The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific and
Cultural Organization

IAEA
International Atomic Energy Agency

1953-34

International Workshop on the Frontiers of Modern Plasma Physics

14 - 25 July 2008

Criteria to define a pair-ion plasma and the role of electrons in nonlinear dynamics.

H. Saleem

Quaid-i-Azam University National Centre for Physics
Islamabad
Pakistan

Criteria to define a pair-ion plasma and the role of electrons in nonlinear dynamics

H. Saleem

1. National Centre for Physics (NCP), Quaid-i-Azam University, Islamabad, Pakistan.

International Workshop on the Frontiers of Modern Plasma Physics 14 – 25 July, 2008 held in

Abdus-Salam International Centre for Theoretical Physics (AS-ICTP), Trieste, Italy

Plan of the Talk

1. *Brief History of the Problem*
2. *Thermal Mode in PI Plasma*
3. *Criteria for Pure PI Plasma*
4. *Some Linear Waves in PI Plasmas*
5. *HM Equation in PI & PIE Plasmas*
6. *KdV-B Equation in PI and PIE Plasmas*
7. *Conclusions*

1. Brief History of the Problem

- Electron-positron $e^{+}-e^{-}$ plasmas are believed to be produced in astrophysical environments.
- For example the pair-production by intense γ - rays in the presence of strong magnetic fields in rotating neutron stars (pulsars). The radiation pulses are speculated to be the result of nonlinear $(e^{-}-e^{+})$ plasma dynamics. The linear and nonlinear waves in $e^{-}-e^{+}$ plasmas have been studied [1,4]. Relativistic and non relativistic pair plasmas are believed to exist in early universe, in quasars and other galaxies.
- Low density $e^{+}-e^{-}$ plasmas have been produced in laboratories as well [5,6].
- Pair plasmas represent a new state of matter with unique thermodynamic properties drastically different from those of ordinary electron-ion (EI) plasmas.

[1]. N. Iwamoto, Phys. Rev. A 39, 4076 (1989); [2] G-P. Zank and R. G. Greaves, Phys. Rev. E51, 6079 (1995).

[3]. P.K. Shukla, N.N. Rao, M.Y.YU, and N.L. Tsintsdze, Phys. Reports 138, 1 (1986).

[4]. A. D. Rogava, S. M. Mahajan and V. I. Berezhiani, Phys. Plasmas 3, 3545 (1996).

[5]. C. M. Surko, M. Leven Thal, and A. Passner, Phys. Rev. Lett. 62, 901 (1989).

[6]. M. Amoretti et.al. Phys. Rev. Lett. 91, 055001 (2003).

- Experimental studies of $e^{-}-e^{+}$ plasmas are complicated due to their short annihilation time. Moreover, the high density $e^{-}-e^{+}$ plasmas are not easy to be produced and confined.
- Therefore attention has been focused on the generation of pair-ion plasmas to investigate the pair plasma behavior.
- The peculiar experimental observations of fullerene $C_{60}^{\pm}$ plasma in 2005 [7] have invoked lot of research interest in this area.
- Several authors tried to explain these experimental observations.
- A series of papers has appeared on linear and nonlinear dynamics of pair – ion plasmas during last a few years [8, 9, 10, 11, 12] and many others.
- The experimental observations showed that the frequency of acoustic wave is larger in pair-ion (PI) plasmas compared to (EI) plasmas [7] at the same temperature.

[7]. W. Oohara, D. Date and R. Hatakeyama, Phys. Rev. Lett. 95, 175003 (2005).
[8]. H. Schamel and A. Luque, New J. Phys. 7, 69 (2005).
[9]. I. Kourakis, A. Esfandyari-Kalegahi, M. Medhipoor, and P.K. Shukla, Phys. Plasmas 13, 052117 (2006).
[10]. J. Vranjes and S. Poedts, Plasma Source Sci. Technol. 14, 485 (2005); J. Vranjes and S. Poedts, Phys. Plasmas 15, 044501 (2008).
[11]. A. Luque, H. Schamel, B. Eliasson, and P.K. Shukla, Plasma Phys. Controlled Fusion 48, 044502 (2006).
[12]. F. Verheest, Phys. Plasmas 13,082301 (2006).

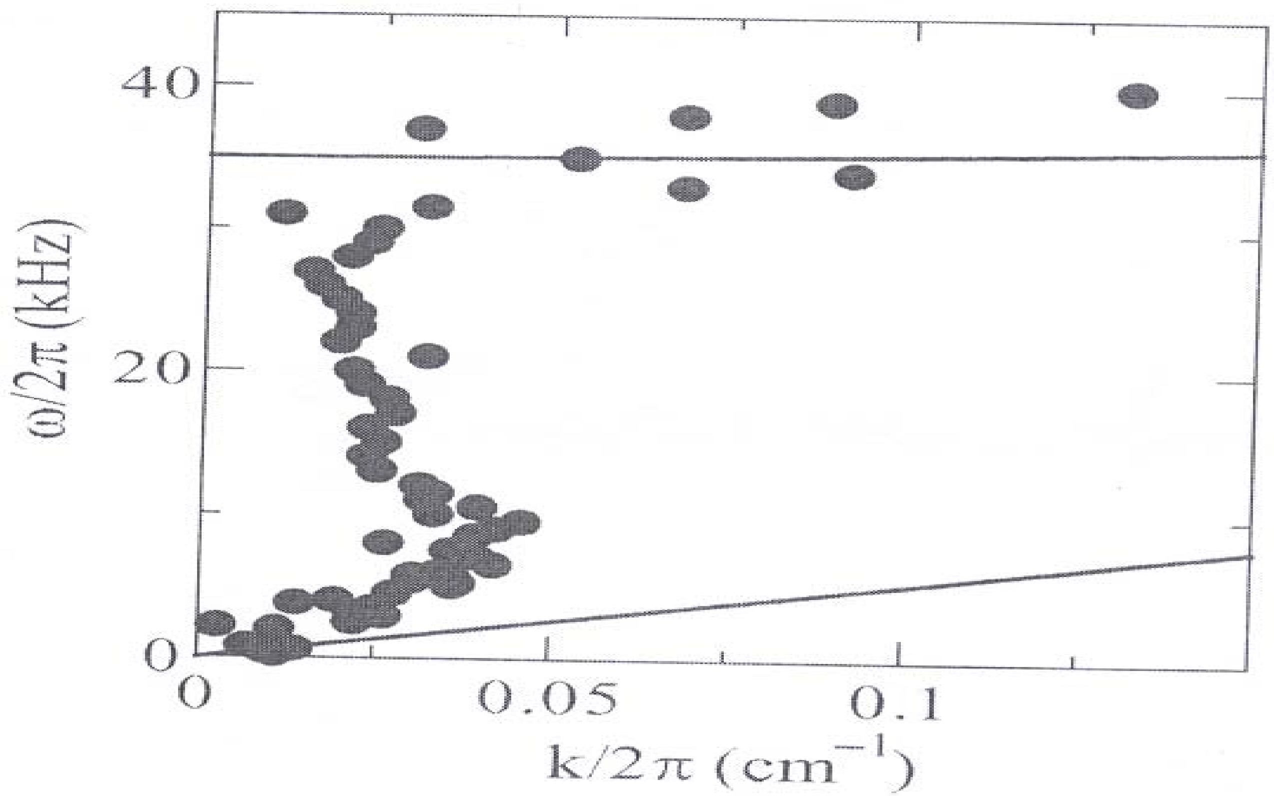

Fig. 1

Fig. 2 of Ref. [7]:

Dispersion relations for electrostatic waves propagating along B-field lines. Solid lines and dots denote results calculated from two-fluid theory and measured experimentally respectively.

