International Workshop on the Frontiers of Modern Plasma Physics

14 - 25 July 2008

MHD turbulence and scaling connections with a wider class of nonlinear phenomena.

S. Chapman
University of Warwick, Centre for Fusion Space and Astrophysics
Coventry
U.K.
MHD turbulence and scaling-connections with a wider class of nonlinear phenomena

Sandra C. Chapman¹,

Thanks to George Rowlands¹, B. Hnat¹, K. Kiyani¹, Nick Watkins²
¹CFSA, Physics Dept., Univ. of Warwick
²Physical Sciences, British Antarctic Survey

➢ Turbulence, MHD turbulence and (formal) dimensional analysis
➢ Scaling and physics- examples from the solar wind
➢ How general is the concept of a Reynolds number?
➢ What turbulence does/ does not have in common with idealized avalanching systems (SOC)

Universality - an example

Pendulum

\[ F = mg, \quad F_t = mg \sin \theta, \quad a_t = l \frac{d^2\theta}{dt^2} \]

\[ F_t = ma_t; \quad \frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin \theta = -\omega^2 \frac{\partial V}{\partial \theta} \]

\[ V(\theta) = 1 - \cos(\theta) \sim \frac{\theta^2}{2} + \ldots \]

same behaviour at

any local minimum in \( V(\theta) \)

(insensetive to details)
Similarity in action...
Similarity and universality

- Different systems, same physical model
- The same function (suitably normalized) can describe them
- This function is universal (the details do not matter)
- The values of the normalizing parameters are not universal
- How can we find the physical model (solution)?
- Particularly useful in nonlinear systems which are ‘hard’ to solve – i.e. turbulence!
- ‘Classical’ inertial range turbulence- self similarity, intermittency…
structures on many length/timescales.

Reproducible, predictable in a statistical sense.

look at (time-space) differences:

\[ y(t, \tau) = x(t + \tau) - x(t) \]

for all available \( t_k \) of the timeseries \( x(t_k) \)

test for statistical scaling i.e

structure functions \( S_p(\tau) = \langle |y(t, \tau)|^p \rangle \propto \tau^{\zeta(p)} \)

we want to measure the \( \zeta(p) \)

fractal (self- affine) \( \zeta(p) \sim \alpha p \)

multifractal \( \zeta(p) \sim \alpha p - \beta p^2 + \ldots \)

would like \( \langle |y(t, \tau)|^p \rangle = \int_{-\infty}^{\infty} |y|^p P(y, \tau) dy \)

finite system/data! conditioning- an estimate is:

\( \langle |y|^p \rangle = \int_{-A}^{A} |y|^p P(y, \tau) dy \) where \( A \sim [10-20] \sigma(\tau) \)
Some Phenomenology....Kolmogorov vz MHD scaling

velocity difference \( d_r v = v(l + r) - v(l) \), energy transfer rate \( \varepsilon_r \sim \frac{d_r v^2}{T} \)

Kolmogorov: simply have \( T \) as the eddy turnover time \( T \sim \frac{r}{d_r v} \) so that \( \varepsilon_r \sim \frac{d_r v^3}{r} \)

MHD: now \( T \) is due to (say) Alfvenic collisions \( T \sim \frac{r}{d_r v} \left( \frac{v_0}{d_r v} \right)^\alpha \) giving \( \varepsilon_r \sim \frac{d_r v^{3+\alpha}}{r} \)

intermittency \( \langle \varepsilon_r^p \rangle \sim \bar{\varepsilon}^p \left( \frac{r}{L} \right)^{\tau(p)} \)

\( \Rightarrow \) Kolmogorov: \( \langle d_r v^p \rangle \sim r^{\frac{p}{3}} \bar{\varepsilon}^\frac{p}{3} \left( \frac{L}{r} \right)^{\tau(p/3)} \sim r^{\zeta(p)} \)

\( \Rightarrow \) MHD: same with \( \frac{p}{3} \rightarrow \frac{p}{(3 + \alpha)} \) intermittency free \( E(k) \sim \left\langle d v^2 \right\rangle / k \sim k^{-(5+\alpha)/(3+\alpha)} \)

\( \langle \varepsilon_r \rangle = \bar{\varepsilon} \) independent of \( r \) (steady state) so \( \tau(1) = 0 \) and \( \zeta(\alpha + 3) = 1 \)

what is \( \alpha \)?

