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Plasma Theory Seminar Note
"scale in the context of nonlinear theory

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— "scale" in the context of nonlinear theory —

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Abstract

We review the notion of "scale" that plays the central role in nonlinear theory of physics.

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1 Topology / Scale

Literally, "topology" indicates "learning of the topos". The "topos" means "scene" or "landscape". The theory of science starts from observing and describing phenomena in a "scene". To put it more clearly, a "topos=scene" is set when we direct our interest to a certain phenomenon and observe it as an "object". The setting of "topos", i.e., the perspective to see an object is subject to our "interest=subject". For example, when we observe and describe the shape of a string thrown out on a table, our recognition is based on a "reference of the difference" (such as connection, linkage, twist, etc.) —this standard defines the "topology" of a string.

In the analytical theory of science, "topos" is defined by "scales". To analyze natural phenomena, we first project nature to the space of geometrical theory —this is the plan of Galilei, i.e., we start analysis of nature by "measuring" a set of parameters, and describe an "object" as a "vector". In the measurement, i.e. in evaluating numbers to sketch an object, we need "units". However, a choice of "unit" is not simply an arbitrary selection of basis, but is a declaration of a scale to describe the object. Here, the problem of "accuracy" intervenes in the notion of scale —accuracy is the standard to describe "difference", which is the very "topology" in the theory of mathematical analysis (in the theory of analysis, the vector space is endowed with the notion of difference based on set theory, and is called a "topological vector space"). For instance, the length of a coast may not be determined if one wants to measure it in the accuracy of [mm]. So-called "fractals" reveals this non-trivial aspect of measurements.

2 Normalization

We critically review how we select scales in describing an object and creating model equations.

We first introduce "normalization" of variables for evaluating a number measuring an object.

Let t and $\boldsymbol{x}$ denote time and spatial position. Selecting certain "units" of time T and length L, we write

$$\check{t} = \frac{t}{T}, \quad \check{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \tag{1}$$

where the variables with $\check{}$ are "dimensionless" numbers. The "dimension" of time and length are carried by the units T and L. We say that the "numbers" $\check{t}$ and $\check{\boldsymbol{x}}$ are normalized by the units T and L, respectively. Here, the notion of normalization is just enumeration of a quantity by giving a unit.

We are going to give a more profound meaning for "normalization" —we select the unit to make the evaluated numbers to have magnitudes typically of order unity. This is equivalent to chose the unit so that it is the "representative". Then, the "normalization" implies the selection of the scale of our interest (definition of our perspective).

For example, the MKS system of units is the normalization based on human scales: The length, weight and time (or velocity of moving) evaluated in the MKS units for bodies or activities of humankind are normally of order unity.

It is often more convenient to select some other units to measure certain specific physical objects. A theorist selects an appropriate set of units for normalization, which is indeed the declaration of topos=scene of the theory to be developed.

3 Scale of phenomenon / Scale of Model

Let us observe how the description of phenomenon change depending on normalization, i.e. selection of scales. We consider a wave propagation such that

$$u(x,t) = u_0(x - Vt), \tag{2}$$

which obeys a wave equation

$$\frac{\partial}{\partial t}u + V\frac{\partial}{\partial x}u = 0. \tag{3}$$

Here, V (assumed to be a positive number) gives the speed of wave propagation. $u_0(x)$ is the wave form at $t = 0$, which propagates with keeping the shape.

Let us select arbitrary scales L and T to normalize independent variables x and t, respectively. For the wave function u, we chose its representative value U for the unit, and normalize as $\check{u} = u/U$. Using these normalized variables, (2) and (3) read as (with the normalized initial wave form $\check{u}_0(\check{x}) = u_0(x/L)/U$)

$$\check{u}(\check{x},\check{t}) = \check{u}_0(\check{x} - \check{V}\check{t}), \tag{4}$$

and

$$\frac{\partial}{\partial \check{t}}\check{u} + \check{V}\frac{\partial}{\partial \check{x}}\check{u} = 0. \tag{5}$$

Here, the wave velocity V is normalized by a unit velocity L/T:

$$\check{V} = \left(\frac{T}{L}\right)V. \tag{6}$$

The T and L represent the scales of the space-time of our interest. They should be chosen in reference to the object of our study $u(x,t)$, i.e., the L and T should be the scales of space-time where the wave $u(x,t)$ exhibits an appreciable variation. For example, in (2), let $u_0(x - Vt) = U\sin[2\pi(x - Vt)/\lambda]$, i.e., a sinusoidal wave with a wavelength λ. Then, L should be of the order of λ, and T should be the period of oscillation λ/V. Then, L, T and V satisfy

$$L = VT, \tag{7}$$

and both $\partial\check{u}/\partial\check{x}$ and $\partial\check{u}/\partial\check{t}$ are of order unity. From (6), we also see that $\check{V}$ is of order unity. In the normalized wave equation (5), both terms have the same order of magnitudes, and the balance of these terms describes the physical law of wave propagation.

