lambda\mu^\epsilon Z_2 \implies \lambda_0=\frac{Z_2}{Z_\phi^2}\lambda\mu^\epsilon\equiv Z_\lambda\lambda\mu^\epsilon$$

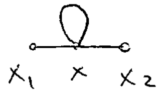
(In $\mathcal{N}=4$ <u>ALSO</u> one would need some $Z\neq 1$ if working in the Wess-Zumino gauge)

To compute the Z's in this simple theory recall: $\Delta(x-y)=\int\frac{d^Dp}{(2\pi)^D}\frac{i\,e^{-ip(x-y)}}{p^2-m^2+i\epsilon}$.

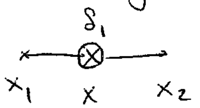
We can work directly in Coordinate-Minko space: Compute δ_1:

$$\langle\phi(x_1)\,\phi(x_2)\rangle=\Delta(x_1-x_2)-\frac{i\lambda\mu^\epsilon}{4!}\cdot 4\cdot 3\int d^Dx\,\Delta(x_1-x)\Delta(x_2-x)\cdot\Delta($$





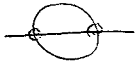
$$-\frac{i}{2}\delta_1 m^2\cdot 2\cdot\int d^Dx\,\Delta(x_1-x)\Delta(x_2-x)+\mathcal{O}(\lambda^2)$$



We need $\frac{\lambda\mu^\epsilon}{2}\cdot\Delta(0)+\delta_1 m^2$ finite

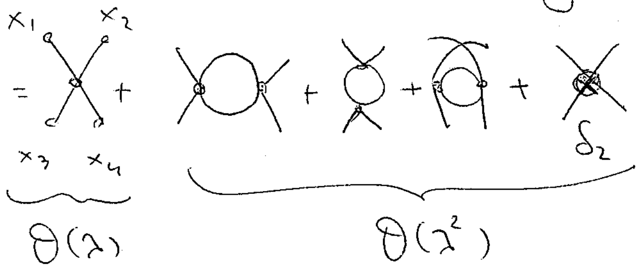
$$\Delta(0)=\int\frac{d^Dp}{(2\pi)^D}\frac{i}{p^2-m^2}\simeq i\,\frac{(-i)}{(4\pi)^2}\cdot\left(-\frac{2}{\epsilon}\right)m^2+\dots$$

$$\implies \delta_1=\frac{\lambda}{(4\pi)^2}\cdot\frac{1}{\epsilon}.$$

δ_ϕ is zero to one loop since the first diagram with a divergence $\propto p^2$ is .

δ_2 can be computed in the same way.

$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle =$



Giving $\delta_2 = \frac{3\lambda}{(4\pi)^2}\cdot\frac{1}{\epsilon}$

EXERCISE: SHOW THIS.

So, to one loop $Z_\lambda = Z_2$

And one could then compute the β-function for the theory: $\lambda = \lambda_0\mu^{-\epsilon}/Z_\lambda$, $Z_\lambda = 1 + \frac{Z_\lambda^{(1)}(\lambda)}{\epsilon} + \cdot$

$$\beta(\lambda,\epsilon) \equiv \mu\frac{\partial}{\partial\mu}\Big|_{Bare}\lambda = -\epsilon\lambda - \frac{\lambda}{Z_\lambda}\mu\frac{\partial}{\partial\mu}\Big|_{Bare} Z_\lambda(\lambda,\epsilon)$$

$$\equiv -\epsilon\lambda - \frac{\lambda}{Z_\lambda}\beta(\lambda,\epsilon)\cdot\frac{\partial}{\partial\lambda}Z_\lambda(\lambda,\epsilon)$$

Thus: $\left(\beta(\lambda,\epsilon) + \epsilon\lambda + \lambda\beta(\lambda,\epsilon)\frac{\partial}{\partial\lambda}\right)Z_\lambda(\lambda,\epsilon) = 0.$

Expanding $\beta(\lambda,\epsilon) = \beta(\lambda) + \epsilon\beta^{(1)}(\lambda) + \epsilon^2\beta^{(2)}(\lambda) + \dots$

we have: $\beta^{(2)} = \beta^{(3)} = \dots = 0$, $\beta^{(1)} = -\lambda$, and:

$$\beta(\lambda) = \epsilon\lambda^2\frac{\partial}{\partial\lambda}Z_\lambda = \lambda^2\frac{\partial}{\partial\lambda}Z_\lambda^{(1)}(\lambda) = +\frac{3\lambda^2}{(4\pi)^2}$$

Of course, the same computation in $\mathcal{N}=4$, to ALLLOOPS would give $\beta \equiv 0$.

Suppose now we want to compute the correlator of a COMPOSITE operator "$\phi^2(x)$"..

Since $\langle\phi(x_1)\phi(x_2)\phi(x_3)\dots\rangle$ is SINGULAR when $x_i \to x_j$ I NEED FURTHER RENORMALIZATIO contants (one for each composite op.).

