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Cylindrical nonlinear Schrödinger equation *versus* cylindrical Korteweg-de Vries equation

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Abstract. A correspondence between the family of cylindrical nonlinear Schrödinger (cNLS) equations and the one of cylindrical Korteweg-de Vries (cKdV) equations is constructed. It associates non stationary solutions of the first family with the ones of the second family. This is done by using a correspondence, recently found, between the families of generalized NLS equation and generalized KdV equation, and their solutions in the form of travelling waves, respectively. In particular, non-stationary soliton-like solutions of the cNLS equation can be associated with non-stationary soliton-like solutions of cKdV equation.

Keywords: nonlinear partial differential equations, cylindrical NLS equations, cylindrical KdV equations, solitons, Madelung's fluid description
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1. INTRODUCTION

Nonlinear wave propagation concerns with a number of problems in many areas of physics that are modelled by appropriate nonlinear partial differential (or integro-differential) equations, ranging from microphysics to macrophysics [1, 2, 3]. Within fluid or plasma physics–based models, two families of nonlinear equations are the most recurrent. They are known as the Korteweg-de Vries (KdV) and the nonlinear Schrödinger (NLS) equations. Typically, they describe very different kind of processes and physical mechanisms, although often some kinds of KdV and NLS equations are together involved in complex nonlinear dynamics ruled by two or more timescales [4].

The general forms of KdV and NLS equations relevant for this paper are, respectively, the following:

$$\frac{\partial u}{\partial \tau} - G[u]\frac{\partial u}{\partial \xi} + a'\frac{\partial^3 u}{\partial \xi^3} + \frac{p}{\tau}u = 0 \tag{1}$$

and

$$i\frac{\partial \Psi}{\partial \tau} + a\frac{\partial^2 \Psi}{\partial \xi^2} - U\left[|\Psi|^2\right]\Psi + i\frac{q}{\tau}\Psi = 0\,, \tag{2}$$

where $G[u]$ and $U[|\Psi|^2]$ are real and hermitian functionals, respectively, of their arguments, a, a', p and q are real constants. Setting $p = q = 0$, eq.s (1) and (2) reduces to the standard forms of generalized Korteweg-de Vries (gKdV) and generalized nonlinear Schrödinger (gNLS) equations, respectively, i.e.,

$$\frac{\partial u}{\partial \tau} - G[u]\frac{\partial u}{\partial \xi} + a'\frac{\partial^3 u}{\partial \xi^3} = 0 \tag{3}$$

and

$$i\frac{\partial \Psi}{\partial \tau} + a\frac{\partial^2 \Psi}{\partial \xi^2} - U\left[|\Psi|^2\right]\Psi = 0\,. \tag{4}$$

When, in particular, $G[u] \propto u^\gamma$ and $U[|\Psi|^2] \propto |\Psi|^{2\beta}$, with $\gamma, \beta > 0$, eq.s (3) and (4) reduces to the standard forms of modified Korteweg-de Vries (mKdV) and modified nonlinear Schrödinger (mNLS) equations, respectively, i.e.,

$$\frac{\partial u}{\partial \tau} + b'u^\gamma\frac{\partial u}{\partial \xi} + a'\frac{\partial^3 u}{\partial \xi^3} = 0 \tag{5}$$

and

$$i\frac{\partial \Psi}{\partial \tau} + a\frac{\partial^2 \Psi}{\partial \xi^2} + b|\Psi|^{2\beta}\Psi = 0\,, \tag{6}$$

where b and b' are real constants, as well. They are encountered in a number of plasma physics problems, especially in the case $\gamma, \beta = 1, 2$ (f.i., see Ref. [5, 6, 7, 8, 9])

Other important classes of KdV and NLS equations are obtained from (3) and (4) setting $G[u] = -b'u$, $U[|\Psi|^2] = -b|\Psi|^2$, and $p = q = 1/2$. They are called cylindrical Korteweg-de Vries (cKdV) and cylindrical nonlinear Schrödinger (cNLS) equations, respectively, i.e.,

$$\frac{\partial u}{\partial \tau} + b'u\frac{\partial u}{\partial \xi} + a'\frac{\partial^3 u}{\partial \xi^3} + \frac{1}{2\tau}u = 0 \tag{7}$$

and

$$i\frac{\partial \Psi}{\partial \tau} + a\frac{\partial^2 \Psi}{\partial \xi^2} + b|\Psi|^2\Psi + \frac{i}{2\tau}\Psi = 0\,. \tag{8}$$

They model several important processes in laboratory, space and astrophysical environments [10, 11, 12, 13], especially in the presence of dust-contaminated plasmas [14, 15].