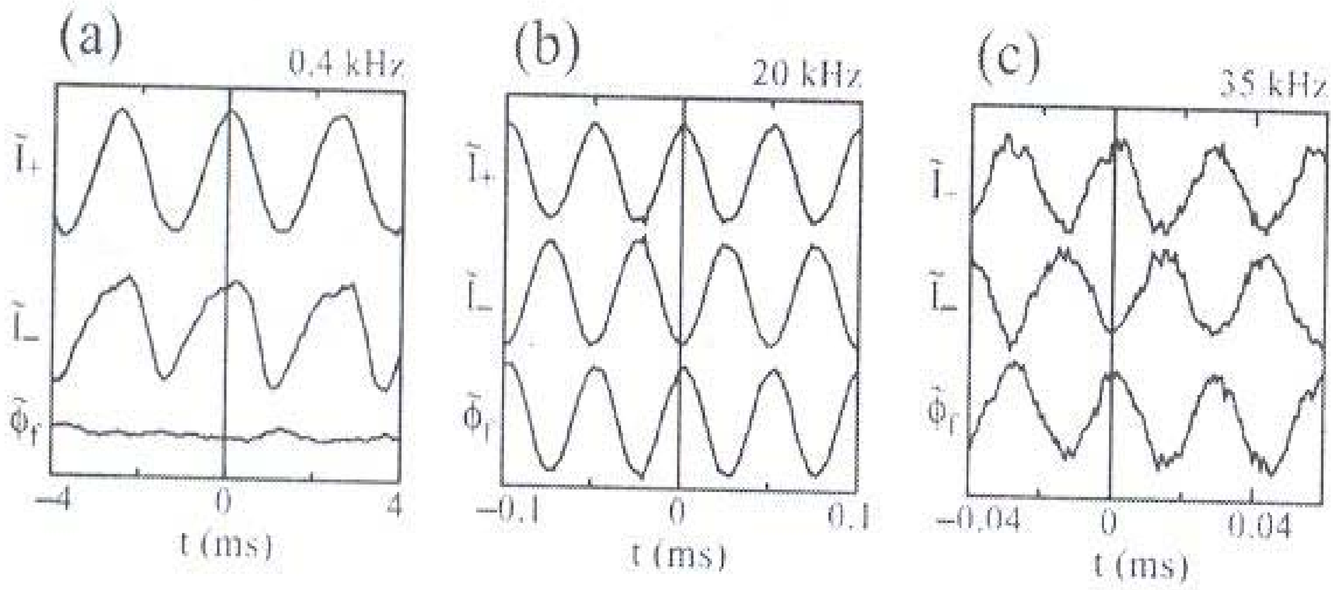

Fig. 2

Fig. 3 of Ref. [7].

Typical temporal variations of the densities and potentials at $\frac{\omega}{2\pi}$ = (a) 0.4, (b) 20, (c) 35KHz. The phase of the negative ion density is in advance of the positive-ion density.

- The IAW observation of Fig. 1 has been explained in a paper [13] using kinetic dispersion relation for ion acoustic wave (IAW). This work suggested that there might be electrons present in the fullerene plasma of Ref. [7]. It has been noticed that the linear dispersion relation of IAW in simplest form turns out to be [13].

$$\omega_s^2 = \frac{(N_0 c_s^2) k^2}{1 + \lambda_{De}^2 k^2} \qquad (1.1)$$

where $c_s^2 = (T_e / m_i)^{1/2}, N_0 = \left(\frac{n_{+0} + n_{-0}}{n_{e0}} \right)$ and $m_+ = m_- = m_i$.

Here, $T_+ = T_- = T_i$ and $T_i << T_e$ have been assumed.

It is obvious that as n_{e0} decreases, the factor $(N_0 c_s^2)$ increases and hence ω_s also.

[7]. W. Oohara, D. Date and R. Hatakeyama, Phys. Rev. Lett. 95, 175003 (2005).
[13]. H. Saleem, Phys. Plasmas 13,044502 (2006).

The quasi-neutrality was used in Ref [14] to study plasma waves. But in PIE plasma the Poisson equation is more suitable. Many basic linear waves have been investigated in Ref. [14] and it was pointed out that electron concentration in PI plasma can increase the frequency of IAW as was observed experimentally [7]. A detailed discussion is in Ref. [13].

Note $\lambda_{De=}\left(\frac{T_e}{4\pi n_{e0}e^2}\right)^{1/2}$ can become very large when n_{e0} goes down and n_{-0} increases. So we can have

$$1 < \lambda_{De}^2 k^2 \qquad (1.2)$$

Then (1.1) gives $\omega^2 = 2\omega_{pi}^2$. It was also pointed out that in the presence of electrons, the Landau damping rate of IAW is reduced, and hence IAW can be easily excited in PIE plasmas for $1 < \lambda_{De}^2 k^2$. The plasma definition is valid for macroscopic wave phenomena for $\lambda_{De}^2 k^2 < 1$ in general. But (1.2) can be a common situation in PIE plasmas.

[7]. W. Oohara, D. Date and R. Hatakeyama, Phys. Rev. Lett. 95, 175003 (2005).
[13]. H. Saleem, Phys. Plasmas 13, 044502 (2006).
[14]. H. Saleem, J. Vranjes, And S. Poedts, Phys. Lett. A350, 375 (2005).