Define: $[\phi^2](x) = Z_{\phi^2}^{-1}\,{}^\times_\times\phi_0(x)\phi_0(x){}^\times_\times \equiv$

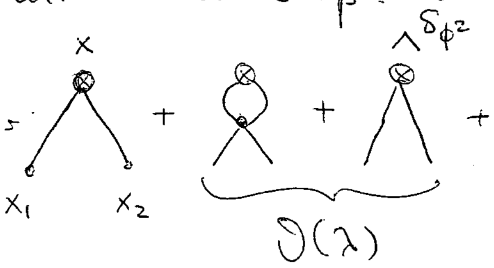
$$\equiv Z_{\phi^2}^{-1}Z_\phi\,{}^\times_\times\phi(x)\phi(x){}^\times_\times \equiv$$

$$\left(\tilde{Z}_{\phi^2} = Z_{\phi^2}/Z_\phi\right) \equiv \tilde{Z}_\phi^{-1}\cdot{}^\times_\times\phi(x)\phi(x){}^\times_\times.$$

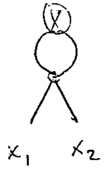
Constructed in such way that $\langle[\phi^2](x)\,\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle$ is finite.

Only $n=2$ is relevant to 1-loop.

$\langle[\phi^2](x)\phi(x_1)\phi(x_2)\rangle =$



$+\ \mathcal{O}(\lambda^2)$



$$= -\frac{i\lambda\mu^{\epsilon}}{4!}\cdot 4\cdot 3\cdot 2\cdot \int d^D y\, \Delta(x_1-y)\Delta(x_2-y)\cdot\Delta(x-y)^2 =$$

$$= -i\lambda\mu^{\epsilon}\cdot\int\frac{d^D p_1\, d^D p_2}{(2\pi)^D}\cdot\frac{e^{-ip_1\cdot(x_1-x)}}{p_1^2-m^2}\;\frac{e^{-ip_2(x_2-x)}}{p_2^2-m^2}\times\underbrace{\int\frac{dk^D}{(2\pi)^D}\frac{1}{(k^2-m^2)((k+p_1+p_2)^2-m^2)}}_{\sim\frac{i}{(4\pi)^2}\cdot\frac{2}{\epsilon}}$$

$$= -2\,\tilde{\delta}_{\phi^2}\,\Delta(x_1-x)\Delta(x_2-x) =$$

$$= 2\,\tilde{\delta}_{\phi^2}\cdot\int\frac{d^D p_1\, d^D p_2}{(2\pi)^D}\;\frac{e^{-ip_1(x_1-x)}}{(p_1^2-m^2)}\;\frac{e^{-ip_2(x_2-x)}}{(p_2^2-m^2)}$$

$$\Rightarrow\; -i\lambda\mu^{\epsilon}\cdot\frac{i}{(4\pi)^2}\cdot\frac{2}{\epsilon}+2\,\tilde{\delta}_{\phi^2}\quad \text{finite} -$$

$$\Rightarrow\; \tilde{\delta}_{\phi^2} = -\frac{\lambda}{(4\pi)^2}\cdot\frac{1}{\epsilon}+\dots \equiv \delta_{\phi^2}$$ since $Z_\phi = 1$ to 1 loop

$$\Rightarrow\; \gamma_{\phi^2} = \mu\frac{\partial}{\partial\mu}\Big|_{\text{Bare}}\log Z_{\phi^2} = (\text{as before}) = \frac{\lambda}{(4\pi)^2}.$$

- These coeff. are the "anomalous dimensions" particularly in the case of $\mathcal{N}=4$ where since $\beta=0$ λ is a true parameter. They will be ZERO for protected ops. But NON-ZERO otherwise -

Quick & Dirty review of IIB sugra compactification on $AdS_5\times S_5$:
(See Kim et al Phys Rev D (1985) 389).

Solution of the b.g. ($\hat{X}$ = background val)

$$\hat{F}_{\mu\nu\rho\sigma\tau} = \frac{1}{L}\hat{\epsilon}_{\mu\nu\rho\sigma\tau} \qquad \left(\epsilon_{01234} = \sqrt{\hat{g}_{AdS}}\right)$$

$$\hat{F}_{\alpha\beta\gamma\delta\epsilon} = \frac{1}{L}\hat{\epsilon}_{\alpha\beta\gamma\delta\epsilon}$$

Volume forms in AdS^5 and S_5 respectively

$$\hat{R}_{\mu\nu\rho\sigma} = -\frac{1}{L^2}\left(\hat{g}_{\mu\rho}\hat{g}_{\nu\sigma}-\hat{g}_{\mu\sigma}\hat{g}_{\nu\rho}\right)$$

$$\hat{R}_{\alpha\beta\gamma\delta} = \frac{1}{L^2}\left(\hat{g}_{\alpha\gamma}\hat{g}_{\beta\delta}-\hat{g}_{\alpha\delta}\hat{g}_{\beta\gamma}\right).$$

metics and curvature on AdS_5 and S_5 respectiv

Bosonic field eqs to @most linearized level:

$$\begin{cases} R_{MN} = -\frac{1}{6}F_{MRPQS}F_N{}^{RPQS} \\ F_{MNRST} = \frac{1}{5!}\epsilon_{MNRSTM'N'R'S'T'}F^{M'N'R'S'T'} \\ D^M\partial_{[M}A_{NP]} = -\frac{2i}{3}F_{NPSTK}D^S A^{TK} \\ D^M\partial_M B = 0 \end{cases}$$

where $M = (\mu, \alpha)$ $\quad \mu = 0..4$ AdS$_5$ index
$\alpha = 1..5$ S_5 index.