Recently, a direct connection between eq.s (3) and (4) and their solutions in the form of travelling waves has been found by employing the Madelung's fluid representation of the gNLS [16, 17].

Let us denote with $g\mathscr{S} = \{u \geq 0\}$ and $g\mathscr{E} = \{\Psi\}$ the sets of all non-negative solutions of the gKdV equation and the set of all the envelope solutions of the gNLS equation, respectively, in the form of travelling waves. Provided that $a' \propto a^2$ and the functionals U and G are related by the equation

$$G[\rho] \;=\; \rho\frac{dU[\rho]}{d\rho} \;+\; 2U[\rho]\,, \tag{9}$$

one can prove, according to Ref.s [16], that the following correspondences between $g\mathscr{S}$ and $g\mathscr{E}$ hold:

$$g\mathscr{F} \ : \ \Psi \in g\mathscr{E} \rightarrow u \in g\mathscr{S} \qquad (10)$$
$$u(\xi - V_0\tau) \ = \ g\mathscr{F}\left[\Psi(\xi,\tau)\right] \ = \ |\Psi(\xi,\tau)|^2 ,$$

and

$$g\mathscr{H} \ : u \in g\mathscr{S} \rightarrow \Psi \in g\mathscr{E} \qquad (11)$$
$$\Psi(\xi,\tau) = g\mathscr{H}\left[u\right] = \sqrt{u(\xi - V_0\tau)}\exp\left\{\frac{i}{2a}\left[\phi_0 - \left(c_0 + V_0^2\right)\tau + V_0\xi + Q(\eta)\right]\right\},$$

where $\eta \equiv \xi - V_0\tau$ (V_0 being a real constant) and $Q(\eta) = A_0 \int u^{-1}(\eta)\, d\eta$. Furthermore, A_0, ϕ_0 and c_0 are arbitrary real constants (for more details, see Ref. [16]).

The above correspondences have been fruitfully used to find travelling waves in the form of solitons or periodic waves for several mKdV and mNLS equations [16, 17, 18] and for some kind of derivative NLS equations [19, 20].

In this paper, we want to find similar correspondences between the families of cKdV and cNLS equations. This is done by using $g\mathscr{F}$ and $g\mathscr{H}$ in the special case of the standard KdV and NLS equations. The paper is organized as follows. In the next section, we find the correspondences between the cNLS and the standard NLS equations that allow us to express an arbitrary solution of one family in terms of a solution of the other one. Similarly, in section 3, we find the correspondences between the cKdV and the standard KdV equations that allow us to express an arbitrary solution of one family in terms of a solution of the other one. Then, in section 4, the correspondences between the standard KdV ($\gamma = 1$ in eq. (5)) and standard NLS equations ($\beta = 1$ in eq. (6)) are obtained by using $g\mathscr{F}$ and $g\mathscr{H}$. Subsequently, in section 5, a direct link between cKdV and the cNLS equations and their solutions is easily found on the basis of the results presented in the previous sections. In section 6, a link between bright and dark cylindrical solitons of cKdV and cNLS equations is given and numerically evaluated. Finally, conclusions and remarks are presented in section 7.

2. CONNECTION BETWEEN NLS AND cNLS EQUATIONS

Let us consider the following cNLS and NLS equations respectively:

$$i\frac{\partial\psi}{\partial t} + a\frac{\partial^2\psi}{\partial x^2} + b\,|\psi|^2\psi + \frac{i}{2t}\psi = 0 \qquad (12)$$

and

$$i\frac{\partial\Psi}{\partial\tau} + a\frac{\partial^2\Psi}{\partial\xi^2} + b\,|\Psi|^2\Psi = 0\,. \qquad (13)$$

We assume that, in general, t and τ (or x and ξ) are not expressed by the same measure unities; alternatively, we may assume that they are all dimensionless quantities.