Problem. *See what happens if we apply different normalizations:*
(1) Set $L/T \gg |V|$.
(2) Set $L/T \ll |V|$.

In the previous example (3), the temporal variation (first term) and the spatial variation (second term) are directly related to derive a unique relation (7) of both scales. However, in a more general system, more than two terms make complex relations whose balances may change in different scale hierarchies.

For example, let us add to the wave equation (3) a term including higher order derivative (physically, implying diffusion effect):

$$\frac{\partial}{\partial t}u + V\frac{\partial}{\partial x}u - D\frac{\partial^2}{\partial x^2}u = 0, \tag{8}$$

where D is a positive constant (called a diffusion coefficient). In the theory of PDEs, the highest order derivatives determine the essential characteristic of the PDE. Our equation (8) includes the second-order derivative with respect to x, while the temporal derivative is first order. This type of PDE is called a parabolic PDE.

If we apply the previous normalization (3), we obtain

$$\frac{\partial}{\partial \check{t}}\check{u} + \check{V}\frac{\partial}{\partial \check{x}}\check{u} - \check{D}\frac{\partial^2}{\partial \check{x}^2}\check{u} = 0, \tag{9}$$

where

$$\check{V} = \left(\frac{T}{L}\right)V, \quad \check{D} = \left(\frac{T}{L^2}\right)D. \tag{10}$$

Here again, we should set L and T to match with spatial and temporal variations of the object $u(x,t)$, and hence, all of the factors $\partial\check{u}/\partial\check{t}$, $\partial\check{u}/\partial\check{x}$, and $\partial^2\check{u}/\partial\check{x}^2$ have magnitudes of order unity.

Unlike the previous example, however, there are two different causals (second and third terms) producing the temporal variation (first term). Each of them are, respectively, scaled by the coefficients V and D, which we call "scale parameters". They control the balances among the three terms. When $V \gg D/L$, the third term is negligible with respect to the second term in the left-hand side of (9), for any choice of T. Hence, the behavior of $u(x,t)$ is determined by the balance of the first and second terms, which may be normalized to be of order unity by setting $T = L/V$. On the contrary, when $V \ll D/L$, the third term is superior to the second term in the left-hand side of (9), for any choice of T. Then, the first term and the third term are normalized to the order unity for $T = L^2/D$.

The above-mentioned observation may be abstracted as follows. Let us consider an equation describing the balance among three different factors A, B, C, i.e.,

$$A + B + C = 0. \tag{11}$$

For these three terms to play their own role in this "topos=scene", each of them must have the same order of magnitude that we normalize to be of order unity. If one of them, say C, has a much smaller magnitude, the equation (11) degenerates into $A + B = 0$.

An interesting change may occur when we observe a different scale hierarchy (shift to a different topos). When we normalize variables to a larger scale, a term A may disappear, while, to a smaller scale, B may —representation of a law (parameterization of an object) may change depending on the normalization.

This statement might cause a confusion, if we miss the deeper meaning of the "normalization". When we write a law in the form of (11), every term of A, B and C must have the same "dimension". Therefore, just changing the units of parameters (for example, shifting from MKS to CGS systems) cannot change the balance of the terms —indeed, any well-posed law must be "unit-invariant". Let us look at (8) as an example: If u is a quantity that has a dimension of $[\mu]$, every term is of dimension $[\mu/T]$ (T is a unit of time). If we change the units as $T \to T' = \alpha T$, $L \to L' = \beta L$ and $\mu \to \mu' = \gamma\mu$ with factors α, β and γ, all terms are divided by the same factor α/γ. Hence, the balance of the terms cannot change.