B is formed from ϕ and C

A_{MN} " B_{MN} & C_{MN}

F_{MNPQR} is the field strenght of C^{+}_{MNPP}.

●

Now decompose all fields in b.g. + fluctuatio:

● Specifying the nature of the indices:

$$\begin{cases} g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu} \\ g_{\alpha\beta} = \hat{g}_{\alpha\beta} + h_{\alpha\beta} \\ g_{\mu\alpha} = \quad h_{\mu\alpha} \quad \text{(pure fluct.)} \end{cases}$$

(Convenient: $h'_{\mu\nu} = h_{\mu\nu} + \frac{1}{3}\hat{g}_{\mu\nu} h^{\alpha}_{\alpha}$)

●

$$\begin{cases} A_{\mu\nu\rho\lambda} = \hat{A}_{\mu\nu\rho\lambda} + a_{\mu\nu\rho\lambda} \\ A_{\mu\nu\rho\alpha} = \quad a_{\mu\nu\rho\alpha} \\ A_{\mu\nu\alpha\beta} = \quad a_{\mu\nu\alpha\beta} \\ A_{\mu\alpha\beta\gamma} = \quad a_{\mu\alpha\beta\gamma} \\ A_{\alpha\beta\gamma\delta} = \hat{A}_{\alpha\beta\gamma\delta} + a_{\alpha\beta\gamma\delta}. \end{cases}$$

(pure fluct.)

$A_{\mu\nu}$, $A_{\mu\alpha}$, $A_{\alpha\beta}$, B pure fluct. keep same symbol.

Now I need to decompose the AdS fields in KK modes.

I can still use diffeo invariance to reduce them to e.g. metric:

$$h'_{\mu\nu} = \sum_I H^I_{\mu\nu}(x)\, Y^I(y),$$

● $$h_{\mu\alpha} = \sum_I B^I_{\mu}(x)\, Y^J_{\alpha}(y)$$

●

$$h_{(\alpha\beta)} = \sum_I \phi^I(x)\, Y^J_{(\alpha\beta)}(y)$$

$$h^{\alpha}_{\alpha} = \sum_I \pi^J(x)\, Y^I(y),$$

the Σ's are all ∞.
I is like a "ℓm" multi

● (In general, before using diffeos one would have more general contributio

● e.g. of the type $\sum_J B^J_\mu(x) D_\alpha Y^J(y)$ to $h_{\mu\alpha}$)

$$\underbrace{Y^I_{(\alpha_1 \dots \alpha_r)}}(y)$$ are "spin r" spherical harmonics on S5.

TRACELESS SYMMETRIC SO(6) irrep ρ_r

I just labels the basis of Y's.

It is perhaps worth spending a word on how they are constructed.

Let $(y^1 .. y^6) \in \mathbb{R}^6$, $\rho^2 = \sum_{i=1}^{6} (y^i)^2$.

The usual (scalar) spherical harmonics are, as well know:

$$\propto \frac{1}{\rho^k} y^{\{i_1} \cdots y^{i_k\}} = \frac{1}{\rho^k} y^{i_1} .. y^{i_k} - \text{traces}.$$

Or, equivalently, introducing a set of constant symmetric traceless tensors in $\{i_1 .. i_k\}$

$$C^I_{\{i_1 \dots i_k\}} \frac{1}{\rho^k} y^{i_1} .. y^{i_k} \equiv Y^I$$

I can be though of as a multiindex containing both orbital and magnetic q.#'s.

The "tensor" harmonics are constructed similarly:

First of all, one can represent a generic tensor $T_{\alpha_1 \dots \alpha_n}$ $\alpha_i = 1 \dots 5$ indices on $S5$ by picking a tensor $\hat{T}_{i_1 \dots i_n}$ $i_i = 1..6$ indices in with the same symmetries and no components ⊥ to S_5, ie:

$$y^{i_k} \hat{T}_{i_1 \dots i_k .. i_n} = 0 \quad \forall k.$$

The metric on the sphere $g_{\alpha\beta}$ can also be written this way: $\hat{g}_{ij} = \delta_{ij} - \frac{y_i y_j}{\rho^2}$

So that $T_{\dots \alpha} V^{\alpha \dots} = \hat{T}_{.. i} \hat{V}^{i \dots}$

To construct the "tensor" harmonics pick a basis $C^I_{i_1 \dots i_p; j_1 \dots j_k}$ where:

•) $j_1 .. j_k$ are in the traceless sym. as before.