It is easy to see that eq.s (12) and (13) are related by the following transformations:

$$\mathscr{R}:\begin{cases} x=\xi/\tau,\ \ t=-1/\tau \\ \Psi(\xi,\tau)=\tau^{-1}\exp\left(i\xi^2/4a\tau\right)\ \psi\left(x=\xi/\tau,\ t=-1/\tau\right)\end{cases} \tag{14}$$

and

$$\mathscr{R}':\begin{cases} \xi=-x/t,\ \ \tau=-1/t \\ \psi(x,t)=-t^{-1}\exp\left(ix^2/4at\right)\ \Psi\left(\xi=-x/t,\ \tau=-1/t\right).\end{cases} \tag{15}$$

Note that $\mathscr{R}$ transforms (12) into (13) whilst $\mathscr{R}'$ transforms (13) into (12). A transformation similar to $\mathscr{R}'$ have been considered in Ref. [13].

3. CONNECTION BETWEEN KdV AND cKdV EQUATIONS

Let us now consider the following KdV and cKdV equations respectively:

$$\frac{\partial u}{\partial \tau}+b'u\frac{\partial u}{\partial \xi}+a'\frac{\partial^3 u}{\partial \xi^3}=0 \tag{16}$$

and

$$\frac{\partial v}{\partial s}+b'v\frac{\partial v}{\partial y}+a'\frac{\partial^3 v}{\partial y^3}+\frac{v}{2s}=0\,, \tag{17}$$

For the variables τ and s (or ξ and y) we take an assumption similar to the one taken in the previuos section. We, in fact, assume that in general they are not expressed by the same unities or, alternatively, they are all dimensionless.

It is easy to see that eq.s (16) and (17) are related by the following transformations:

$$\mathscr{T}:\begin{cases} \xi=s^{-1/2}y,\ \ \tau=-2s^{-1/2} \\ v(y,s)=s^{-1}\left[u\left(\xi=s^{-1/2}y,\tau=-2s^{1/2}\right)+y/2b'\right]\end{cases} \tag{18}$$

and

$$\mathscr{T}':\begin{cases} y=-2\xi/\tau,\ \ s=4/\tau^2 \\ u(\xi,\tau)=\tau^{-1}\left[4\tau^{-1}v\left(y=-2\xi,s=4/\tau^2\right)+\xi/b'\right]\end{cases}. \tag{19}$$

Note that $\mathscr{T}$ transforms (16) into (17) whilst $\mathscr{T}'$ transforms (17) into (16). A transformation similar to $\mathscr{T}'$ have been first introduced in Ref. [22].

4. CONNECTION BETWEEN THE STANDARD NLS EQUATION AND THE STANDARD KdV EQUATION

Let us now apply eq.s (10) and (11) to the special case of standard NLS and KdV equations, namely eq.s (13) and (16). It is easily seen that the envelope solutions $\Psi(\xi,\tau)$

in the form of travelling waves of (13) correspond to travelling wave solutions of (16) by means of the transformation:

$$\begin{cases} \mathscr{F} : \Psi \in \mathscr{E} \to u = |\Psi|^2 \in \mathscr{S}, \\ G[u] = \left[6ab|\Psi|^2\right]_{|\Psi|^2=u}, \end{cases} \tag{20}$$

and, viceversa, non-negative travelling solutions $u(\xi - V_0\tau)$ of (16) correspond to envelope solutions of (13) in the form of travelling waves by means of the transformation:

$$\begin{cases} \mathscr{H} : u \in \mathscr{S} \to \Psi = \sqrt{u}\exp\left\{\frac{i}{2a}\left[\phi_o + V_o\xi - \left(E + V_0^2/2\right)\tau + A_0\int d\eta/u(\eta)\right]\right\} \in \mathscr{E} \\ U[|\Psi|^2] = [(b'/3)u]_{u=|\Psi|^2}, \end{cases} \tag{21}$$

where $\mathscr{S}$ and $\mathscr{E}$ are the sets of all non-negative travelling wave solutions of KdV and all travelling wave envelope solutions of NLS equations, respectively. Additionally, $u = u(\eta)$, $\eta = \xi - V_0\tau$ and E is the energy eingenvalue corresponding to the following eigenvalue problem in the η-space, i.e.,

$$\begin{aligned} -2a^2\frac{d^2u^{1/2}}{d\eta^2} + W[u]\,u^{1/2} &= Eu^{1/2} \\ W[u] &= -2abu + \frac{A_0^2}{2u^2}, \end{aligned} \tag{22}$$

with A_0 arbitrary real constant. Note that the coefficients a, b, a', b' are linked each other by

$$a' = -a^2V_0/2E,\; b' = -3abV_0/E\,. \tag{23}$$

Note that eq.s (23) imnply the following simple relationship:

$$\frac{b'}{a'} = \frac{6b}{a}\,. \tag{24}$$

Localized solutions, in the form of bright and dark solitons, are possible for $A_0 = 0$, whilst grey solitons are determined for $A_0 \neq 0$ [17].