Kolmogorov Obukhov (1941) hydrodynamic: \( \alpha = 0 \)
Iroshnikov Kraichnan (1964) weak isotropic MHD \( \alpha = 1 \),
Goldreich Sridhar (1994-5) strong MHD \( \alpha_\perp = 0 \)
Boldyrev (2005) strong, background field anisotropic MHD \( \alpha_\perp = 1 \)
Velocity fluctuations parallel and perpendicular to the *local* B field direction

Exponents $\zeta(p)$ for $\langle |\delta v_{\parallel,\perp}|^p \rangle \sim \tau^{\zeta(p)}$ for

$$\delta v_{\parallel} = \delta v \cdot \hat{b}$$ and its remainder $\delta v_{\perp} = \sqrt{\delta v \cdot \delta v - (\delta v \cdot \hat{b})^2}$

$\zeta(3 + \alpha) = 1$ determines phenomenology

$$\bar{B} = B(t) + \ldots + B(t + \tau'), \quad \hat{b} = \frac{B}{|B|}, \text{here } \tau' = 2\tau \text{ and } \delta v = v(t + \tau) - v(t)$$

ACE 64s av. 1998-2001


---

**Diagram Description:**

- **Left Panel:**
  - Chart showing $\xi(p)$ against moment $p$.
  - Data points for remainder $\delta v_{\parallel}$ and $\delta v \cdot <B>_{2\tau}$.
  - ACE 64s av. 1998-2001

- **Right Panel:**
  - Chart showing $\xi(m)$ against moment $m$.
  - Data points for $\rho, \rho v, \rho v^2, B^2$.

---

**Legend:**

- $\delta v_{\parallel}$, $\delta v_{\perp}$
- $\rho, \rho v, \rho v^2, B^2$
- $\nu, B$

---

**Centre for Fusion, Space and Astrophysics (CFSA)**

**The University of Warwick**
Distinguishing self-affinity (fractality) and multifractality

Levy flight -- Fractal

Kiyani et al, PRL (2007)

P-model -- Multifractal
Solar cycle variation WIND -- $|B|^2$

2000 - Solar max  
1996 - Solar min

Fractal signature ‘embedded’ in (multifractal) solar wind inertial range turbulence coincident with complex coronal magnetic topology

ULYSSES- north polar pass at solar minimum

ULYSSES 60s averages
July-Aug 1995, \( \sim 8.5 \times 10^4 \) points, selected as a quiet interval
-Multifractal
-Fractality coincides with topologically complex coronal fields?

The Sun Approaching Solar Maximum
Solar and Heliospheric Observatory, Extreme ultraviolet Imaging Telescope

Early 1997  Mid 1998  Late 1999

centre for fusion, space and astrophysics
Similarity in action…

Buckingham $\pi$ theorem

System described by $F(Q_1...Q_p)$ where $Q_{1..p}$ are the relevant macroscopic variables

$F$ must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$

if there are $R$ physical dimensions (mass, length, time etc.)

there are $M = P - R$ distinct dimensionless groups.

Then $F(\pi_{1..M}) = C$ is the general solution for this universality class.

To proceed further we need to make some intelligent guesses for $F(\pi_{1..M})$

See e.g. Barenblatt, Scaling, self-similarity and intermediate asymptotics, CUP, [1996]
also Longair, Theoretical concepts in physics, Chap 8, CUP [2003]
Example: simple (nonlinear) pendulum

System described by $F(Q_1...Q_p)$ where $Q_k$ is a macroscopic variable
$F$ must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$
if there are $R$ physical dimensions (mass, length, time etc.) there are $M = P - R$ dimensionless groups

Step 1: write down the relevant macroscopic variables:

<table>
<thead>
<tr>
<th>variable</th>
<th>dimension</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_0$</td>
<td>-</td>
<td>angle of release</td>
</tr>
<tr>
<td>$m$</td>
<td>[M]</td>
<td>mass of bob</td>
</tr>
<tr>
<td>$\tau$</td>
<td>[T]</td>
<td>period of pendulum</td>
</tr>
<tr>
<td>$g$</td>
<td>[L][T]$^2$</td>
<td>gravitational acceleration</td>
</tr>
<tr>
<td>$l$</td>
<td>[L]</td>
<td>length of pendulum</td>
</tr>
</tbody>
</table>