•) $i_1 .. i_p$ are in whatever irrep we are discussing (eg also traceless sym. for the graviton).

•) symmetrization of any i_k with $j_1 .. j_k$ yields zero

Thus $C^I_{i_1 \dots i_p; j_1 .. j_k} \frac{1}{\rho^k} y^{j_1} .. y^{j_k}$ describes $Y^I_{\alpha_1 .. \alpha_p}$

One does a similar thing for C_4:

Fluctuations:

$$a_{\mu\nu\rho\sigma} = \sum b^{I}_{\mu\nu\rho\sigma}(x)\, Y^{I}(y)$$

$$\vdots$$

$$a_{\alpha\beta\gamma\delta} = \sum b^{I}(x)\, \underset{\underset{\epsilon_{\alpha\beta\gamma\delta\epsilon} D^{\epsilon} Y^{I}(y)}{\equiv}}{Y^{I}_{[\alpha\beta\gamma\delta]}}(y) .$$

($b^{I}_{\mu\nu\rho\sigma}$ and $b^{I}_{\mu\nu\rho}$ have no d.o.f.)

Plug and collect the terms $\propto$ each Y to get a <u>long</u> list of eqs (see paper)

Picking one of the simplest ones for illustration purpose:

$$\left[\left(\Delta_{AdS_5} + \Delta_{S5}\right) b^{I} + \frac{1}{2L^2} H^{I\mu}_{\mu} - \frac{4}{3L^2}\pi^{I}\right] Y^{I} = 0 .$$

Δ_{S^5} "goes through" b^{J} and "hits" Y^{I} to give the quantization:

$$\Delta_{S5} Y^{I} = -\frac{1}{L^2} k(k+4)\, Y^{I}$$

$I = (k, \vec{m})$ multiindex.

(equivalent to $\Delta_{S^2} Y_{\ell,m} = -\frac{1}{r^2}\ell(\ell+1) Y_{\ell m}$ in QM)

The action of Δ on the other Ys can be obtained by brute force or (better) by group theory (various casimirs).

$$\Delta Y^{I}_{[\alpha\beta]} = -\frac{1}{L^2}(k+2)^2\, Y^{I}_{[\alpha\beta]}$$

$$\Delta Y^{I}_{\alpha} = -\frac{1}{L^2}(k+1)(k+3)\, Y^{I}_{\alpha}$$

$$\Delta Y^{I}_{(\alpha\beta)} = -\frac{1}{L^2}(k^2+4k+8)\, Y^{I}_{(\alpha\beta)} .$$

$$\vdots$$

Now we are left w/ a bunch of linear coupled II order PDE in AdS only that can be brought into a diagonal form yielding all masses

e.g: <u>SCALARS:</u> $m^2 L^2 = \downarrow$

$h^{\alpha}_{\alpha}, a_{\alpha\beta\gamma\delta}$	$\begin{cases}(k+2)(k-2)\\(k+4)(k+8)\end{cases}$
B	$k(k+4)$
$h_{(\alpha\beta)}$	$(k+2)(k+6)$
$A_{\alpha\beta}$	$\begin{cases}(k-1)(k+3)\\(k+3)(k+7)\end{cases}$

For convenience I shifted the formula so that $k \geq 0$ <u>everywhere</u>

Also noteworthy is the spin 2 and (one of the) spin 1 KK towers:

SPIN 2: $h^{1}_{\mu\nu}$: $m^2L^2 = k(k+4)$ $k \geq 0$

SPIN 1 : $h_{\alpha\mu}, a_{\mu\alpha\beta\gamma}$ $m^2L^2 = k(k+2)$ $k \geq 0$.

- It is possible to make a consistent truncation of the spectrum (both fermi & bose) to a set of low laying mass eigenstates and reproduce "gauged $\mathcal{N}=8$ $d=5$ SUGRA".

There is a PERFECT MATCH.

Let's look at the scalars:

m^2L^2	$\Delta = 2+\sqrt{4+m^2L^2}$	$\mathcal{O}$
$(k-2)(k+2)$	$k+2$	$\mathcal{O}_{k+2} \sim tr(X^{\{i_1} \ldots X^{i_{k+2}\}})$ CHIRAL PRIMARY
$(k-1)(k+3)$	$k+3$	$Q^2\mathcal{O}_{k+2} \sim tr\, \lambda^{\alpha}\lambda_{\alpha} X^k$
$k(k+4)$	$k+4$	$Q^4\mathcal{O}_{k+2} \sim tr\, f^{\alpha\beta} f_{\alpha\beta} X^k$
$(k+2)(k+6)$	$k+6$	$Q^2\bar{Q}^2\mathcal{O}_{k+4} \sim tr\, \lambda^{\alpha}\lambda_{\alpha}\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} X^k$
$(k+3)(k+7)$	$k+7$	$Q^4\bar{Q}^2\mathcal{O}_{k+4} \sim tr\, f^{\alpha\beta} f_{\alpha\beta}\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} X$
$(k+4)(k+8)$	$k+8$	$Q^4\bar{Q}^4\mathcal{O}_{k+4} \sim tr\, f^{\alpha\beta} f_{\alpha\beta}\bar{f}^{\dot{\alpha}\dot{\beta}}\bar{f}_{\dot{\alpha}\dot{\beta}} X$