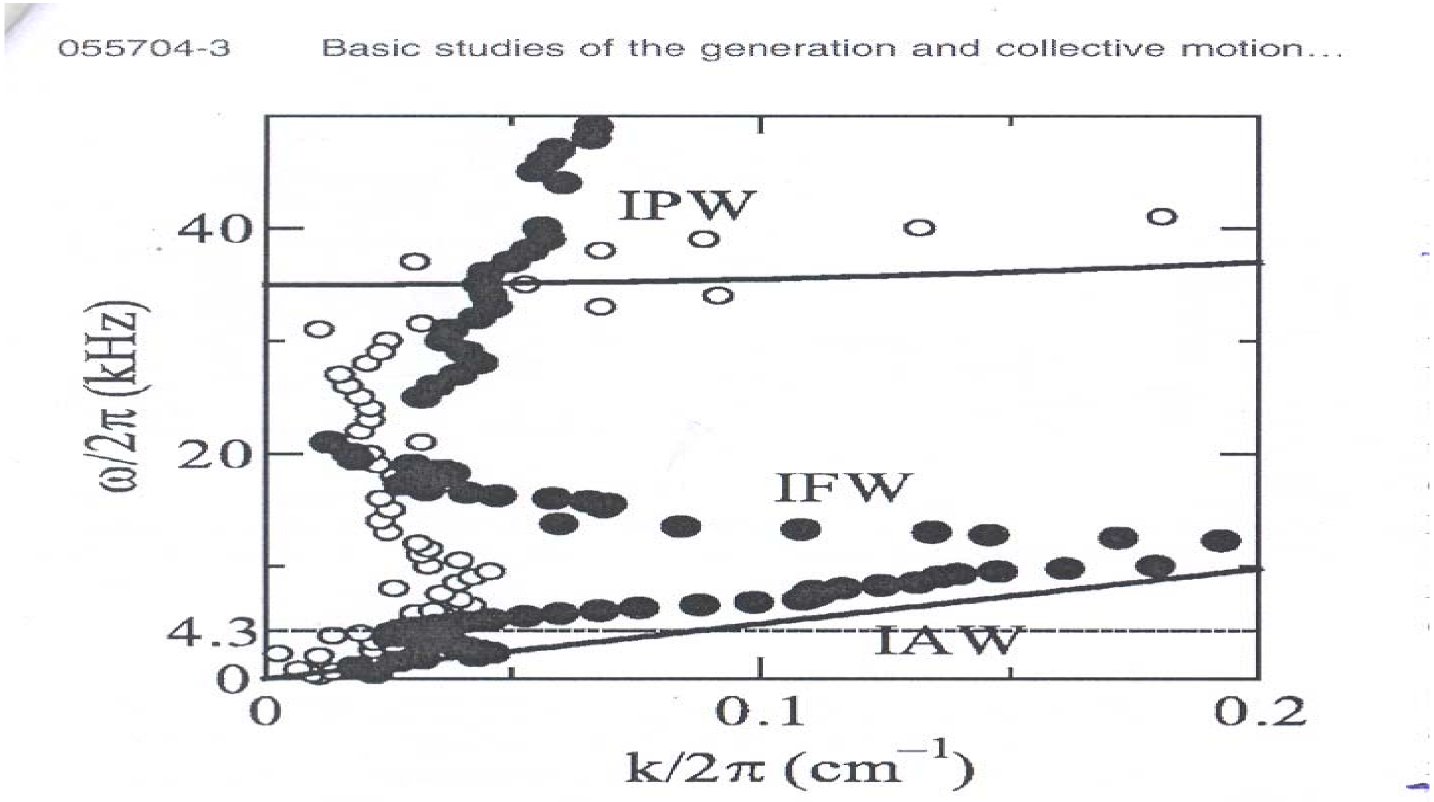

Fig. 3

In the recent observations of fullerene PI plasma [16], the measured dispersion relation of IPW is considerably changed compared with the previous result The author's honestly expressed that it is not understood at present.

It may also be noted that the wavelength of IAW turns out to be very long (of the order of system's length) in Figs. 1 and 3 and hence linear theory does not seem to work properly.

The experimental observations are not discussed here further.

But we discuss the PI and PIE plasma theoretically.

Note that in Fig. 3, there is a frequency larger than ω_{pi} which seems to be the effect of $n_{e0} \neq 0$.

It is important to note that IAW does not exist in pure Pi plasma. On the other hand, the observation of IAW is an indication of the fact that electrons are present in the system with significant concentration, in our opinion.

If $T_e \simeq T_i$ is assumed, then Figs. 1 and 3 show that $c_s k < \omega_s$ which is predicted by Eq. (1.1) because $1 < N_0$ for $n_{e0} \neq 0$.

The simplest derivation of (1.1) using fluid theory is as follows:

$$\partial_t n_+ + n_{+0}\partial_z v_{+z} = 0\,;\partial_t n_- + n_{-0}\partial_z v_{-z} = 0$$

$$\partial_t v_{\pm z} = \mp\frac{e}{m_i}\partial_z\varphi$$

$$n_e \simeq n_{e0} e^{\frac{e\varphi}{T_e}} \simeq n_{e0}\frac{e\varphi}{T_e}; \qquad \text{for } \frac{e\varphi}{T_e} < 1 \tag{1.3}$$

$$-\nabla^2\varphi = 4\pi e(n_+ - n_- - n_e) \tag{1.4}$$

Then continuity equations yield,

$$\partial_t^2(n_+ - n_-) - (n_{+0} + n_{e0})\frac{e}{m_i}\partial_z^2\varphi = 0 \tag{1.5}$$

which becomes,

$$\partial_t^2\left\{n_{e0}\frac{e\varphi}{T_e} + \frac{k^2}{4\pi e}\varphi\right\} = (n_{+0} + n_{-0})\frac{e}{m_i}\partial_z^2\varphi \tag{1.6}$$

and hence one obtains (1.1).

- It is important to note that one cannot say with certainty that a plasma has exactly zero electron density.
- It is the limit on the ratio of electron density to positive ion density $\frac{n_{e0}}{n_{+0}}$ which can decide if the role of electrons is negligible in the plasma dynamics and it can be treated as a pure PI plasma.
- In 2007 [15], the criteria for PI plasma was presented. It was pointed out that an interesting situation may arise in PIE plasmas as follows,

$$\omega_{pe} << \omega_{pi\pm} \tag{1.7}$$

where

$$\omega_{pj} = \left(\frac{4\pi n_{j0} e^2}{m_j} \right)^{1/2}$$

Then PIE plasma can behave as a pure PI plasma.

[15]. H. Saleem Phys. Plasmas 14,014505 (2007).

- Note that in dusty plasmas n_{e0} can also be negligible. But it does not change the plasma character drastically as in case of PIE plasma.
- In classical magnetized case,

$$\lambda_{De} < \rho_s \quad \text{due to} \quad v_A < c \tag{1.8}$$

always holds, where $\rho_s = c_s / \Omega_i$. Fluid equations are applicable for $\rho_s^2 k_\perp^2 \leq 1$.

- Therefore, to use fluid model we need the condition,

$$\lambda_{De}^2 k^2 < \rho_s^2 k_\perp^2 \leq 1 \tag{1.9}$$

even for PIE plasmas

- More experimental papers have appeared in 2007 [16,17] on PI plasmas. In Ref. [17], the efforts have been made to produce H^+ and H^- plasma. At the same time it has been theoretically proposed that it will be more suitable to use lighter atoms and molecules for producing pure PI plasma [15]. Therefore it is very important to discus PIE & PI plasmas in linear and nonlinear regimes.
- Here, the possible criteria to define pair-ion plasma is presented. In addition to it, the electrostatic shocks, solitons and vortices in PIE and PI plasmas are investigated.

[15]. H. Saleem Phys. Plasmas 14,014505 (2007).
[16]. W. Oohara, Y. Kuwabara, and R. Hatakayama, Phys. Rev. E 75, 056403 (2007).14
[17]. W. Oohara and R. Hatakeyama, Phys. Plasmas 14, 055704 (2007).

2. Thermal Mode in PI Plasma

Let us assume, $\mathrm{B} = B_0 z$, perturbation $\alpha e^{i(k_z z - \omega t)}$ i.e. no effect of B_0 on waves.

If $T_+ = T_- = T_i$, then only possible electrostatic mode with $\boldsymbol{k} \parallel \boldsymbol{B}_0$ ideal PI plasma can be the ion plasma wave. IAW in this case does not seem to exist.

Let us consider the simple eqs. of motion for singly charged positive and negative ions, as

$$n_{+0}\partial_t v_{+z} = \frac{e}{m_i} n_{+0} E_z - \frac{T_+}{m_+} \partial_z n_+ \qquad (2.1)$$

$$n_{-0}\partial_t v_{-z} = -\frac{e}{m_i} n_{-0} E_z - \frac{T_-}{m_-} \partial_z n_- \qquad (2.2)$$

Assume $E_{z1} \neq 0$. Then (2.1) minus (2.2) gives,

$$n_0 \partial_t (v_{+z} - v_{-z}) \simeq 2 \frac{e}{m_i} n_0 E_z - v_{Ti}^2 \partial_z (n_+ - n_-) \qquad (2.3)$$

where $n_{+0} = n_{0-} = n_0$.

If charge separation is assumed, Poisson eq. becomes,

$$-\nabla^2 \varphi = 4\pi e(n_+ - n_-) \qquad (2.4)$$

Using continuity equations, one obtains

$$\partial_t (n_+ - n_-) + n_0 \partial_z (v_{+z} - v_{-z}) = 0 \qquad (2.5)$$

The linear dispersion relation turns out to be

$$\omega^2 = \omega_{pi}^2 + v_{Ti}^2 k^2 \qquad (2.6)$$

If $n_+ = n_-$ (quasi-neutrality holds), then (for $T_+ = T_- = T_i$)

(2.1) + (2.2) $\Longrightarrow$

$$n_0 \partial_t (v_{+z} + v_{-z}) = v_{Ti}^2 \partial_z (n_+ + n_-) \simeq 2 v_{Ti}^2 \partial_z n \qquad (2.7)$$

where $n = n_+ = n_-$. No reason to assume $v_{+z} \neq v_{-z}$.