Then, why do we claim that the balance of terms changes by varying the "normalization"? It is because, as we emphasized at the beginning, the "normalization" is not only a selection of units, but it is a determination of the "scales" of our observation. The variables of our interest (in the above example, the independent variables $\check{t}$, $\check{x}$ and dependent variable $\check{u}$) are set to be of order unity; this is the determination of the scale. When we change the scales of interest (the representative values of the variables), the newly normalized variables must be again of order unity. This condition may cause changes in the "scale parameters" (in the previous example, $\check{V}$ and $\check{D}$), because each term consists of the variables we want to normalize and a scale parameter, which split dimensions differently. In the example of (10), the scale parameters transform as $\check{V} \to \check{V}/(\beta/\alpha)$, $\check{D} \to \check{D}/(\beta^2/\alpha)$. It is these "scale parameters" that bring about changes in the balance of the terms in a law.

4 Nonlinearity and Singularity

Up to now, we discussed linear systems where the scale of interest can be easily prescribed. This is primarily because the estimates of the scale parameters depend only on the independent variables —in (6) or (10), for example, only L and T are involved, while they are independent of U.

In a nonlinear system, however, the scales of interest depend above all on dependent=unknown variables —so to say, nature chooses the scale.

Let us examine a simple example. We replace the V in (3) by u to make a nonlinear wave equation

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = 0, \tag{12}$$

where the dependent variable u itself is the propagation speed. If the representative values of time T, length L and the dependent variable U (having the dimension of velocity) satisfy a relation

$$L = UT, \tag{13}$$

we can normalize (12) as

$$\frac{\partial}{\partial \check{t}}\check{u} + \check{u}\frac{\partial}{\partial \check{x}}\check{u} = 0. \tag{14}$$

Here, we remark that the condition (13) includes the scale U of the unknown u. Hence, if we fix L, for instance, the appropriate time scale for observation must be altered as the unknown u changes, which is in sharp contrast to the linear case (7).

Another interesting difference from the linear example (5) is that the normalized equation (14) does not include any scale parameter, and hence, it is totally equivalent to the original equation (12). This fact implies that the nonlinear equation (12) is "scale invariant" with respect to the transformation $t \to \check{t}$, $x \to \check{x}$ and $u \to \check{u}$ that satisfy the relation (13). This cannot happen in the linear equation (3), where scale transformations of T and L cause, with no way out of altering U, change the scale parameter $\check{V}$.

Nature does choose the scale —this is enabled by "nonlinearity". On the other hand, the scale is a subjective "perspective" that we select to observe the phenomenon. The ambivalent notion of "scale" connotes a possibility of causing a serious discrepancy between the subjectivity of observer and the object. And this discrepancy can be actualized as a creation of "singularity".

Here, a singularity means a place (in time-space) where variations (derivatives) of fields (functions) diverge (or becomes undefinable). The scale L of the independent variable x that yields a variation of a function $f(x)$ is evaluated as

$$L = \frac{|f(x)|}{|df(x)/dx|}. \tag{15}$$

In the neighborhood of a singularity, the denominator $|df(x)/dx|$ diverges, and the scale L shrinks to zero. And if L is really zero, L cannot be magnified to be a finite number. In this sense, a singularity is a special configuration of scale-invariance, and it may be created by some scale-invariant mechanism.

Let us see how a singularity can be created by invoking the previous nonlinear model (12). We assume an initial distribution such as

$$u_0(x) = \begin{cases} -a & (x \leq -1) \\ ax & (-1 < x < 1) \\ a & (1 \leq x), \end{cases} \tag{16}$$

where a is a real number of order unity. The solution of (12) is

$$u(x,t) = \begin{cases} -a & (x \leq -1 - at) \\ ax/(1+at) & (-1 - at < x < 1 + at) \\ a & (1 + at \leq x). \end{cases} \tag{17}$$

For $a < 0$, we must restrict $t < -1/a$.

If $a > 0$, $u(x,t)$ is made smooth with time. On the other hand, if $a < 0$, the slope of $u(x,t)$ gets steeper with time, and at $t = -1/a$, it becomes infinite at $x = 0$.

Observing (17), the length scale of the variation of $u(x,t)$ is estimated $1 + at$ that changes with t, For $a < 0$, the length scale becomes zero at a finite time $-1/a$.

In a nonlinear system, the scale changes autonomously, and sometimes, infinitely small scale=singularity is created. Creation of singularity ruins the model (differential equation). Then, we should not persist in the original selection of the scale, but we have to gazes at smaller scale. By changing the perspective, we will be able find structures created by the nonlinearity.

5 Reductive Perturbation Method

Description of nonlinear phenomena depends strongly on the choice of scales of both independent and dependent variables. Here, we study the so-called "reductive perturbation" that is a strong method to chose an "optimal scales" that highlight simple structures in a nonlinear system.

