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Inflation and Cosmological Perturbations.

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Inflation and cosmological perturbations

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Definitions and formulae

References

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1 Metric perturbations

1.1 FRW background

Infinitesimal displacement between two events in a homogeneous background spacetime described by a spatially flat FRW background metric with scale factor a:

$$\begin{aligned} ds^2 &= -dt^2 + a^2 \delta_{ij} dx^i dx^j \,, \\ &= a^2 \left[-d\eta^2 + \delta_{ij} dx^i dx^j \right] \,, \end{aligned} \tag{1.1}$$

where t is cosmic time, corresponding to proper time measured by observers at fixed spatial coordinates, and η is conformal time.

The Hubble expansion rate is

$$H = \frac{\dot{a}}{a} \,, \tag{1.2}$$

and the conformal Hubble rate is

$$\mathcal{H} = \frac{a'}{a} = aH \,. \tag{1.3}$$

1.2 Perturbed metric

Allowing for arbitrary inhomogeneous perturbations we have

$$\begin{aligned} ds^2 = a^2 \Big[&-(1+2A)d\eta^2 + 2(B_{,i} - S_i)dx^i d\eta \\ &+ \left\{ (1-2\psi)\delta_{ij} + 2E_{,ij} + 2F_{(i,j)} + h_{ij} \right\} dx^i dx^j \Big] \,, \end{aligned} \tag{1.4}$$

where $B_{,i} = \partial B / \partial x^i$. My notation is the same as that in [4] and only differs from [2] in the signature of the metric and the perturbed lapse function A is ϕ in [2].

The intrinsic Ricci scalar curvature of constant time hypersurfaces is given by

$${}^{(3)}R = \frac{4}{a^2} \partial^2 \psi \,, \tag{1.5}$$

where $\partial^2 \equiv \delta^{ij} (\partial / \partial x^i)(\partial / \partial x^j)$ is the spatial Laplacian. Hence we refer to ψ as the curvature perturbation.

1.3 Gauge transformations

Under a first-order coordinate transformation

$$\eta \rightarrow \eta + \delta\eta \,, \tag{1.6}$$

$$x^i \rightarrow x^i + \delta^{ij} \delta x_{,j} + \bar{\delta x}^i \,, \tag{1.7}$$

scalar field perturbations transform as

$$\delta\varphi \rightarrow \delta\varphi - \varphi' \delta\eta \,. \tag{1.8}$$

- Scalar metric perturbations transform as

$$
\begin{aligned}
A &\rightarrow A - \mathcal{H}\delta\eta - \delta\eta' \,, &(1.9)\\
B &\rightarrow B + \delta\eta - \delta x' \,, &(1.10)\\
E &\rightarrow E - \delta x \,, &(1.11)\\
\psi &\rightarrow \psi + \mathcal{H}\delta\eta \,. &(1.12)
\end{aligned}
$$

Note that the scalar shear

$$\sigma = E' - B \,, \tag{1.13}$$

is independent of the spatial gauge

$$\sigma \rightarrow \sigma - \delta\eta \tag{1.14}$$

- Vector metric perturbations transform as

$$
\begin{aligned}
S_i &\rightarrow S_i + \bar{\delta x}_i' \,, &(1.15)\\
F_i &\rightarrow F_i - \bar{\delta x}_i \,, &(1.16)
\end{aligned}
$$

so that the vector shear $F_i' + S_i$ is gauge independent.

- Tensor metric perturbation h_{ij} is gauge independent and describes gravitational waves.

1.4 Gauge invariant variables

- Longitudinal gauge ($\tilde{\sigma} = 0$):

$$
\begin{aligned}
\Phi &= A - \mathcal{H}\sigma - \sigma' \,, &(1.17)\\
\Psi &= \psi + \mathcal{H}\sigma \,. &(1.18)
\end{aligned}
$$

- Uniform-density gauge ($\tilde{\delta\rho} = 0$):

$$\zeta = -\psi - \frac{\mathcal{H}}{\rho'}\delta\rho \,. \tag{1.19}$$

- Comoving orthogonal gauge ($\tilde{v} + \tilde{B} = 0$):

$$
\begin{aligned}
\mathcal{R} &= \psi - \mathcal{H}(v + B) \,, &(1.20)\\
\delta\rho_{\rm com} &= \delta\rho + \rho'(v + B) \,. &(1.21)
\end{aligned}
$$