The last 3 ops require at least the chiral primary $\mathcal{O}_4 \sim tr\, X^{\{i_1}X^{i_2}X^{i_3}X^{i_4\}}$ or otherwis they would vanish.

One could do more :

- One could check that the SU(4) quantum #'s work out as well.

- Note that the matching goes beyond susy. This can easily be seen because the spectrum of chiral primaries depends on which gauge group I pick and the IIB side does not !.

- Two more notable sequences are

SPIN 2 $m^2L^2 = k(k+4)$ for which the "massless" $(k=0)$ mode has $\Delta = 4$ and corresponds, in the bdry theory, to the STRESS ENERGY tensor $T_{\mu\nu} = \mathrm{tr}(F_{\mu\rho}F_\nu{}^\rho + \dots)$.

SPIN 1 $m^2L^2 = k(k+2)$ for which the massless $(k=0)$ mode has $\Delta = 3$ and corresponds to the R-Symmetry current

$$J^\mu = \mathrm{tr}(\bar{\lambda}\sigma^\mu\lambda + \dots)$$

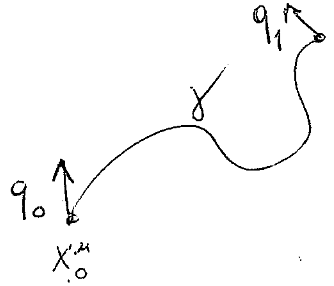
WILSON LOOP

Short introduction :

Consider a (say SU(N)) non abelian gauge field defined in a region of space-time $\mathbb{R}^{3,1}$. Think of it classically at first.

Suppose I take a "quark" q_0 at point $(\vec{x}_0, t_0) = X_0$ (That is a vector pointing in some direction in color space) and I want to "parallel transport" it along a curve,

$$\gamma : \quad \tau \in [0,1] \longrightarrow X^\mu(\tau) \qquad \begin{matrix} X^\mu(0) = X_0^\mu \\ X^\mu(1) = X_1^\mu \end{matrix}$$



This can be accomplished by finding a solution to : $\begin{cases} D_\tau q(\tau) = 0 \\ q(0) = q_0 \end{cases}$ and setting $q_1 = q(1)$.

where $D_\tau = \partial_\tau - i\dot{X}^\mu(\tau)A_\mu(X(\tau))$ (The "−" is conventional, I can reverse it by $\tau \to 1-\tau$)

If I think of τ as "time" I can think of $+i\dot{x}A_\mu$ as the "hamiltonian" and q as a "wave function".

Equivalently I can set $g(\tau) = W(\tau)\, \psi_0$ and solve : $\begin{cases} D_\tau W(\tau) = 0 \\ W(0) = \mathbb{1} . \end{cases}$

Just in the same way as I solve for the time evolution operator in QM. Here the concept of "time order" is of course replaced by "PATH ORDER" : $W(\tau) = P\, e^{+i\int_0^\tau d\tau\, \dot{x}^\mu(\tau) A_\mu(x(\tau))}$

Or, more concisely:

$$W(x_1, x_0; \gamma) = P\, e^{+i\int_\gamma dx^\mu A_\mu}$$

<u>EXERCISE</u> Show that, under a gauge transformation $U(x)$

$$W(x_1, x_0; \gamma) \to U(x_1)\, W(x_1, x_0; \gamma)\, U^\dagger(x_0).$$

Hint to do this, assume that W satisfies $D_\tau W$ and show that the gauge transformed object obeys the gauge transformed eq.

An object like $W(x_1, x_0; \gamma)$ is useful e.g. because it allows the making of (NON LOCAL) gauge inv. ops such as e.g

$$\mathrm{tr}\left(F_{\mu\nu}(x)\, W(x,y,\gamma)\, F^{\mu\nu}(y)\, W(y,x,\gamma')\right)$$

If I want to make a GAUGE INVARIANT object I simply take a CLOSED LOOP and take the TRACE:

$$W(\gamma) = \mathrm{tr}\left(P\, e^{+i\oint_\gamma dx^\mu A_\mu}\right)$$

This is still a classical object depending on the field A_μ which we thought of as a classical matrix so far.