5. LINK BETWEEN cNLS EQUATION AND cKdV EQUATION

The results presented in the previous sections allow us to construct a non-stationary solution of the cKdV (cNLS) equation starting from a non-stationary envelope solution of the cNLS (cKdV) equation. To this end, let us denote with $c\mathscr{E}$ the set of all the envelope solutions of the cNLS equation. Let us also denote with $c\mathscr{S}$ the set of all the solutions of the cKdV equation. For instance, by using the correspondences $\mathscr{R}$, $\mathscr{F}$ and $\mathscr{T}$, one can associate a solution $\psi(x,t) \in c\mathscr{E}$ with a solution $v(y,s) \in c\mathscr{S}$, i.e.,

$$v = \mathscr{T} \circ \mathscr{F} \circ \mathscr{R}\,\psi\,. \tag{25}$$

It is easy to see that eq. (25) leads to the following explicit relationship between $\psi(x,t)$ and $v(y,s)$:

$$v(y,s) = \left| \frac{1}{2}\psi\left(x = -\frac{1}{2}y,\, t = \frac{1}{2}s^{1/2}\right)\right|^2 + \frac{y}{2b's}\,. \tag{26}$$

Furthermore, the composition of $\mathscr{T}'$, $\mathscr{H}$ and $\mathscr{R}'$ allows us to associate a solution $v(y,s) \in c\mathscr{S}$ with a solution $\psi(x,t) \in c\mathscr{E}$, i.e.,

$$\psi = \mathscr{R}' \circ \mathscr{H} \circ \mathscr{T}'\, v\,, \tag{27}$$

which leads to the explicit expression:

$$\psi(x,t) = \sqrt{4v\,(y = -2x,\, s = 4t^2) - \frac{E}{3abV_0}\frac{x}{t^2}}\; \exp\left\{\frac{i}{2a}\left[\Phi_0 + \frac{(x-V_0)^2 + 2E}{2t}\right] + Q(x,t)\right\}, \tag{28}$$

where Φ_0 is an arbitrary real constant and $Q(x,t) = A_0\,[\int d\eta/u(\eta)]_{\xi=-x/t,\,\tau=-1/t}$.

6. CYLINDRICAL SOLITONS

In the special case $A_0 = 0$, one can find travelling wave solutions $\Psi(\xi,\tau)$ and $u(\xi,\tau)$ of the standard NLS and KdV equations, respectively, in the form of bright or dark solitons, linked by means of the correspondences $\mathscr{F}$ and $\mathscr{H}$ [17].

(i). For bright solitons:

$$u(\xi,\tau) = u_m\,\mathrm{sech}^2\left[\sqrt{\frac{b'u_m}{12a'}}\,(\xi - V_0\tau)\right], \tag{29}$$

where $u_m > 0$ is the soliton's amplitude, $a'b' > 0$, $V_0 = b'u_m/3$, and

$$\Psi(\xi,\tau) = \sqrt{-\frac{E}{ab}}\,\mathrm{sech}\left[\sqrt{-\frac{E}{2a^2}}\,(\xi - V_0\tau)\right] \exp\left\{\frac{i}{2a}\left[\phi_0 + V_0\xi - \left(E + V_0^2/2\right)\tau\right]\right\}, \tag{30}$$

where $E < 0$ and $ab > 0$. According to the mappings $\mathscr{F}$ and $\mathscr{H}$, $u_m = \sqrt{-E/ab}$. Thus, making use of (18)-(23), it is easily seen that the following non-stationary solutions of the cKdV and cNLS equations hold, respectively:

$$v(y,s) = \frac{1}{b's}\left\{\frac{1}{2}y + b'u_m\,\mathrm{sech}^2\left[\sqrt{\frac{b'u_m}{12a's}}\,(y + 2V_0)\right]\right\}, \tag{31}$$

and

$$\psi(x,t) = \sqrt{-\frac{E}{ab}}\,\frac{1}{t}\,\mathrm{sech}\left[\sqrt{-\frac{E}{2a^2}}\left(\frac{x - V_0}{t}\right)\right] \exp\left\{\frac{i}{2a}\left[\Phi_0 + \frac{(x-V_0)^2 + 2E}{2t}\right]\right\}. \tag{32}$$