Step 2: form dimensionless groups: $P = 5, R = 3$ so $M = 2$
$\pi_1 = \theta_0, \pi_2 = \frac{\tau^2 l}{g}$ and no dimensionless group can contain $m$
then solution is $F(\theta_0, \frac{\tau^2 l}{g}) = C$

Step 3: make some simplifying assumption: $f(\pi_1) = \pi_2$ then the period: $\tau = f(\theta_0)\sqrt{\frac{l}{g}}$
NB $f(\theta_0)$ is universal ie same for all pendula-
we can find it knowing some other property eg conservation of energy..
Example: fluid turbulence, the Kolmogorov '5/3 power spectrum'

System described by $F(Q_1...Q_p)$ where $Q_k$ is a macroscopic variable
$F$ must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$
if there are $R$ physical dimensions (mass, length, time etc.) there are $M = P - R$ dimensionless groups

Step 1: write down the relevant variables (incompressible so energy/mass):

<table>
<thead>
<tr>
<th>variable</th>
<th>dimension</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(k)$</td>
<td>$[L]^3[T]^{-2}$</td>
<td>energy/unit wave no.</td>
</tr>
<tr>
<td>$\varepsilon_0$</td>
<td>$[L]^2[T]^{-3}$</td>
<td>rate of energy input</td>
</tr>
<tr>
<td>$k$</td>
<td>$[L]^{-1}$</td>
<td>wavenumber</td>
</tr>
</tbody>
</table>

Step 2: form dimensionless groups: $P = 3, R = 2, \text{ so } M = 1$

$$\pi_1 = \frac{E^3(k)k^5}{\varepsilon_0^2}$$

Step 3: make some simplifying assumption:

$F(\pi_1) = \pi_1 = C$ where $C$ is a non universal constant, then: $E(k) \sim \varepsilon_0^{2/3}k^{-5/3}$
Buchingham $\pi$ theorem (similarity analysis)
universal scaling, anomalous scaling
System described by $F(Q_1...Q_p)$ where $Q_k$ is a relevant macroscopic variable
$F$ must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$
if there are $R$ physical dimensions (mass, length, time etc.) there are $M = P - R$ dimensionless groups
Turbulence:

<table>
<thead>
<tr>
<th>variable</th>
<th>dimension</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(k)$</td>
<td>$L^3T^{-2}$</td>
<td>energy/unit wave no.</td>
</tr>
<tr>
<td>$\varepsilon_0$</td>
<td>$L^2T^{-3}$</td>
<td>rate of energy input</td>
</tr>
<tr>
<td>$k$</td>
<td>$L^{-1}$</td>
<td>wavenumber</td>
</tr>
</tbody>
</table>

introduce another characteristic speed....

<table>
<thead>
<tr>
<th>variable</th>
<th>dimension</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(k)$</td>
<td>$L^3T^{-2}$</td>
<td>energy/unit wave no.</td>
</tr>
<tr>
<td>$\varepsilon_0$</td>
<td>$L^2T^{-3}$</td>
<td>rate of energy input</td>
</tr>
<tr>
<td>$k$</td>
<td>$L^{-1}$</td>
<td>wavenumber</td>
</tr>
<tr>
<td>$v$</td>
<td>$LT^{-1}$</td>
<td>characteristic speed</td>
</tr>
</tbody>
</table>

$M = 1, \pi_1 = \frac{E^3(k)k^5}{\varepsilon_0^2}, E(k) \sim \varepsilon_0^{2/5}k^{-5/3}$

$M = 2, \pi_1 = \frac{E^3(k)k^5}{\varepsilon_0^2}, \pi_2 = \frac{v^2}{E_k}$ let $\pi_1 \sim \pi_2^a, E(k) \sim k^{-(5+a)/(3+a)}$
Turbulence and ‘degrees of freedom’

- System is driven on one lengthscale ($L$) and dissipates on another ($\eta$) – forward cascade
- Inverse cascade - same thing, just the other way around
- System has many degrees of freedom i.e. structures on many lengthscales (eddies here)
- System is scaling - structures, processes can be rescaled to ‘look the same on all scales’
- These structures transmit some dynamical quantity from one lengthscale to another that is, over all the d.o.f.
- There is conservation of flux of the dynamical quantity - here energy transfer rate
- Steady state (not equilibrium) means energy injection rate balances energy dissipation rate on the average
Homogeneous Isotropic Turbulence and Reynolds Number