Continuity eqs. are

$$\partial_t n_\pm + n_0 \partial_z v_\pm = 0 \qquad (2.8)$$

and,

hence we obtain thermal mode [or only acoustic (sound) mode of neutral fluids],

$$\omega^2 = v_{Ti}^2 k^2 \qquad (2.9)$$

3. Criteria for Pure PI Plasma

- Kinetic linear dispersion relation using Poision Eq. for $E = -\nabla\varphi$ and $B_0 = 0$ can be written as

$$1 + \sum_j \frac{1}{k^2 \lambda_{Dj}^2}\{1 + \imath\sqrt{\pi} Z_j W(Z_j)\} = 0 \qquad (3.1)$$

where $\lambda_{Dj}^2 = T_j / 4\pi n_{j0} e^2; Z_j = \omega / \sqrt{2} k v_{Tj}; v_{Tj} = (T_j / m_j)^{1/2}$ and $W(Z_j)$ is the plasma dispersion function for the jth species $(j = \pm, e)$. In equilibrium, the quasi-neutrality demands $n_{-0} + n_{e0} = n_{+0}$ where both the positive and negative ions are assumed to be singly charged. Let $T_i < T_e$ and $\upsilon_{Te} \ll \omega / k$ such that $1 \ll |Z_e| \ll |Z_i|$ which allows us to use an asymptotic expansion of $W(Z_j)$ to study the perturbation analytically.

Assuming $m_+ = m_- = m_i$ Eq. (3.1) can be expressed as,

$$\omega^2 - P_n^0 \omega_{pi+}^2 - \frac{3}{\omega^2}\{k^2 v_{Te}^2 \omega_{pe}^2 + k^2 v_{Te-}^2 \omega_{pi-}^2 + k^2 v_{Ti+}^2 \omega_{pi+}^2\}$$
$$+ i\sqrt{\pi}\omega^2 \left\{ \frac{Z_e}{k^2 \lambda_{De}^2} e^{-Z_e^2} + \frac{Z_{i-}}{k^2 \lambda_{Di-}^2} e^{-Z_{i-}^2} + \frac{Z_{i+}}{k^2 \lambda_{Di+}^2} e^{-Z_{i+}^2} \right\} = 0 \qquad (3.2)$$

where $P_n^0 = (1 + \frac{n_{-0}}{n_{+0}} + \frac{n_{e0}}{n_{+0}} \frac{m_i}{m_e})$. Assuming $\omega = \omega_r - i\gamma$

we have,

$$\omega_r^2 \simeq P_n^0 \omega_{pi+}^2 + \frac{3}{P_n^0}\left[v_{Ti+}^2 k^2 + \left(\frac{\omega_{pi-}^2}{\omega_{pi+}^2} k^2 \lambda_{Di-}^2 \right) \omega_{pi-}^2 + \left(\frac{\omega_{pe}^2}{\omega_{pi+}^2} k^2 \lambda_{De}^2 \right) \omega_{pe}^2 \right] \qquad (3.3)$$

and

$$\gamma \simeq \sqrt{\frac{\pi}{4}} \left[\frac{Z_{i+}}{k^2 \lambda_{Di+}^2} \exp\{-Z_{i+}^2\} + \frac{Z_{i-}}{k^2 \lambda_{Di-}^2} \exp\{-Z_{i-}^2\} \right.$$
$$\left. + \frac{Z_e}{k^2 \lambda_{De}^2} \exp\{-Z_e^2\} \right] \omega_r \qquad (3.4)$$

- If $\omega_{pe} < \omega_{pi+}$ and $v_{Te}k < \omega$ then electron plasma wave turns into an ion plasma wave.

The thermal correction terms are smaller than the value of ω_{pi+} for plasma wave because $\lambda_{De}^2 k^2 \leq 1$ in general. In electron-ion (EI) plasma, we know that $n_{-0} = 0.$ Then for $T_i \ll T_e$ and $v_{te}k < \omega$, the above equation yields the electron plasma wave [18]

with $\omega = \omega_r - i\gamma$

$$\omega_r^2 = \omega_{pe}^2 (1 + 3k^2 \lambda_{De}^2) \qquad (3.5)$$

and

$$\gamma_{(k)} \simeq \sqrt{\frac{\pi}{8}} \frac{\omega_{pe}}{k^3 \lambda_{De}^3} \exp\left[-\left(\frac{1}{2k^2 \lambda_{De}^2} + \frac{3}{2} \right) \right] \qquad (3.6)$$

The ions are assumed to form a stationary background of positive charge in which electrons are oscillating with frequency ω_{pe} and these oscillations can propagate if $T_e \neq 0.$

[18]. A. I. Akhiezer, I. A. Akhiezer, R. V. Polovin, A. G. Stenko, and K. N. Stepanov, Plasma Electrodynamics, translated by D. ter Haar (Pergamon, Oxford 1975) Vol. 1.

- In case of (1.3) we cannot assume ions to be static. If $T_i < T_e$, the comparison between the two terms of Eq. (3.2) $k^2 v_{Te}^2 \omega_{pe}^2$ and $k^2 v_{Ti+}^2 \omega_{pi+}^2$ can be very important to decide if the plasma can be called a pure (PI) system. Note that if $T_+ = T_- = T_i$ and $m_{i+} = m_{i-} = m_i$ then $k^2 v_{Te}^2 \omega_{pe}^2 \ll k^2 v_{Ti+}^2 \omega_{pi+}^2$ provided that

$$\frac{n_{0e}}{n_{+0}} \ll \frac{T_i}{T_e}\left(\frac{m_e}{m_i}\right)^2 \qquad (3.7)$$

- In this case electron contribution to the plasma dynamics can be neglected only for a very small value of the ratio n_{e0}/n_{i+} Then the plasma can be called a pure (PI) plasma.
- In the case of fullerene plasma. $m_i \simeq 720 m_p$ (where m_p is mass of the proton) and $m_e/m_p \sim 1/1836$. Therefore we have $m_e/m_i = 7.56\times10^{-7}$ If $T_i \le T_e$ is assumed, then the fullerene plasma discussed in Ref. [7] can be called a pure pair-ion plasma only if the following limit holds:

$$\frac{n_{e0}}{n_{i+}} \ll (7.56\times10^{-7})^2 \frac{T_i}{T_e} \qquad (3.8)$$

- In the experimental observations of excited waves in Fig. 2 of Ref. [7] $n_{i0} \sim 10^7 cm^{-3}$ has been used. It means that this system can become a pure (PI) plasma only if there is almost no electron in the system which is very unlikely physically. Fortunately we have a better situation than (3.7) to call the plasma a pure (PI) plasma. That is $\omega_{pe}^2 \ll \omega_{pi\pm}^2$ which replaces (3.7) by a new limit as follows [12],

$$\frac{n_{e0}}{n_{0+}} \simeq \alpha \frac{m_e}{m_i} \tag{3.9}$$

where α is an arbitrary constant and it must be small, i.e. $\alpha \ll 1$. But we need to choose a value of α. Correspondingly the condition on electron thermal correction term in (3.3) is,

$$k^2 v_{Te}^2 \frac{\omega_{pe}^2}{\omega_{pi+}^2} \ll \omega_{pi+}^2 \tag{3.10}$$