(ii). For dark solitons:

$$u(\xi,\tau) = u_m \tanh^2\left[\sqrt{-\frac{b'u_m}{12a'}}\,(\xi - V_0\tau)\right], \tag{33}$$

where $u_m > 0$, $a'b' < 0$, $V_0 = 2b'u_m/3$, and

$$\Psi(\xi,\tau) = \sqrt{-\frac{E}{2ab}}\left|\tanh\left[\sqrt{\frac{E}{4a^2}}\,(\xi - V_0\tau)\right]\right| \exp\left\{\frac{i}{2a}\left[\phi_0 + V_0\xi - \left(E + V_0^2/2\right)\tau\right]\right\}, \tag{34}$$

where $E > 0$ and $ab < 0$. According to the mappings $\mathscr{F}$ and $\mathscr{H}$, $u_m = \sqrt{-E/2ab}$. In this case, eq.s (18)-(23) lead to the following non-stationary solutions of the cKdV and cNLS equations, respectively:

$$v(y,s) = \frac{1}{b's}\left\{\frac{1}{2}y + b'u_m \tanh^2\left[\sqrt{-\frac{b'u_m}{12a's}}\,(y + 2V_0)\right]\right\}, \tag{35}$$

and

$$\psi(x,t) = \sqrt{-\frac{E}{2ab}}\,\frac{1}{t}\left|\tanh\left[\sqrt{\frac{E}{4a^2}}\left(\frac{x - V_0}{t}\right)\right]\right| \exp\left\{\frac{i}{2a}\left[\Phi_0 + \frac{(x - V_0)^2 + 2E}{2t}\right]\right\}. \tag{36}$$

Due to the presence of a linear term in y, Eq.s (31) and (35) describe non-stationary "tilted" bright and dark solitons, respectively, whilst eq.s (32) and (36) describe non-stationary bright and dark envelope solitons, respectively. Solutions (31) and (35) of the cKdV equation are linked to solutions (32) and (36), respectively, by means of (26) and (28).

Solutions (31) and (35) reproduces the results given in Ref. [12], whilst (32) is compatible with the one shown in Ref. [13].

Figure 1 displays the s-evolution of the cylindrical bright and dark solitons given by eq.s (31) and (35), respectively. The presence of a tilt at the early stage of the evolution is clearly evident and results in a substantial distortion of the soliton peak amplitudes. As s increases, the solitons flatten.

Figure 2 shows the t-evolution of the modulus of the cylindrical bright and dark solitons (32) and (36), corresponding to (31) and (35), respectively. The amplitude of the solitons decreases as t increases and the profile flattens.

7. CONCLUSIONS AND REMARKS

In this paper, a direct link between solutions of the cKdV and the cNLS equations has been constructed. To this end, suitable connections between the cNLS and the standard NLS equations and the cKdV and the standard KdV equations, respectively, have been