For a scalar field $v + B = -\delta\varphi/\varphi'$ and hence

$$\mathcal{R} = \psi + \frac{\mathcal{H}}{\varphi'}\delta\varphi \,. \tag{1.22}$$

- Spatially-flat gauge ($\psi = 0$):

$$Q = \delta\varphi + \frac{\varphi'}{\mathcal{H}}\psi = \frac{\varphi'}{\mathcal{H}}\mathcal{R} \,. \tag{1.23}$$

2 Field equations perturbed at first-order

2.1 Scalar perturbations

- Einstein equations
  - Energy and momentum constraints

$$3\mathcal{H}\left(\psi' + \mathcal{H}A\right) - \nabla^2\left[\psi + \mathcal{H}\sigma\right] = -4\pi G a^2 \delta\rho, \tag{2.24}$$

$$\psi' + \mathcal{H}A = -4\pi G a^2 (\rho + P)(v + B)\,, \tag{2.25}$$

  - Evolution equations (trace and trace-free parts)

$$\psi'' + 2\mathcal{H}\psi' + \mathcal{H}A' + \left(2\mathcal{H}' + \mathcal{H}^2\right)A = 4\pi G a^2 \left(\delta P + \frac{2}{3}\nabla^2 \Pi\right)\,, \tag{2.26}$$

$$\sigma' + 2\mathcal{H}\sigma + \psi - A = 8\pi G a^2 \Pi\,, \tag{2.27}$$

  where Π is the scalar part of the (tracefree) anisotropic stress.

- Energy and momentum conservation equations

$$\delta\rho' + 3\mathcal{H}\left(\delta\rho + \delta P\right) - 3\left(\rho + P\right)\psi' + (\rho + P)\nabla^2\left(v + E'\right) = 0\,, \tag{2.28}$$

$$(v + B)' + (1 - 3c_{\rm s}^2)\mathcal{H}(v + B) + \phi + \frac{1}{\rho + P}\left(\delta P + \frac{2}{3}\nabla^2\Pi\right) = 0\,, \tag{2.29}$$

  where $c_{\rm s}^2 = P'/\rho'$ is the adiabatic speed of sound.

2.2 Vector perturbations

- Einstein equations
  - Momentum constraint

$$\nabla^2\left(F_i' + S_i\right) = -16\pi G a^2 \delta q_i\,, \tag{2.30}$$

  relates the vector shear perturbation to the vector part of the 3-momentum

$$\delta q_i = (\rho + P)(\bar{v}_i - S_i)\,. \tag{2.31}$$

- Momentum conservation (equivalent to Einstein evolution equation for vector shear)

$$\delta q_i' + 4\mathcal{H}\delta q_i = -\partial^2 \Pi_i\,, \tag{2.32}$$

  where the vector part of the anisotropic stress is given by $\Pi_{(i,j)}$.

2.3 Tensor perturbations

The transverse, tracefree part of the Einstein evolution equations yields a simple wave equation

$$h_{ij}'' + 2\mathcal{H}h_{ij}' - \partial^2 h_{ij} = 8\pi G_N a^2 \Pi_{ij}\,, \tag{2.33}$$

where $\Pi_{ij}^{(TT)}$ is the transverse and tracefree part of the anisotropic stress. Tensor metric perturbations are not subject to constraint equations and describe the free part of the gravitational field.

3 Perturbations from inflation

3.1 Massless field in unperturbed FRW

Klein-Gordon for inhomogeneous massless field

$$\varphi = \varphi_0(t) + \delta\varphi(t, \mathbf{x}) \,, \tag{3.34}$$

in unperturbed FRW spacetime

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} + \frac{k^2}{a^2}\delta\varphi = 0 \,. \tag{3.35}$$

Change to conformal time, η, and rescaled field $v = a\delta\varphi$ gives

$$v'' + \left(k^2 - \frac{a''}{a}\right) v = 0 \,. \tag{3.36}$$

where it is useful to note that

$$\frac{a''}{a} = (2 - \varepsilon) a^2 H^2 \,, \tag{3.37}$$

where $\varepsilon = -\dot{H}/H^2$ is bounded to be $0 \leq \varepsilon < 1$ during inflation.

Equation (3.36) easiest to solve in two limits

- Large scales: $k^2 \ll a^2 H^2$ (super-Hubble)

$$v'' - \frac{a''}{a} v \approx 0 \,. \tag{3.38}$$

  Growing mode solution: $v \propto a \;\Rightarrow\; \delta\varphi \to$ constant ("frozen-in").