We invoke a well-known example of plasma physics, that is the ion acoustic soliton. The governing equations are, in a dimension-less units,

$$\begin{cases} \partial_t n + \partial_x(un) = 0, \\ \partial_t u + u\partial_x u + \partial_x \phi = 0, \\ \partial_x^2 \phi = e^{\phi} - n, \end{cases} \tag{18}$$

where n is the ion density, u is the ion flow velocity (one dimensional), ϕ is the electrostatic potential. In the system (18), the first equation is the mass conservation law, the second one is the momentum equation with the electrostatic force $\partial_x \phi$, and the third one is the Poisson equation with the electron density having the Maxwell distribution e^{ϕ}.

We have a trivial solution $n = 1, u = 0, \phi = 0$. Our interest is a small (but nonlinear) perturbation on this stationary state. Introducing a small parameter ε, we expand the dependent variables:

$$\begin{cases} n = 1 + \varepsilon n^{(1)} + \varepsilon^2 n^{(2)} + \cdots, \\ u = \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \cdots, \\ \phi = \varepsilon \phi^{(1)} + \varepsilon^2 \phi^{(2)} + \cdots. \end{cases} \tag{19}$$

All new variables $n^{(k)}, u^{(k)}, \phi^{(k)}$ $(k = 1, 2, \cdots)$ are assumed to be of order unity. Plugging (19) into (18) yields determining equations of the new variables, in which the number of unknown variables have been increased infinitely, so they are not useful at this stage (note that the nonlinearity causes couplings of different scales, producing unseparable hierarchy of equations).

However, we can choose new scales of independent variables that simplify the system of equation, i.e., that organize a hierarchy of separated scales in the order of the powers of ε.

Let us transform the independent variables $x$and t as

$$\xi = \varepsilon^{1/2}(x - t), \quad \tau = \varepsilon^{3/2} t. \tag{20}$$

The deferential operators are written as

$$\begin{aligned} \partial_t &= \frac{\partial \xi}{\partial t}\partial_\xi + \frac{\partial \tau}{\partial t}\partial_\tau = -\varepsilon^{1/2}\partial_\xi + \varepsilon^{3/2}\partial_\tau, \\ \partial_x &= \frac{\partial \xi}{\partial x}\partial_\xi + \frac{\partial \tau}{\partial x}\partial_\tau = \varepsilon^{1/2}\partial_\xi. \end{aligned}$$

Using these new variables in (18), we obtain

$$\begin{aligned}
&\varepsilon^{3/2}\left[\partial_\xi(-n^{(1)}+u^{(1)})\right] \\
&\qquad +\varepsilon^{5/2}\left[\partial_\tau n^{(1)}+\partial_\xi(n^{(1)}u^{(1)})+\partial_\xi(u^{(2)}-n^{(2)})\right]+\cdots=0, \\
&\varepsilon^{3/2}\left[\partial_\xi(-u^{(1)}+\phi^{(1)})\right] \\
&\qquad +\varepsilon^{5/2}\left[\partial_\tau u^{(1)}+u^{(1)}\partial_\xi u^{(1)}+\partial_\xi(\phi^{(2)}-u^{(2)})\right]+\cdots=0, \\
&\varepsilon^{3/2}\left[\partial_\xi(-\phi^{(1)}+n^{(1)})\right] \\
&\qquad +\varepsilon^{5/2}\left[\partial_\xi^3\phi^{(1)}-\phi^{(1)}\partial_\xi\phi^{(1)}+\partial_\xi(n^{(2)}-\phi^{(2)})\right]+\cdots=0.
\end{aligned}$$

Summing up the terms multiplied by $\varepsilon^{3/2}$, we obtain

$$n^{(1)}=u^{(1)}=\phi^{(1)}. \tag{21}$$

From the order of $\varepsilon^{5/2}$, we obtain three equation, which are summed up as, using (21),

$$\partial_\tau\phi^{(1)}+\phi^{(1)}\partial_\xi\phi^{(1)}+\frac{1}{2}\partial_\xi^3\phi^{(1)}=0. \tag{22}$$

The first order variable $\phi^{(1)}(\xi,\tau)$ (simultaneously, $n^{(1)}, u^{(1)}$ have the common waveforms) is determined (separately from the higher-order terms) by this "KdV equation".