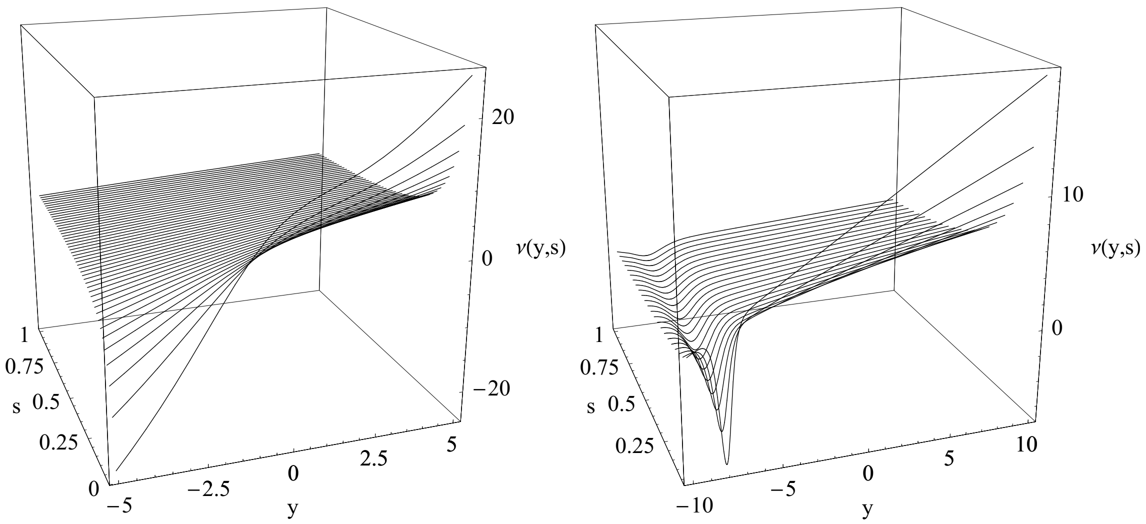

FIGURE 1. Plot of s-evolution of cylindrical bright (left) and dark (right) solitons given by eq.s (31) and (35), respectively. $a' = 1$ (bright), $a' = -1$ (dark), $b' = 6$, $V_0 = 2$ (bright), $V_0 = 4$ (dark), $u_m = 1$.

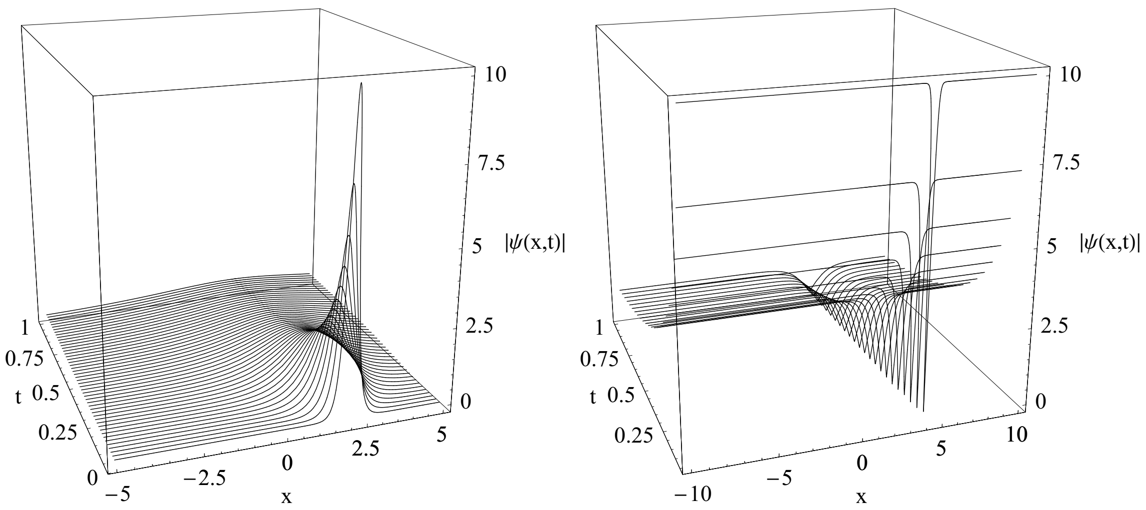

FIGURE 2. Plot of t-evolution of the modulus of cylindrical bright (left) and dark (right) solitons (32) and (36), corresponding to (31) and (35), respectively. $a = 1$ (bright), $a = -1$ (dark), $b = 1$, $V_0 = 2$ (bright), $V_0 = 4$ (dark), $E = -1$ (bright), $E = 2$ (dark).

determined. Then, to complete the construction, a link between the standard KdV and NLS equations, and their travelling wave soltions, have been used. In particular, non-stationary envelope bright and dark soliton-like solutions of the cNLS equation have been linked with non-stationary "tilted" bright and dark soliton-like solutions of the cKdV equation, respectively.

We would like to point out that the crucial role of the above construction is played by the use of $\mathscr{F}$ and $\mathscr{H}$ correspondences that, however, are limited to the cases of travelling waves. It is reasonable (but not yet proven) that this restricts the families of solutions that may be linked by means of correspondences between $c\mathscr{E}$ and $c\mathscr{S}$. Therefore, an effort to investigate on possible extensions of the present construction to wider families of solutions would be important.

Other important investigations finalized to extend the above construction to the case of modified-cylindrical and derivative-cylindrical KdV and NLS equations are under way.

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