Since $\omega_{pe}^2 / \omega_{pi+}^2 \simeq \alpha$ therefore (3.10) suggests that α should at least satisfy,

$$c_s^2 k^2 \ll \alpha k^2 v_{Te}^2 \ll \omega_{pi+}^2 \tag{3.11}$$

So we can choose α such that $m_e / m_i \ll \alpha \ll 1.$

- A smaller value of $\alpha (i.e\, \alpha < m_e / m_i)$ is preferable. If n_{e0} is so small that α is almost zero, then we will have

$$\alpha v_{Te}^2 k^2 \ll c_s^2 k^2 <<< \omega_{pi+}^2 \qquad (3.12)$$

- In this case the IAW remains almost non-existent. That is n_{e0} is so small that electron pressure contribution is negligible.
- The condition (3.11) is in agreement with the fact that the ion acoustic wave should not appear in the pure (PI) plasmas. The only normal mode of the system with $\mathbf{k} \,||\, \mathbf{B}_0$ is the ion plasma wave which may have a negligible contribution from the small number of hot electrons. For the case of helium (He) plasma $m_e / m_p \simeq 10^{-4}$ and if $T_i < T_e$ is assumed, then it will become almost a pure (PI) plasma if $n_{0e} / n_{i+} \ll 10^{-5}$ holds. Therefore we conclude that it is more suitable to try to produce (PI) plasma of lighter atoms (or molecules) if other physical conditions like the ionization, recombination rates, and the electron attachment cross section can be controlled.

4. Some Linear Waves in PI Plasma

- The equation of motion of j-the species can be written as,

$$m_j n_j \partial_t \mathbf{v}_j = n_j q_j \left(E + \frac{1}{c} \mathbf{v}_j \times B_0 \hat{\mathbf{z}} \right) - \nabla p_j . \tag{4.1}$$

here subscript $j = \pm$ denotes positive and negative ions. We assume $v_{-j0} = 0, E = -\nabla \varphi$ and $p_j = n_j T_j$. The above equation yields

$$\left(\partial_t^2 + \Omega_j^2 \right) \mathbf{v}_{j\perp} = \frac{q_j}{m_j} (\partial_t \mathbf{E} + \Omega_j \mathbf{E}_\perp \times \mathbf{z}) - \frac{\Omega_j}{m_j} \frac{\nabla_\perp p_j \times \mathbf{z}}{n_j} - \partial_t \frac{\nabla_\perp p_j}{m_j n_j} \tag{4.2}$$

and

$$\partial_t v_{jz} = \frac{q_j}{m_j} E_z - \frac{\partial_z p_j}{m_j n_j} \tag{4.3}$$

- The continuity equation can be written as

$$\partial_t n_j + n_{j0}\nabla_\perp.\mathbf{v}_{j\perp} + n_{j0}\partial_z v_{jz} = 0. \tag{4.4}$$

Eqs. (4.2) – (4.4) give,

$$\begin{aligned}&\{\omega^2(\omega^2 - \Omega_j^2) - v_{Tj}^2 k^2\omega^2 + v_{Tj}^2 k_z^2\Omega_j^2\} n_j \\ &- \frac{n_{j0}q_j}{m_j} k_\perp^2\omega^2\varphi - \frac{n_{j0}q_j}{m_j}(\omega^2 - \Omega_j^2) k_z^2\varphi = 0\end{aligned} \tag{4.5}$$

Writing Eq. (4.5) for $j = \pm$ and then subtracting one equation from the other, we obtain

$$\begin{aligned}&[\omega^2(\omega^2 - \Omega_i^2) - v_{Ti}^2 k^2\omega^2 + v_{Ti}^2 k_z^2\Omega_i^2](n_+ - n_-) \\ &-(n_{+0} + n_{-0})\frac{q}{m_i}k_\perp^2\omega^2\varphi \\ &-(n_{+0} + n_{-1})\frac{q}{m_i}k_z^2(\omega^2 - \Omega_i^2)\varphi = 0,\end{aligned} \tag{4.6}$$

- Here, γ_i is the ratio of specific heats for the adiabatic ions. The Poisson equation reads

$$(n_+ - n_-) = -\frac{1}{4\pi e}(\nabla^2 \varphi) \qquad (4.7)$$

We assume Boltzmann density distribution for electrons

$$n_e \approx n_{e0} \exp\left(\frac{e\varphi}{T_e}\right). \qquad (4.8)$$

When n_{e0}, k_z and T_i are assumed to be zero, and the Poisson equation is used instead of quasi-neutrality in Eq.(4.6), one obtains $\omega^2 = \Omega_i^2 + 2\omega_{pi}^2$ which is the analogue of the upper hybrid oscillations in electron-ion plasmas.

If $n_{e0} = 0$, and we use the Poisson equation along with $k_\perp = 0$ we obtain the ion plasma wave dispersion relation $\omega^2 = 2\omega_{pi}^2 + v_{Ti}^2 k_z^2$.

- Now we show that the ion cyclotron wave dispersion relation will be modified. Let us assume that the plasma is quasi-neutral in the presence of Boltzmann electrons. For simplicity we ignore the ion temperature effects. Eqs. (4.6) , (4.8) then yield,

$$\omega^4 - \left(\Omega_i^2 + \frac{q}{e} N_0 c_s^2 k^2 \right) \omega^2 + \frac{q}{e} N_0 c_z^2 k_z^2 \Omega_i^2 = 0 \qquad (4.11)$$

If $n_{e0} = 0$, then pair plasma convective cell (PPCC) can exist in such systems in the quasi-neutral approximation. In this situation we obtain from Eq. (4.6), using $n_{+0} \simeq n_{-0}$, a linear dispersion relation as,

$$\omega^2 = \frac{k_z^2}{k^2} \Omega_i^2 \simeq \frac{k_z^2}{k_\perp^2} \Omega_i^2 \qquad (4.12)$$

- It may be noted that this mode requires the condition $k_z << k_\perp$ and hence, $\omega << \Omega_i$.
- This mode exists, in electron positron plasmas as well.
- In electron-ion plasma case $N_0 = 1$ and for the ion cryotron wave we have $k_z << k_\perp$, therefore Eq. (4.11) yields the well-known dispersion relation $\omega^2 = \Omega_i^2 + c_s^2 k^2$
- In the present situation, $1 < N_0$ is possible along with $\omega^2 < N_0 c_s^2 k_z^2$ Therefore we retain the last term in Eq. (4.11). It gives for $q = e$ in PIE plasma,

$$\omega^2 = \frac{1}{2}\Big[\left(\Omega_i^2 + N_0 c_s^2 k^2\right)$$

$$\pm \left(\left(\Omega_i^2 + N_0 c_s^2 k^2\right)^2 - 4 N_0 c_s^2 k_z^2 \Omega_i^2 \right)^{1/2} \Big] \qquad (4.13)$$

For very large number, plasma must be treated as a PI plasma.

- It may also be noted that the low frequency electromagnetic Alfven waves are not dispersive in a pair-plasma. If we assume $n_{eo} = 0$ and use quasi-neutrality because of the low frequency limit along with $\mathrm{E} = -\nabla\phi - \frac{1}{c}(\partial A_z / \partial t)\mathrm{z}$, then we obtain the following Alfven wave dispersion relation

$$\omega^2 = v_A^2 k_z^2 / 2 \qquad (4.14)$$

where

$$v_A^2 = B_0^2 / \left(4\pi n_{+0} m_i\right)$$

Dispersive effects will appear if Poision Eq. is used as is the case of $e^+ + e^-$ plasmas.