The WILSON LOOP is the vev of this object:

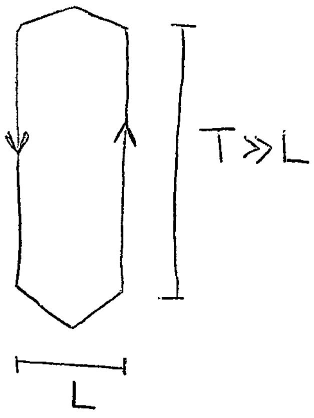
$$\langle W(\gamma) \rangle = \int \mathcal{D}A_\mu \cdots\, e^{-S_{YM}}\, W(\gamma) .$$

and it is a FUNCTIONAL of $\gamma : S^1 \to \mathbb{R}^{3,1}$ ONLY.

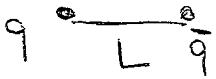
•) The expectation value of W DEPENDS on the REPRESENTATION in which the matrix A_μ (and thus W itself) is taken.

•) It is an important "ORDER PARAMETER" for ANY gauge theory (not just $\mathcal{N}=4$!).

Consider the standard configuration:
(usually in Euclidean space)



This can be thought as putting a $q, \bar{q}$ pair at a distance L and computing the interaction potential and $\int$ over time



V(L).

Thus $\langle W(\square)\rangle \sim e^{-V(L)\cdot T}$ (in Minkowski $e^{iV(L)T}$)

If the charges are <u>confined</u>: $V(L) \sim \sigma L$
(σ is called the "STRING TENSION")
and $\langle W\rangle \sim e^{-\sigma \times \text{Area}}$.
This happens, eg. in pure YM w/ $q \in \square$
(But NOT for $q \in \text{Adj}$ since the gluons can screen the charge).

If there is no confinement <u>or</u> it is more advantageous to create $q\bar{q}$ pairs from the vacuum (they must be present in the theory, of course)

Then: $V(L) \sim V_0$ and
$\langle W\rangle \sim e^{-V_0 \frac{1}{2} \text{Perimeter}}$ (Where Perimeter $= 2T + 2L \approx 2$

A third possibility is that the theory is <u>CONFORMAL</u> (eg $\mathcal{N}=4$) Then $V(L) \sim \frac{1}{L}$ and
$\langle W\rangle \sim e^{-f(\lambda)\cdot T/L}$
where $f(\lambda)$ is a funct. of the coupling constant which is now a parameter of the theory ( ) called it λ because) think of the large N limit, but it is true in general).

The T/L dependence is forced by conformal invariance, since there can be dimensionfull parameters such as V_0 or σ.

Notice that for a circular loop ⓡ there is only one dimensionfull paramet and the $\langle W\rangle$ of $\mathcal{N}=4$ cannot depend on i

$\therefore \langle W(\bigcirc)\rangle \sim e^{-f(\lambda)\cdot 1}$

In an $\mathcal{N}=4$ theory one can also couple to the scalar fields.

From the 10 dim point of view one has

$$\mathrm{tr}P\, e^{i\oint A_M \dot{Z}^M d\tau} = \mathrm{tr}P\, e^{i\oint\left(A_\mu(x,\cancel{y})\dot{x}^\mu + X_i(x,\cancel{y})\dot{y}^i\right)d\tau}$$

no y's.

- in Minkowski space. $\begin{pmatrix} Z^M = x^\mu, y^i \\ A_M = A_\mu, X_i \end{pmatrix}$.
- $$\delta_\epsilon A_M = \left\{ \begin{array}{l} \delta_\epsilon A_\mu = \bar\Psi \Gamma_\mu \epsilon \\ \delta_\epsilon X_i = \bar\Psi \Gamma_i \epsilon \end{array} \right\} \equiv \underset{\uparrow}{\bar\Psi}\, \Gamma_M \epsilon$$

10 dim spinor.

Then: $\delta_\epsilon W = \mathrm{tr}P\, i\oint d\tau\, \bar\Psi\left(\Gamma_\mu \dot{x}^\mu + \Gamma_i \dot{y}^i\right)\epsilon\, e^{i\oint A_M d\dot{Z}^M}$

- To preserve some susy we need
- $(\Gamma_\mu \dot{x}^\mu + \Gamma_i \dot{y}^i)\epsilon = 0$ for some ϵ.

Squaring it we get the condition $\dot{x}^{\mu 2} + \dot{y}^{i2} = 0$ which has solutions in Minkowski but not Euclidean space.

To go to Euclidean space and still ☺ have a chance of getting a susy loop we must change the definition to:

$$W = \mathrm{tr}P\, e^{\oint\left(iA_\mu \dot{x}^\mu + \underset{\uparrow}{X_i} \dot{y}^i\right)ds}.$$

No i!

- which gives $|\dot{x}|^2 = |\dot{y}|^2$ which) can
- solve LOCALLY (that is for $\tau = \tau_0$ fixed).

by taking $\dot{y}^i(\tau_0) = \theta^i(\tau_0)\,|\dot{x}^i(\tau_0)|$

where $|\theta^i|^2 = 1$ defines a pt. on a $S5$.