- Small scales: $k^2 \gg a^2 H^2$ (sub-Hubble)

$$v'' + k^2 v \approx 0 \,. \tag{3.39}$$

  Free oscillations $v \propto e^{\pm ik\eta} \;\Rightarrow\; \delta\varphi \propto e^{\pm ik\eta}/a$.

During inflation $aH = \dot{a}$ increases, i.e., comoving Hubble scale shrinks and modes that start in vacuum state on sub-Hubble scales at early times are stretched to super-Hubble scales at late times.

Zero-point fluctuations of quantum vacuum state on sub-Hubble scales

$$\langle \delta\varphi_{\mathbf{k}} \delta\varphi_{\mathbf{k}'} \rangle = \frac{1}{2ka^2} (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') \,. \tag{3.40}$$

sets the amplitude of growing mode perturbations on super-Hubble scales

$$\langle \delta\varphi_{\mathbf{k}} \delta\varphi_{\mathbf{k}'} \rangle_{k<aH} = \left(\frac{1}{2ka^2}\right)_{k=aH} (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') \,. \tag{3.41}$$

Power spectrum on super-Hubble scales:

$$P_{\delta\varphi}(k) = \left(\frac{1}{2ka^2}\right)_{k=aH} = \left(\frac{H^2}{2k^3}\right)_{k=aH} . \tag{3.42}$$

Alternative power spectrum (variance) per logarithmic range of k:

$$\mathcal{P}_{\delta\varphi}(k) = \frac{4\pi k^3}{(2\pi)^3} P_{\delta\varphi}(k) = \left(\frac{H}{2\pi}\right)^2_{k=aH} . \tag{3.43}$$

Scale invariant spectrum if H at Hubble-crossing is independent of time.

3.2 Single massive field in perturbed FRW

Linearly perturbed Klein-Gordon equation including metric perturbations

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} + \left(\frac{k^2}{a^2} + V''\right)\delta\varphi = -2V'A + \dot{\delta\varphi}\left(\dot{A} + 3\dot{\psi} + \frac{k^2}{a^2}\sigma\right) , \tag{3.44}$$

where $V' = dV/d\varphi$.

Work with the field perturbation in the spatially-flat gauge

$$Q = \delta\varphi + \frac{\dot{\varphi}}{H}\psi , \tag{3.45}$$

and use constraint equation to eliminate remaining metric perturbations, gives

$$\ddot{Q} + 3H\dot{Q} + \left[\frac{k^2}{a^2} + V'' - \frac{8\pi G}{3}\frac{d}{dt}\left(\frac{a^3\dot{\varphi}^2}{H}\right)\right] Q = 0 . \tag{3.46}$$

Change to conformal time, η, and rescaled field $v = a\delta\varphi$ gives

$$v'' + \left(k^2 - \frac{z''}{z}\right) v = 0 . \tag{3.47}$$

where $z = a\dot{\varphi}/H$ and in slow-roll

$$\frac{z''}{z} \simeq (2 + 5\epsilon_V - 3\eta_V)a^2H^2 . \tag{3.48}$$

For slow-roll inflaton field ($\eta_V \ll 1$), equation (3.47) easiest to solve in two limits

- Large scales: $k^2 \ll a^2H^2$ (super-Hubble)

$$v'' - \frac{z''}{z} v \approx 0 . \tag{3.49}$$

Growing mode solution: $v \propto z \ \Rightarrow\ Q \propto \dot{\varphi}/H \ \Rightarrow$ comoving curvature $\mathcal{R} \to$ constant.

- Small scales: $k^2 \gg a^2 H^2$ (sub-Hubble)

$$v'' + k^2 v \approx 0\,. \tag{3.50}$$

  Free oscillations $v \propto e^{\pm ik\eta} \;\Rightarrow\; \delta\varphi \propto e^{\pm ik\eta}/a$.

Zero-point fluctuations of quantum vacuum state on sub-Hubble scales sets the amplitude of growing mode perturbations on super-Hubble scales

$$\langle Q_{\mathbf{k}} Q_{\mathbf{k}'} \rangle_{k<aH} \simeq \left(\frac{1}{2ka^2} \right)_{k=aH} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k}')\,. \tag{3.51}$$

Power spectrum on super-Hubble scales:

$$P_{\mathcal{R}}(k) \simeq \left(\frac{H^2}{2ka^2\dot{\varphi}^2} \right)_{k=aH} = \left(\frac{H^4}{2\dot{\varphi}^2 k^3} \right)_{k=aH}\,. \tag{3.52}$$