Let us analyze what was the essential ingredient that derived this simplified equation. KdV is the most simple equation describing the balance between the nonlinearity (that tends to produce singularity) and the dispersion (that works to prevent singularity). In the system (18), the Poisson equation (including a higher-order derivative ∂_x^2) is the origin of the dispersion. The point of the ordering (20) is that the second order spatial derivative ∂_x^2 scales as $\varepsilon^1\partial_\xi^2+\cdots$ to make a balance with a second-order nonlinearity that appears as multiplication of $\varepsilon^1\phi^{(1)}+\cdots$. Therefore, at the length scale $\varepsilon^{1/2}x$, the second-oder nonlinearity produced by perturbations of the scale ε can make a balance with the dispersion term.

Problem. *Try a different scaling*

$$\xi=\varepsilon(x-t),\quad \tau=\varepsilon^2 t,$$

and see what happens.

6 Co-existence of Different Scales —turbulence theory

The analysis of the former section was to pick up (and separate) a specific scale hierarchy where a nonlinearity and a higher-order effect (singular perturbation) can make a balance. In a general nonlinear system, however, "co-existence" and complex interactions of many different scales are the most essential nature —a typical example is the "turbulence". In this section, we study how we can estimate balances of nonlinearity and singular perturbation.

Let us consider an incompressible ($\nabla \cdot \boldsymbol{V} = 0$) viscous fluid obeying the Navier-Stokes equation

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = \mu \Delta \boldsymbol{V} - \nabla h + \boldsymbol{F}, \tag{23}$$

where h is the enthalpy, μ is the kinematic viscosity and F is a certain external force.

The "Reynolds number" is the ration of the average magnitudes of the nonlinear term $(\boldsymbol{V} \cdot \nabla)\boldsymbol{V}$ and the viscous damping term $\mu \Delta \boldsymbol{V}$:

$$Re = \frac{|(\boldsymbol{V} \cdot \nabla)\boldsymbol{V}|}{|\mu \Delta \boldsymbol{V}|} \approx \frac{VL}{\mu}, \tag{24}$$

where V and L are the typical velocity and length scale.

We immediately notice that Re depends on the scale L of our interest (or the system of subject). Denoting by F_k the magnitude of a quantity F in the scale hierarchy of k ($= L^{-1}$: typical wave number), we should write

$$Re(k) = \frac{V_k}{k\mu} \tag{25}$$

to specify the Reynolds number of the subject system of our interest.

If V_k is not an increasing function of k (we assume that $\boldsymbol{F}$ exists only in a very large scale), a smaller scale has a smaller $Re(k)$. The scale where $Re(k) = 1$ is called the "Kolmogorov scale" where the viscous damping starts to work. On the other hand, in a larger scale hierarchy with $Re(k) \gg 1$, the viscosity term may be neglected –this is the "inertial range".

By this definition, the Kolmogorov scale k_D is given by

$$k_D = \frac{V_{k_D}}{\mu}. \tag{26}$$

To estimate (eliminate) V_k, we have to invoke the "energy transfer rate" ε that is (assumed to be) scale invariant and equal to the energy input rate (by the external forcing $\boldsymbol{F}$ at a large scale) as well as the "energy damping rate":

$$\varepsilon = \mu k^2 V_k^2|_{k=k_D} = \frac{1}{\mu} V_{k_D}^4. \tag{27}$$

Solving for V_{k_D}, we obtain

$$V_{k_D} = \varepsilon^{1/4}\mu^{1/4}.$$

Substituting this into (26) yields

$$k_D = \varepsilon^{1/4}\mu^{-3/4}. \tag{28}$$

This relation shows that the "dissipative length scale" (k_D^{-1}) is determined by not only the viscosity μ but also by the energy input rate ε.

Remark. In the inertial range, the energy transfer rate is purely dominated by the nonlinear term. By Kolmogorov's locality ansatz, ε is estimated by (in view of NS equation)

$$\varepsilon = \langle (\boldsymbol{V}\cdot\nabla)\boldsymbol{V}\cdot\boldsymbol{V}\rangle|_k \approx V_k^3 k,$$

which must be independent of k. Hence, we may estimate $V_k \propto \varepsilon^{1/3}k^{-1/3}$. The energy in the range of scale $(k, k+\delta)$ is $\mathcal{E}(k, k+\delta) := V_k^2\delta = C\varepsilon^{2/3}k^{-2/3}\delta$. The energy spectrum density $E(k)$ is defined by

$$\mathcal{E}(k) = \int_k^{k+\delta} E(k)\, dk.$$

We thus find

$$E(k) = C'\varepsilon^{2/3}k^{-5/3}, \quad (C' = (9/8)C).$$

which is the Kolmogorov spectrum.