However, the susy preserved at different

- points might be different and this
- is not an entirely satisfactory state

off affairs (see Zarembo hep-th/0205160

EXERCISE Show that for $\theta^i \equiv$ const. the only truly global susy solution is $\gamma =$ straight lin

One can ~~do~~ better by setting

$\theta^i = M^i_\mu \frac{\dot{x}^\mu}{|x|}$ for some CONSTANT 4×6 matrix M obeying $\sum_{i=1}^{6} M^i_\mu M^i_\nu = \delta_{\mu\nu}$.

This is a bit redundant, one might just write $\dot{y}^i = M^i_\mu \dot{x}^\mu$

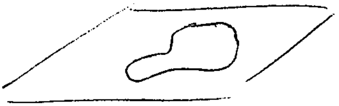
- The GLOBAL susy now becomes (dropping
- the overall $\dot{x}^\mu$ factor):

$$(\Gamma_\mu - i M^i_\mu \Gamma_i)\epsilon = 0$$

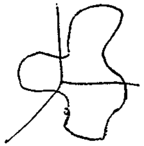
which is an algebraic eq. whose sol's can be classified. Typically this

- depend on how many dimensions the
- loop "moves" in

eg.



2d => $\frac{1}{4}$ susy



3d => $\frac{1}{8}$ susy

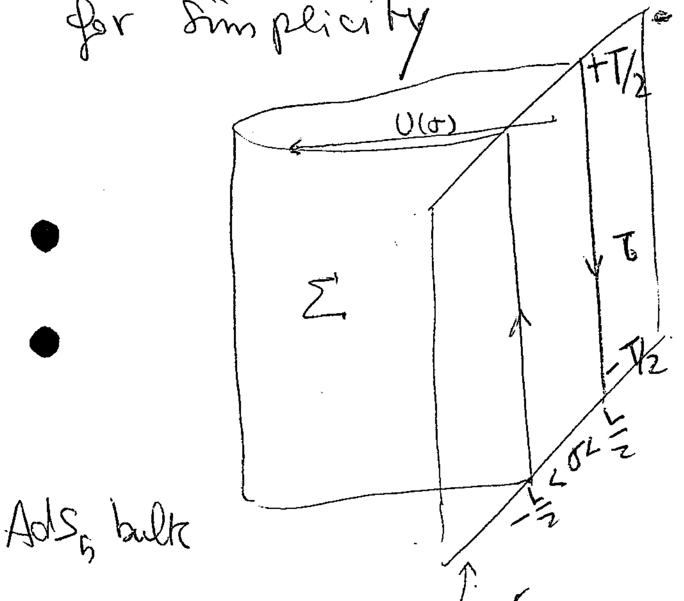
(generically (all 4d) => $\frac{1}{16}$ susy)

The AdS/CFT PRESCRIPTION is very natural.

- "Energy" coordinate $U = r/\alpha'$
- $ds^2 = \alpha' \left(\frac{U^2}{R^2}(dx^{\mu 2}) + R^2 \frac{dU^2}{U^2} + R^2 d\Omega_5^2 \right)$

- where $R = \frac{L}{\sqrt{\alpha'}} = \lambda^{\frac{1}{4}}$ is the radius
- in α' units.

- Consider $\theta^i = $ const. (drop all S^5 dependen
for simplicity



$U(\pm\frac{L}{2}) = U_\infty$

Notice that by SYMMETRY $U(\sigma, \cancel{\tau})$

$U_{min} = U(0) \equiv U$

$U = U_\infty$ (Bdry at $U_\infty \to \infty$)

$$S_{WS} = \frac{1}{2\pi\alpha'} \int_{-T/2}^{+T/2} d\tau \int_{-\frac{L}{2}}^{L/2} d\sigma \sqrt{\det(G_{MN} \partial_\alpha X^M \partial_\beta X^N)}$$

We define $\langle W \rangle = e^{-S_{ws}}$.

where S_{ws} is the value of the Nambu-Goto action EVALUATED AT the minimal surface Σ bounded by the loop.

There will be two trivial divergences:

°) $S_{ws} \propto T \to \infty$ simply because of time translation

°) $S_{ws} \propto U_\infty \to \infty$ but this is the "bare mass" of the quark that can be easily subtracted out.

Thus we take for our world sheet:

$$X^0 = \tau$$

$$U = U(\sigma)$$

$$X^1 = \sigma$$

The induced metric whose determinant enters the NG action:

$$h_{\alpha\beta} = G_{MN}\, \partial_\alpha X^M\, \partial_\beta X^N$$

has the following components:

$$h_{\tau\tau} = G_{00}\, \partial_\tau X^0 \partial_\tau X^0 = G_{00} = \alpha' \frac{U^2}{R^2}$$

$$h_{\sigma\tau} = h_{\tau\sigma} = 0$$

$$h_{\sigma\sigma} = G_{11}\, \partial_\sigma X^1 \partial_\sigma X^1 + G_{UU}\, \partial_\sigma U \partial_\sigma U = \alpha'\left(\frac{U^2}{R^2} + \frac{R^2}{U^2} U'^2\right)$$