Alternative power spectrum (variance) per logarithmic range of k:

$$\mathcal{P}_{\mathcal{R}}(k) \simeq \left(\frac{H^2}{2\pi\dot{\varphi}} \right)^2_{k=aH} \simeq \left(\frac{8V}{3\epsilon_V M_{\rm Pl}^4} \right)_{k=aH}\,. \tag{3.53}$$

Because $\mathcal{R}$ is constant on super-Hubble scales for adiabatic perturbations, this can be identified with the amplitude of primordial curvature perturbations measured by WMAP

$$\mathcal{P}_{\mathcal{R}}(k = 0.002\mathrm{Mpc}^{-1}) \approx 2.5 \times 10^{-9} \tag{3.54}$$

3.3 Tensor perturbations

Amplitude of tensor metric perturbation

$$h_{ij} = h(t) q_{ij}(\mathbf{k}) e^{i\mathbf{k}.\mathbf{x}}\,. \tag{3.55}$$

where q_{ij} is transverse and tracefree polarisation tensor, obeys same equation of motion as a massless field in unperturbed FRW spacetime

$$\ddot{h} + 3H\dot{h} + \frac{k^2}{a^2} h = 0\,. \tag{3.56}$$

Change to conformal time, η, and rescaled field $v = ah/\sqrt{32\pi G}$ (which has same effective action as v for a canonical massless field) gives

$$v'' + \left(k^2 - \frac{a''}{a} \right) v = 0\,. \tag{3.57}$$

Two limits

- Large scales: $k^2 \ll a^2 H^2$ (super-Hubble)

$$v'' - \frac{a''}{a} v \approx 0\,. \tag{3.58}$$

  Growing mode solution: $v \propto a \;\Rightarrow\; \delta\varphi \to \text{constant}$ ("frozen-in").

- Small scales: $k^2 \gg a^2H^2$ (sub-Hubble)

$$v'' + k^2 v \approx 0\,. \tag{3.59}$$

Free oscillations $v \propto e^{\pm ik\eta} \;\Rightarrow\; h \propto e^{\pm ik\eta}/a$.

Zero-point fluctuations of quantum vacuum state on sub-Hubble scales sets the amplitude of growing mode perturbations on super-Hubble scales

$$\langle h_{\mathbf{k}} h_{\mathbf{k}'}\rangle_{k<aH} = \left(\frac{32\pi G}{2ka^2}\right)_{k=aH} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k}')\,. \tag{3.60}$$

Power spectrum for tensor perturbations on super-Hubble scales:

$$P_T(k) = 2\left(\frac{32\pi G}{2ka^2}\right)_{k=aH} = 2\left(\frac{32\pi G H^2}{2k^3}\right)_{k=aH}\,. \tag{3.61}$$

where extra factor of 2 for 2 polarisations of tensor perturbations. Alternative power spectrum (variance) per logarithmic range of k:

$$\mathcal{P}_T(k) = \frac{4\pi k^3}{(2\pi)^3} P_T(k) = 64\pi G \left(\frac{H}{2\pi}\right)^2_{k=aH}\,. \tag{3.62}$$

Gives tensor-scalar ratio

$$r = \frac{P_T}{P_{\mathcal{R}}} \simeq 16\epsilon_V\,. \tag{3.63}$$

WMAP5 constrains $r < 0.2$ and hence $\epsilon_V < 0.01$.

3.4 Scale-dependence

- Tensors:

$$\begin{aligned} n_T &= \frac{d\ln \mathcal{P}_T}{d\ln k}\,, \\ &\simeq \frac{d}{d\ln k}\left(\ln H^2\right)_{k=aH}\,, \\ &\simeq \frac{dt}{d\ln(aH)}\frac{d\ln H^2}{dt}\,, \\ &\simeq -2\epsilon_V\,. \end{aligned} \tag{3.64}$$

Note $n_T < 0$ [inflation is falsifiable!] and there is a consistency equation

$$r \simeq -16\epsilon_V \simeq -8n_T\,. \tag{3.65}$$

- Scalars:

$$\begin{aligned} n_{\mathcal{R}} - 1 &= \frac{d\ln \mathcal{P}_{\mathcal{R}}}{d\ln k}\,, \\ &\simeq -6\epsilon_V + 2\eta_V\,. \end{aligned} \tag{3.66}$$

WMAP5 constrains $n_{\mathcal{R}} = 0.97 \pm 0.02$ [first evidence for slow-roll dynamics?]