5. HM Eq. in PI & PIE Plasmas

- Here we derive Hasegawa-Mima (HM) equation in magnetized nonuniform PI and PIE plasmas. To study nonlinear plasma dynamics, we use fluid model. Now we consider the linear and nonlinear theory of drift waves and look at some interesting new features which can appear because of the presence of negative ions in the usual (EI) plasmas.
- If $T_i \neq 0$, then the perpendicular drift velocities for ions can be written as,

$$\mathbf{v}_{j\perp} = \frac{c}{B_0}\boldsymbol{E}_\perp \times \hat{\boldsymbol{z}} - \frac{\nabla p_j \times \hat{\boldsymbol{z}}}{\Omega_j m_j n_j} - \frac{1}{\Omega_i}(\nabla_t + \mathbf{v}_j.\nabla)\mathbf{v}_j \times \hat{\boldsymbol{z}} \qquad (5.1)$$

$$= \mathbf{v}_E + \mathbf{v}_{Dj} + \mathbf{v}_{pj}$$

- Here $j = \pm$ and $\Omega_j = q_j B_0 / m_j c$. For electrons, we have

$$\mathbf{v}_{e\perp} = \frac{c}{B_0}\mathbf{E}_\perp \times \hat{\mathbf{z}} + \frac{\nabla p_j \times \hat{\mathbf{z}}}{\Omega_e m_e n_e} = \mathbf{v}_E + \mathbf{v}_{De} \qquad (5.2)$$

where $|\partial_t| \ll \Omega_e = eB_0 / m_e c$ has been used.

- The continuity equations of the ions yield,

$$\partial_t(n_+ - n_-) + \frac{c}{B_0}\nabla n_{e0}.(\hat{\mathbf{z}}\times\nabla_\perp\varphi) - \frac{c}{B_0\Omega_i}(n_{+0}+n_{-0})$$
$$\times(\partial_t + \mathrm{v}_E.\nabla)\nabla^2\varphi \simeq n_{-0}\partial_z v_{z-} - n_{+0}\partial_z v_{z+} \tag{5.3}$$

and the parallel equation of motion becomes,

$$(\partial_t + \mathrm{v}_E.\nabla)(n_{-0}v_{z-} - n_{+0}v_{z+}) = \frac{e}{m_i}(n_{+0}+n_{-0})\partial_z\varphi. \tag{5.4}$$

Assuming Boltzmann density distribution for electrons $n_e \sim n_{e0}e^{e\varphi/T_e}$ and using the Poisson equation.

$$\nabla.\mathbf{E} = 4\pi e(n_+ - n_- - n_e) \tag{5.5}$$

the nonlinear Eqs. (5.3) and (5.4) can be written, respectively as,

$$\partial_t\left\{-\lambda_{De}^2\nabla^2\Phi+\Phi\right\}+D_e\kappa_{ne}.(\hat{\mathbf{z}}\times\nabla_\perp\Phi)-N_0\rho_s^2 \times\left(\partial_t+D_e\hat{\mathbf{z}}\times\nabla_\perp\Phi.\nabla\right)\nabla_\perp^2\Phi=\partial_z V \tag{5.6}$$

and

$$\left(\partial_t+D_e\hat{\mathbf{z}}\times\nabla_\perp\Phi.\nabla\right)V=c_s^2N_0\partial_z\Phi \tag{5.7}$$

where $V=(n_{-0}v_{-z}-n_{+0}v_{+z})/n_{e0},\ N_0=(n_{+0}+n_{-0})/n_{e0},\Phi=e\varphi/T_e,$

$D_e=cT_e/eB_{0.},\ \rho_s^2=c_s^2/\Omega_i^2$ and $c_s=(T_e/m_i)^{1/2}$. Equation (5.6)

is the Hasegawa-Mima equation for drift waves in (PIE) plasmas if the RHS is ignored (for $v_{iz}\to 0$) and $\lambda_{De}^2k^2\ll 1$ is assumed.

- A stationary solution of the above nonlinear equations can be obtained in the (η, x) frame where $\eta = y + \mu z - \mu t$ coordinate is moving with speed u in the yz plane, $\mu = u_y / u_z$ and $u = (u_y^2 + u_x^2)^{1/2}$. In this moving frame, the HM equation for the case of (PIE) plasmas including parallel ion motion can be written as,

$$C_1 d_\eta \nabla_\perp^2 \Phi + u_0 d_\eta \Phi + \{\nabla_\perp^2 \Phi, \Phi\} = 0 \qquad (5.8)$$

where $u_0 = (u - v_0^* - \mu L_0)/a_0^2,\ a_0^2 = N_0 \rho_s^2 D_e, C_1 = -u/a_0^2(\lambda_{De}^2 + N_0 \rho_s^2),\ v_0^* = |\kappa_{ne}| D_e,\ \kappa_{ne} = -\hat{x}\left(1/n_{0e})(dn_{0e}/dx\right)$

and $L_0 = (C_0 - \alpha N_0 c_s^2 / u)$. Here C_0 is an arbitrary constant.

These equations give the coupled linear dispersion relation of drift wave and IAW in (PIE) plasmas as,

$$G_0 \omega^2 - \omega_e^* \omega - N_0 c_s^2 k_z^2 = 0. \qquad (5.9)$$

where

$$G_0 = (1 + \lambda_{De}^2 k^2 + N_0 \rho_s^2 k_\perp^2), \omega_e^* = \mathbf{v}_0^* . \mathbf{k} = v_0^* k$$

- If $N_0 c_s^2 k_\perp^2 \ll \omega_e^*$ holds, then we obtain only the drift wave dispersion relation as,

$$\omega = \frac{\omega_e^*}{(1 + \lambda_{De}^2 k^2 + N_0 \rho_s^2 k_\perp^2)} \qquad (5.10)$$

- In (PIE) plasma the quasi-neutrality can break down for IAW in the limit $1 \ll \lambda_{De}^2 k^2$ because n_{e0} can be very small. On the other hand the inequality $\lambda_{De} < \rho_s$ always holds.

Since $\rho_s^2 k^2 < 1$ in the fluid model, therefore $\lambda_{De}^2 k^2$ should not be much larger than 1. This means in magnetized plasmas, the IAW cannot have wavelengths shorter than λ_{De} within fluid theory framework because the limit

$$\lambda_{De}^2 k^2 < \rho_s^2 k_\perp^2 \ll 1 \qquad (5.11)$$

must be satisfied.

- It is important to note that as n_{e0} decreases, N_0 increases to have $\lambda_{De}^2 k^2 \ll N_0 \rho_s^2 k_\perp^2$ and hence Eq. (5.9) gives the pair plasma convective cell (PPCC) mode (4.12),

$$\omega^2 = \frac{k_z^2}{k_\perp^2} \Omega_i^2$$

- As n_{e0} decreases, the IAW converts into the PPCC mode. In between these two limits, the electron drift wave couples with IAW and PPCC. The nonlinear dynamics of (PI) plasma are described by Eqs. (5.3) and (5.4) in the limit $1 \ll N_0$ and they reduce, respectively, to the following equations:

and $$(\partial_t + D_i \hat{\mathbf{z}} \times \nabla_1 \Phi . \nabla) \nabla_\perp^2 \Phi = -\frac{1}{2\rho_i^2} \partial_z V \tag{2.12}$$

$$(\partial_t + D_i \hat{\mathbf{z}} \times \nabla_\perp \Phi . \nabla) V = 2 v_{Ti}^2 \partial_z \Phi \tag{2.13}$$

where $\Phi = e\varphi / T_i, D_i = cT_i / eB_0, \rho_i = v_{Ti} / \Omega_i$ and $V = (v_{z-} - v_{z+})$.