$$\Rightarrow \sqrt{\det h} = \alpha' \left(U'^2 + \frac{U^4}{R^4}\right)^{\frac{1}{2}}$$

$$\Rightarrow S_{ws} = \frac{1}{2\pi} \int_{-\frac{T}{2}}^{+\frac{T}{2}} d\tau \int_{-\frac{L}{2}}^{+\frac{L}{2}} d\sigma \sqrt{U'^2 + \frac{U^4}{R^4}} =$$

$$= \frac{T}{2\pi} \int_{-\frac{L}{2}}^{+\frac{L}{2}} d\sigma \sqrt{U'^2 + \frac{U^4}{R^4}}$$

INDEPENDEN on α' as it should in a CFT (note that $R^2 \propto \sqrt{\lambda}$ is dimensionless).

To minimize, we treat $\sqrt{U'^2+\frac{U^4}{R^4}}$ as a "Lagrangian". $\Pi_U = \frac{\partial \mathcal{L}}{\partial U'} = \frac{U'}{\sqrt{U'^2+\frac{U^4}{R^4}}}$

Conserved "Hamiltonian": $H = \Pi_U U' - \mathcal{L} =$

$$= \frac{U'^2}{\sqrt{U'^2+\frac{U^4}{R^4}}} - \sqrt{U'^2+\frac{U^4}{R^4}} = \frac{U^4/R^4}{\sqrt{U'^2+\frac{U^4}{R^4}}}$$

This value will be constant through the world sheet and in particular it will be the same at $\sigma=0$ where, by symmetry $U'(0)=0$ and we have chosen $U(0)\equiv U_0$.

$$\therefore \frac{U^4/R^4}{\sqrt{U'^2+U^4/R^4}} = U_0^2/R^2$$

$$\Rightarrow \frac{dU}{d\sigma} = \frac{U^2}{R^2}\sqrt{\frac{U^4}{U_0^4}-1}$$

$$\Rightarrow \int_0^\sigma d\tilde\sigma = \int_{U_0}^{U} \frac{R^2\, d\tilde U}{\tilde U^2\sqrt{\frac{\tilde U^4}{U_0^4}-1}} \qquad \text{// set } \tilde U = U_0\cdot y$$

$$\Rightarrow \sigma = \frac{R^2}{U_0}\cdot\int_1^{U/U_0}\frac{dy}{y^2\sqrt{y^4-1}}.$$

Where U_0 is determined by setting $\sigma = \frac{L}{2}$ and $\frac{U}{U_0} = \frac{U_\infty}{U_0} \simeq \infty$.

$$\frac{L}{2} = \frac{R^2}{U_0}\int_1^\infty \frac{dy}{y^2\sqrt{y^4-1}} \sim \frac{R^2}{U_0}$$

(It makes sense, the wider the loop, the deeper will the w.s. penetrate into AdS_5

$$U_0 \sim \frac{R^2}{L} \sim \frac{\sqrt{\lambda}}{L}$$

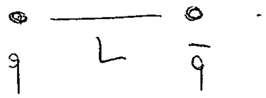
Plugging the solution into the NG action:

$$S_{ws} = \frac{T}{\cancel{2}\pi}\cdot\cancel{2}\int_0^{L/2} d\sigma\sqrt{U'^2+\frac{U^4}{R^4}} =$$

$$= \frac{T}{\pi}\int_{U_0}^{U_\infty} \underbrace{dU\,\frac{\cancel{R^2}}{U^2\sqrt{\frac{U^4}{U_0^4}-1}}}_{d\sigma}\cdot\underbrace{\frac{U^4}{U_0^2\cancel{R^2}}}_{\mathcal{L}}. \qquad \text{// set } U = U_0 y$$

$$= \frac{T}{\pi}\cdot U_0\cdot\int_1^{U_\infty/U_0}\frac{dy\cdot y^2}{\sqrt{y^4-1}}$$

The integral is linearly divergent in $U_\infty \to \infty$ but that's ok. It represents the large mass of the two quarks and should not be included into the $V(L)$ representing the energy of the configuration



Subtracting it:

$$S_{WS} \to \frac{T}{\pi} U_0 \cdot \int_1^{U_\infty/U_0} dy \left(\frac{y^2}{\sqrt{y^4-1}} - 1 \right) \to$$

$$\frac{T U_0}{\pi} \cdot \int_1^\infty dy \left(\frac{y^2}{\sqrt{y^4-1}} - 1 \right) \sim$$

$$\sim T U_0 \sim \sqrt{\lambda} \cdot \frac{T}{L} .$$

EXERCISE Do the integrals and find the numerical contants.

Final result : $\langle W(\square) \rangle \sim e^{-\text{Const.} \sqrt{\lambda} \frac{T}{L}}$

- Agrees w/ CFT prediction
- Extra prediction : $\sqrt{\lambda}$ behavior @ large λ.