Step 1: write down the relevant variables:

<table>
<thead>
<tr>
<th>variable</th>
<th>dimension</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_0$</td>
<td>$[L]$</td>
<td>driving scale</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$[L]$</td>
<td>dissipation scale</td>
</tr>
<tr>
<td>$U$</td>
<td>$[L][T]^{-1}$</td>
<td>bulk (driving) flow speed</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$[L]^2[T]^{-1}$</td>
<td>viscosity</td>
</tr>
</tbody>
</table>

Step 2: form dimensionless groups: $P = 4, R = 2$, so $M = 2$

$\pi_1 = \frac{UL_0}{\nu} = R_E, \pi_2 = \frac{L_0}{\eta}$ and importantly $\frac{L_0}{\eta} = f(N), \text{where } N \text{ is no. of d.o.f}$

Step 3: d.o.f from scaling ie $f(N) \sim N^\alpha$ here $\frac{L_0}{\eta} \sim N^3$, or $N^{3\beta}$ or $\frac{L_0}{\eta} \sim \lambda^{N/3}$ or ...

Step 4: assume steady state and conservation of the dynamical quantity, here energy...

transfer rate $\varepsilon_r \sim \frac{u_r^3}{r}$, injection rate $\varepsilon_{\text{inj}} \sim \frac{U^3}{L_0}$, dissipation rate $\varepsilon_{\text{diss}} \sim \frac{\nu^3}{\eta^4}$ - gives $\varepsilon_{\text{inj}} \sim \varepsilon_r \sim \varepsilon_{\text{diss}}$

this relates $\pi_1$ to $\pi_2$ giving: $R_E = \frac{UL_0}{\nu} \sim \left(\frac{L_0}{\eta}\right)^{3/3} \sim N^\alpha, \alpha > 0$ thus $N$ grows with $R_E$
Generalize the idea of a Reynolds Number
... a control parameter for the onset of 'disorder'
(turbulence, burstiness)
The above is true for other systems with:

\[ P = 4, R = 2 \ (L, T), \ \text{so} \ M = 2 \]
\[ \pi_1 = R_E \ \text{the Reynolds Number} \]
\[ \pi_2 = f (N) \] where \( N \) is the number of degrees of freedom

flux of some dynamical quantity is conserved- steady state scaling so \( f (N) \sim N^\alpha \)
gives \( \pi_1 = f (\pi_2) \) or \( R_E = f (N) \)
Avalanching systems and scaling behaviour

Avalanche models: add grains slowly, redistribute only if local gradients exceed a critical value. 

Suggested as a model for bursty transport and energy release in plasmas- solar corona, magnetotail, edge turbulence in tokamaks (L-H), accretion disks.

Avalanching systems:
• Threshold for avalanching
• Avalanches are much faster than feeding rate
• Avalanches on all sizes, no characteristic size
• Feeding rate=outflow rate on average only
• System moves through many metastable states- rather than toward an equilibrium
Statistics of avalanches (rice)

Shown: Statistics of energy dissipated per avalanche
- Power law - no characteristic event size: scaling
- ‘finite size scaling’ - Normalize to the size of the box


- Dynamical quantity - rice
- Flux is conserved
- d.o.f. are the possible avalanche (sizes/topplings)
Avalanche model (Self Organized Criticality and all that...)

Step 1:

<table>
<thead>
<tr>
<th>variable</th>
<th>dimension</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_0$</td>
<td>[L]</td>
<td>system size</td>
</tr>
<tr>
<td>$\delta l$</td>
<td>[L]</td>
<td>grid size</td>
</tr>
<tr>
<td>$h$</td>
<td>[S][T]$^{-1}$</td>
<td>average driving rate per node</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>[S][T]$^{-1}$</td>
<td>system average dissipation/loss</td>
</tr>
</tbody>
</table>

Step 2: form dimensionless groups: $P = 4, R = 2, \text{ so } M = 2$

$$\pi_1 = \frac{h}{\epsilon} = R_A, \pi_2 = \frac{L_0}{\delta l} = f(N) \text{ where } N \text{ is no. of d.o.f.}$$