- In the stationary (η, x) frame, the coupled Eqs. (5.12) and (5.13) can be written as,

$$C_1 d_\eta \nabla_\perp^2 \Phi + V_0 d_\eta \Phi + \{\nabla_\perp^2 \Phi, \Phi\} = 0 \tag{5.14}$$

where $C_1 = -u / D_i$, $V_0 = \mu L_0 / 2\rho_i^2 D_i$, $L_0 = (C_0 - 2\mu v_{Ti}^2 / u)$ and C_0 is an arbitrary constant. The important point to note is that the form of Eq. (5.14) is similar to the Hasegawa- Mima equation but the physics of this equation is completely different. The set of nonlinear equations (5.11) - (5.14) is valid as well for electron-positron plasmas in the classical limit. But these equations do not contain the drift wave and the ion acoustic mode. They describe the nonlinear dynamics of (PPCC) mode. In the linear limit these equations yield (PPCC) mode. The T_i can be cancelled out in Eqs. (2.12) and (2.13), because PPCC mode does not require $T_j \neq 0$ to exist.

6. KdV-B Equation in PIE Plasmas

Let $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ constant, $T_i \ll T_e (i = +, -)$ and $\Omega_i = \dfrac{|q_i| B_0}{m_i c}$.

υ_{+n} = collision freq. of + ve ions with neutrals
υ_{-n} = collision freq. of - ve ions with neutrals

Neutral density is assumed to be large so that $\upsilon_{ei}, \upsilon_{ie} \ll \upsilon_{+n}, \upsilon_{-n}$. This is the case of weakly ionized PIE plasma.

The effect of ion-neutral collisions is to give a finite width to the shock wave. In fully ionized ideal plasma, $\upsilon_{+n} = \upsilon_{-n} = 0$ and we shall have just wave breaking [19], if dissipation is neglected. Let us consider partially ionized plasma, neglecting electron-ion collisions which give rise to linear drift dissipative instability.

[19]. H. Tasso, Phys. Lett. A24,618 (1967).

Then equations of motion give for +ve and –ve ions, respectively, the perpendicular velocities in drift ordering $\left|\partial_t\right| < \Omega_i$ as

$$\mathbf{v}_{+\perp} = -\frac{c}{B_0}\nabla_\perp\varphi\times\hat{\mathbf{z}} - \frac{1}{\Omega_i}(\partial_t + \mathbf{v}_+.\nabla)\mathbf{v}_+\times\hat{\mathbf{z}} - \frac{\upsilon_{+n}}{\Omega_i}(\mathbf{v}_+\times\hat{\mathbf{z}}) \quad (6.1)$$

$$= \mathbf{v}_E + \mathbf{v}_{+P} + \mathbf{v}_{+c}$$

and

$$\mathbf{v}_{-\perp} = -\frac{c}{B_0}\nabla_\perp\varphi\times\hat{\mathbf{z}} + \frac{1}{\Omega_i}(\partial_t + \mathbf{v}_-.\nabla)\mathbf{v}_-\times\hat{\mathbf{z}} - \frac{\upsilon_{-n}}{\Omega_i}(\mathbf{v}_-\times\hat{\mathbf{z}})$$

$$= \mathbf{v}_E + \mathbf{v}_{-P} + \mathbf{v}_{-c} \quad (6.2)$$

The continuity equations give,

$$\partial_t(n_+ - n_-) + \nabla_\perp.\{(n_+ - n_-)\mathrm{v}_E\} + \nabla_\perp.\{\mathrm{v}_{+p} - \mathrm{v}_{-p}\}$$
$$+\nabla_\perp.\{\mathrm{v}_{+c} - \mathrm{v}_{-c}\} = 0 \quad (6.3)$$

where $|\partial_t| \ll |\nabla_\perp|$ and $|V_{iz}\partial_z| << |V_E.\nabla_\perp|$ has been used. Since $\partial_z v_{iz}$ has been neglected, the ion acoustic wave will not appear. Poisson equation gives

$$(n_+ - n_-) \simeq n_{e0}\frac{e\varphi}{T_e} - \frac{1}{4\pi e}\nabla_\perp^2\varphi \tag{6.4}$$

for

$$n_e \simeq n_{e0}e^{e\varphi/T_e} \simeq n_{e0}(1 + e\varphi/T_e) \tag{6.5}$$

Then (6.3) becomes,

$$\partial_t\{n_{e0}\frac{e\varphi}{T_e} - \frac{1}{4\pi e}\nabla_\perp^2\varphi\} + \frac{c}{B_0}n_{e0}\{(\kappa_{en}^v \times \hat{\mathbf{z}}.\nabla_\perp)\varphi - \frac{e\varphi}{T_e}(\kappa_{eT}^v \times \hat{\mathbf{z}}.\nabla_\perp)\varphi\}$$
$$-\frac{\upsilon}{\Omega_i}\frac{c}{B_0}\nabla_\perp^2\varphi - \frac{2}{\Omega_i}\frac{c}{B_0}\partial_t\nabla_\perp^2\varphi = 0 \tag{6.6}$$

where $\kappa_{en}^v = \frac{1}{n_{e0}}\left(\nabla n_{e0}\right), \kappa_{eT}^v = \frac{1}{T_e}\nabla T_e$ and $\upsilon = (\upsilon_{+n} + \upsilon_{-n})$.

Eq. (6.6) can be written as,

$$\partial_t \Phi + v_D^* \partial_y \Phi - (\lambda_{De}^2 + 2\rho_s^2)\partial_t \partial_y^2 \Phi + \frac{1}{2} V_T^* \partial_y \Phi^2 - \upsilon \rho_s^2 \partial_y^2 \Phi = 0 \tag{6.7}$$

where $v_D^* = -D_e \kappa_{en}$; $D_e = \dfrac{cT_e}{eB_0}$;

$$\kappa_{en} = \left| \frac{1}{n_{e0}} \frac{dn_{e0}}{dx} \right| ; v_T^* = D_e \kappa_{eT} \ ; \ \kappa_{eT} = \left| \frac{1}{T_e} \frac{dT_e}{dx} \right| ; \ \rho_s = \frac{c_s}{\Omega_i}$$

and $c_s = (T_e / m_i)^{1/2}$.

If dissipation is ignored, (6.7) becomes

$$-\partial_t \Phi + D_e \kappa_{en} \partial_y \Phi - \frac{1}{2} V_T^* \partial_y \Phi^2 = 0 \tag{6.8}$$

which is Tasso's equation(4) [16] for $n_- = 0$.

Eq (6.8) predicts wave breaking.

