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# Summer School in Cosmology

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## Gravitational waves
## Lecture 2
## Additional notes

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Waves on a curved background:

Putting $g_{\alpha\beta} = \hat{g}_{\alpha\beta} + h_{\alpha\beta}$ is illustrative, but highly restrictive. More generally,

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + h_{\alpha\beta}$$

A slowly varying background: universe, black hole, solar system, ...

We assume there exists a separation of scales:

$$\lambda \equiv \text{scale of radiation}$$
$$\ll \mathcal{L} \equiv \text{scale of background}$$

$$\tau \equiv \text{timescale of radiation } (= \lambda/c)$$
$$\ll \mathcal{T} \equiv \text{timescale of background.}$$

Can then separate B.G. from wave by an appropriate <u>averaging</u> procedure:

$$\hat{g}_{\alpha\beta} \equiv \langle g_{\alpha\beta} \rangle$$
$$h_{\alpha\beta} \equiv g_{\alpha\beta} - \langle g_{\alpha\beta} \rangle$$

where $\langle\,\rangle$ means to average enclosed quantities over a scale $L$, where $\lambda \ll L \ll \mathcal{L}$. Result is unique up to errors $O(\lambda^2/\mathcal{L}^2)$.

For simplicity, restrict ourselves to vacuum spacetimes:

$$G_{\mu\nu} = 0$$

Consider our spacetime to have a form

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + \varepsilon h_{\alpha\beta} + \varepsilon^2 j_{\alpha\beta} + \mathcal{O}(\varepsilon^3)$$

$\varepsilon$ is a formal order counting parameter.

Develop the Einstein tensor for this spacetime, expanding the result in powers of $\varepsilon$:

$$G_{\mu\nu}(g_{\alpha\beta}) = 0$$

$$\rightarrow \quad 0 = G^0_{\mu\nu}(\hat{g}_{\alpha\beta}) + \varepsilon G^1_{\mu\nu}(h_{\alpha\beta}) + \varepsilon^2 G^2_{\mu\nu}(h_{\alpha\beta}) + \varepsilon^2 G^1(j_{\alpha\beta})$$

We require this equation to hold at each order in $\varepsilon$:

$\varepsilon^0$ : $G^0_{\mu\nu}(\hat{g}_{\alpha\beta}) = 0 \rightarrow$ Statement that our background is a vacuum solution.

$\varepsilon^1$: $G^1_{\mu\nu}(h_{\alpha\beta}) = 0 \rightarrow$ wave equation for $h_{\alpha\beta}$ on background. Expand, we find

$$-\frac{1}{2}\hat{\Box}\bar{h}_{\alpha\beta} - \underset{\substack{\uparrow \\ \text{Riemann of b.g.}}}{\hat{R}_{\alpha\mu\beta\nu}}\bar{h}^{\mu\nu} = 0$$

$\hat{\Box} \equiv \hat{g}^{\mu\nu}\hat{\nabla}_\mu\hat{\nabla}_\nu$ ; $\hat{\nabla}_\mu \equiv$ covariant derivative.

$\varepsilon^2$ : $\quad G^1_{\mu\nu}(j_{\alpha\beta}) = - G^2_{\mu\nu}(h_{\alpha\beta})$

$G^2_{\mu\nu}$ is a (messy) tensor quadratic in $h_{\alpha\beta}$.

Schematically,

$$G^2_{\mu\nu}(h_{\alpha\beta}) \sim h\hat{\nabla}^2 h + (\hat{\nabla} h)^2$$

To proceed, it is useful to use separation of lengthscales on 2ND-order perturbation:

$$j_{\alpha\beta} \equiv \langle j_{\alpha\beta}\rangle + \delta j_{\alpha\beta}$$

varies on $\mathcal{L}$ $\qquad$ oscillatory, varies on $\lambda$.

Note metric now has form

$$g_{\alpha\beta} = g^{\mathcal{L}}_{\alpha\beta} + \varepsilon h_{\alpha\beta} + \varepsilon^2 \delta j_{\alpha\beta}$$

where $g^{\mathcal{L}}_{\alpha\beta} \equiv \hat{g}_{\alpha\beta} + \varepsilon^2 \langle j_{\alpha\beta}\rangle$ is a "new background," including a 2ND order piece.

→ New background is all pieces of metric that vary on background scale $\mathcal{L}$.

To see where modification of background comes from, average Einstein:

$$0 = G^0_{\mu\nu}(\hat{g}_{\alpha\beta}) + \varepsilon \, G^1_{\mu\nu}(\cancel{\langle h_{\alpha\beta} \rangle}) \xrightarrow{} 0$$

can take averaging inside Einstein operator $G^1$ due to linearity

$$+ \ \varepsilon^2 \langle G^2_{\mu\nu}(h_{\alpha\beta}) \rangle + \varepsilon^2 G^1_{\mu\nu}(\langle j_{\alpha\beta} \rangle)$$

or

$$\boxed{G_{\mu\nu}\left(\hat{g}_{\alpha\beta} + \varepsilon^2 \langle j_{\alpha\beta} \rangle\right) = -\varepsilon^2 G^2_{\mu\nu}(h_{\alpha\beta})}$$

The second-order Einstein tensor acting on $h_{\alpha\beta}$ is effectively a <u>source</u> modifying our background spacetime!

Definition:

$$T^{GW}_{\mu\nu} \equiv \frac{-1}{8\pi G} G^2_{\mu\nu}(h_{\alpha\beta})$$

$$= \frac{1}{32\pi G} \langle \hat{\nabla}_\mu h_{\alpha\beta} \hat{\nabla}_\nu h^{\alpha\beta} \rangle$$

Using the quadrupole solution we found earlier,

$$\frac{dE}{dt} = r^2 \int d\Omega \, T^{0i} n_i$$

$$= \frac{32 G}{5} \dddot{I}_{ij} \dddot{I}_{ij}$$