Step 3: d.o.f from scaling ie $f(N) \sim N^\alpha, N \sim \left(\frac{L_0}{\delta l}\right)^\alpha$ with Euclidean dimension $D \geq \alpha > 0$

Step 4: assume steady state and conservation of the dynamical quantity, here sand...S

conservation of flux of sand gives $h \times (\text{no of nodes}) \sim \epsilon$

so $h \left(\frac{L_0}{\delta l}\right)^D \sim \epsilon$ this relates $\pi_1$ to $\pi_2$ giving $R_A = \frac{h}{\epsilon} \sim \left(\frac{\delta l}{L_0}\right)^D \sim N^{-\alpha D}$

this is in the opposite sense to fluid turbulence, $N$ is maximal when $R_A \to 0$
How is SOC different to turbulence? consider...

Intermediate driving (or what happens as we change $R_A \sim \frac{h}{\varepsilon}$):

Suggest two conditions for avalanching transport:

$h\delta t \ll g\delta l$ - takes many timesteps $\delta t$ to make a cell go unstable

$h\delta t \ll g\delta l \left( \frac{L_0}{\delta l} \right)^D$ - takes many timesteps to swamp the system

where $g$ is average critical gradient, $D$ is Euclidean dimension. These are both satisfied for SDIDT $(h \to 0, \varepsilon \to 0)$

If $L_0 \gg \delta l$ we can consider intermediate behaviour $gL_0 \gg h\delta t > g\delta l$

where the smallest avalanches are swamped, but large avalanches persist. Corresponds to:

reducing the available d.o.f. by increasing $h$, and hence $R_A$.
Two runs of the BTW (Bak et al, PRL, [1987]) sandpile driven at the ‘top’ corner formed by two adjacent closed boundaries, the other boundaries are open. The box is 100x100 and $h$ is 4 (●) and 16 (X). Left: raw results; Right: the $h=16$ run is rescaled $S \rightarrow S/16$. 
Two runs of the BTW (Bak et al, PRL, [1987]) sandpile
Box $100 \times 100$, $h=4$ (●); box $400 \times 400$ and $h=16$ (X).
Left: raw results; Right: the $h=16$ run is rescaled $S \rightarrow S/16$.
$h=16$, $400 \times 400$ run has same scaling, dynamic range as $h=4$, $100 \times 100$
To Conclude..

- Scaling - a manifestation of universal behaviour of disordered systems
- Intermittency free scaling in MHD turbulence
- Outlined a general framework for identifying a Reynolds number $R$
- $R$ is the control parameter for a broad range of systems that are many coupled d.o.f., driven, dissipating and on average in steady state
- Scaling, flux conservation relates the Reynolds number to the number of d.o.f.
- Discussed avalanche models for bursty dynamics and turbulence
- Avalanche models - maximal d.o.f. (SOC) when $R \to 0$, in the opposite sense to fluid turbulence, crossover to laminar flow as we increase $R$ but if the system is large enough, we still see ‘SOC’ over a range of $R$ - so applicable to real systems
- Speculate that there are applications elsewhere - level of complexity of ecosystems, of individual organisms, of organizations...
A Reynolds number for ecosystems

- d.o.f. are ‘meta-species’ i.e. any (interchangeable) species that occupies a particular niche in the web
- Species all linked by predation/consumption which processes some dynamical quantity (energy, biomass..)
- System driven by ‘bottom’ species introducing energy/biomass and top predators removing it
- It does not matter what the dynamical quantity is as long as we can conserve flux
  - still ok if there are losses i.e. a fixed fraction is passed from one species to the next, or if there is recycling (bottom species feeding off dead top predators)
- Steady state: timescale over which we change $R_B$ is slow compared to timescale for d.o.f. to propagate the dynamical quantity through the web (recycling time)
Velocity fluctuations parallel and perpendicular to the local $B$ field direction

Exponents $\zeta(p)$ for $\langle |\delta v_{\parallel\perp}|^p \rangle \sim \tau^{\zeta(p)}$ for

$\delta v_{\parallel} = \delta v \cdot \hat{b}$ and its remainder $\delta v_{\perp} = \sqrt{\delta v \cdot \delta v - (\delta v \cdot \hat{b})^2}$

$\bar{B} = B(t) + ... + B(t + \tau')$, $\hat{b} = \frac{\bar{B}}{B}$, here $\tau' = 2\tau$ and $\delta v = v(t + \tau) - v(t)$