Eq. (6.7) is Korteweg de-Vriers-Burgers (KdV-B) equation. It' s solutions in a moving frame are discussed. Let us define a frame moving with constant velocity u as

$$\eta = (y - ut) \tag{6.9}$$

In this frame Eq. (6.7) becomes,

$$-u d_\eta \Phi + \frac{1}{2}\beta d_\eta \Phi^2 - \gamma d_\eta^2 \Phi + \mu d_\eta^3 \Phi = 0 \tag{6.10}$$

where

$$\beta = \frac{v_T^*}{g} \quad ; \quad g = (1 + v_0^* / u),$$

$$v_0^* = -v_D^* = D_e \kappa_{en} \quad , \quad \mu = \frac{1}{g}(\lambda_{De}^* + 2\rho_s^2)u,$$

and

$$\gamma = \frac{1}{g}\upsilon\rho_s^2$$

- # 6a. Soliton (dissipation negligible)

(6.10) becomes

$$-ud_{\eta}\Phi + \frac{\beta}{2} d_{\eta}\Phi^{2} + \mu d_{\eta}^{3}\Phi = 0$$

(6.11)

This is KdV-equation and it admits soliton solution,

$$\Phi = \Phi_{m} \ Sech^{2} (\eta / \delta) \tag{6.12}$$

where $\Phi_{m} = \frac{3u}{\beta}$ is amplitude and

$\delta = \left(\frac{4\mu}{\beta} \right)^{1/2}$ is width of the pulse

- 6b. Monotonic Shock (dispersion neglected)

If concentration of neutrals is large, (6.10) become,

$$-u d_\eta \Phi + \frac{1}{2}\beta d_\eta \Phi^2 - \gamma d_\eta^2 \Phi = 0 \qquad (6.13\text{ a})$$

It may be expressed as,

$$-\alpha d_\eta \Phi + \frac{1}{2} d_\eta \Phi^2 - \Gamma d_\eta^2 \Phi = 0 \qquad (6.13\text{ b})$$

where $\alpha = \frac{u}{\beta}$ and $\Gamma = \gamma / \beta$

Solution of (5.13b) is

$$\Phi = \alpha\left[1 - tanh\left(\alpha / 2\Gamma\right)\right] \qquad (6.14)$$

where α and $\alpha / 2\Gamma$ are the height and thickness of the shock, respectively.

- ## 6c. Oscillatory Shocks

Eq. (6.10) can be written as,

$$-\alpha\, d_\eta \Phi + \frac{1}{2} d_\eta \Phi^2 - \Gamma d_\eta^2 \Phi + D d_\eta^3 \Phi = 0 \qquad (6.15)$$

where

$$D = \frac{\mu}{\beta}$$

An analytical solution of (6-15) can be obtained in a limit in the form of an oscillatory shock [20].

First few oscillations at the wave front will be close to solitons moving with velocity u. If the dissipation increases above a critical value say Γ_{cr} , then the solitary structure assumes the form of a monotonic shock.

[20]. V. I. Karpman, Nonlinear Waves in Dispersive Media (pergamon, Oxford, 1975).

Integrate (6.15) with B.C. $\eta \to -\infty$,

$$\Phi = d_\eta \Phi = d_\eta^2 \Phi = 0 \Rightarrow$$ constant = 0

and then one obtains

$$D d_\eta^2 \Phi - \Gamma d_\eta \Phi + \left(\frac{1}{2} \Phi^2 - \alpha \Phi \right) = 0 \qquad (6.16)$$

Let $P(\Phi) = \left(\frac{\Phi^3}{6} - \frac{\alpha}{2} \Phi^2 \right)$ (6.17)

be the potential. Then (5.11) is analogue of an oscillator equation under force $F = \frac{-dP}{d\Phi}$ and with dissipation Γ. The role of (t) is played by $(-\eta)$.

Then $\dfrac{dP(\Phi)}{d\Phi} = 0$

gives the extreme value of $\Phi = 2\alpha = \Phi_1$ (say). Let us assume at $\eta = +\infty$,

$$\Phi = \Phi_1 + f \qquad (6.18)$$

where f is a small perturbation to Φ_1 s.t.

$$f << \Phi_1$$

Substitute (6.18) into (6.16) and linearizing w.r.t. f, we obtain,

$$Dd_\eta^2 f - \Gamma d_\eta f + \alpha f = 0 \qquad (6.19)$$

Note $e^{P\eta}$ is solution of (6.14) for

$$P = \frac{\Gamma}{2D} \pm \left(\frac{\Gamma^2}{4D^2} - \frac{\alpha}{D} \right)^{\frac{1}{2}} \qquad (6.20)$$

Let $\Gamma_{cr} = \sqrt{4D\alpha}$. Then for $\dfrac{\Gamma^2}{2D^2} << \dfrac{\alpha}{D}$

i.e. $\Gamma << \Gamma_{cr}$, the real part of the solution becomes,

$$f = f_0 e^{\left(\frac{\Gamma}{2D}\right)\eta} \cos\left(\sqrt{\frac{\alpha}{D}}\eta\right) \tag{6.21}$$

where f_0 is a constant. We find,

$$\Phi = \Phi_1 + f_0 e^{\left(\frac{\Gamma}{2D}\right)\eta} \cos\left(\sqrt{\frac{\alpha}{D}}\eta\right) \tag{6.22}$$

This gives an oscillatory shock.

For $\Gamma_{cr} < \Gamma$, the solution is like a monotonic shock because P becomes a real number.

7. Conclusions

1. IAW is not a normal mode of pure PI plasma. The observation of electrostatic IAW itself is an indication of the presence of electrons in the system.
2. The frequency of linear IAW in PIE plasma is larger than it is in EI plasma for the same n_{+0}. Therefore experimetnal observation of IAW indicates $n_{e0} \neq 0.$
3. Criteria for pure PI plasma have been presented.
4. Some linear waves in magnetized and unmagnetized PIE and PI plasmas have been discussed.
5. Hasegawa – Mima equation and vortex solution in PIE and PI plasmas have been investigated.
6. Shocks and Solitons in inhomogeneous PIE plasmas have been studied in the presence of neutrals.
7. The observation $\omega_{pi} < \omega$ in a recent experiment [17], again indicates $n_{e0} \neq 0.$

References

1. N. Iwamoto, Phys. Rev. A 39, 4076 (1989).
2. G. P. Zank and R. G. Greaves, Phys. Rev. E 51, 6079 (1995).
3. P.K. Shukla, N.N. Rao, M.Y. Yu and N.L. Tsintsadze, Phys. Reports 138, 1 (1986).
4. A. D. Rogava, S. M. Mahajan and V. I. Berezhiani, Phys. Plasmas 3, 3545 (1996).
5. C. M. Surko, M. Leven Thal, and A. Passner, Phys. Rev. Lett. 62, 901 (1989).
6. M. Amoretti et.al. Phys. Rev. Lett. 91, 055001 (2003).
7. W. Oohara, D. Date and R. Hatakeyama, Phys. Rev. Lett. 95, 175003 (2005).
8. H. Schamel and A. Luque, New J. Phys. 7, 69 (2005).

9. P. K. Shukla and M. Khan, Phys. Plasmas 12, 014504 (2005).
10. I. Kourakis, A. Esfandyari-Kalegahi, M. Medhipoor, and P.K. Shukla, Phys. Plasmas 13, 052117 (2006).
11. J. Vranges and S. Poedts, Plasma Source Sci. Technol. 14, 485 (2005); J. Vranges and S. Poedts, Phys. Plasmas 15, 044501 (2008).
12. A. Luque, H. Schamel, B. Eliasson, and P.K. Shukla, Plasma Phys. Controlled Fusion 48, 044502 (2006).
13. F. Verheest, Phys. Plasmas 13, 082301 (2006).
14. H. Saleem, Phys. Plasmas 13, 044502 (2006).
15. H. Saleem J. Vranjes, and S. Poedts, Phys. Lett. A 350, 375 (2006).
16. H. Saleem, Phys. Plasmas 14, 014505 (2007).
17. W. Oohara, Y. Kuwabara, and R. Hatakayama, Phys. Rev. E 75, 056403 (2007).50
18. W. Oohara and R. Hatakayama, Phys. Plasmas 14, 055704 (2007).
19. A.I. Akhiezer, I.A. Akhiezer, R.V. Polovin, A.G. Stenko, and K.N. Stepanov, Plasma Electrodynamics, translated by D. ter Haar (Pergamon, OX tord 1975) vol.1
20. H. Tasso, Phys. Lett. A 24, 618 (1967).