Workshop on Topics in Quantum Turbulence

16 - 20 March 2009

Computations based on the Gross-Pitaevskii (GP) model

M. Tsubota

Osaka City University, Dept. of Physics Osaka
Japan
Computations based on the Gross-Pitaevskii (GP) model

M. Tsubota (Osaka City University, Japan)

Thanks to M. Kobayashi, M. Machida, R. Numasato

1. Introduction -Brief history-
2. Quantum turbulence in trapped Bose gas
3. Big simulation towards the classical-quantum crossover
4. Two-dimensional quantum turbulence
1. Introduction - Brief history -

How to describe the vortex dynamics

Vortex filament model (Schwarz)

\[ \mathbf{v}_s(r) = \frac{\kappa}{4\pi} \int \frac{(s - r) \times ds}{|s - r|^3} \]

A vortex makes the superflow of the Biot-Savart law, and moves with this local flow. At a finite temperature, the mutual friction should be considered.

The Gross-Pitaevskii (GP) model for the macroscopic wave function

\[ \Psi(r) = \sqrt{n_0(r)} e^{i\theta(r)} \]

\[ i\hbar \frac{\partial \Psi(r,t)}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(r) + g|\Psi(r,t)|^2 \right] \Psi(r,t) \]
They showed directly vortex reconnections for the first time.

They showed sound emission at vortex reconnections.

FIG. 4. Density cross-sections in the $x = 0$ plane at $t = 54$ for ring radii $R = 6$ with offset $D = 8$ (same parameters as in Fig. 3). The sound pulse appears as two arcs with radius of curvature, 25, suggesting that the reconnections occurred at $z = \pm 7$ and $t = 29$, consistent with Fig. 3. The angular spread is approximately equal to the reconnection angle ($\theta = 96^\circ$ for this example). The white lines indicate the positions of the reconnections, $z = \pm 7$, the reconnection angle $\theta$ and the expected position of the sound pulse. Grey scale: black 0.95; white 1.025.
By using the GP model, they obtained a vortex tangle with starting from the Taylor-Green vortices.
In order to study the Kolmogorov spectrum, it is necessary to decompose the total energy into some components. (Nore et al., 1997)

Total energy

\[
E = \frac{1}{\int dx} \int dx \rho \left[ -\nabla^2 + \frac{g}{2} |\Phi|^2 \right] \Phi = \sqrt{\rho} \exp(i \theta)
\]

\[
E = E_{\text{int}} + E_q + E_{\text{kin}}
\]

The kinetic energy \( E_{\text{kin}} = \frac{1}{\int dx} \int dx \rho \int dx (|\Phi| \nabla \theta)^2 \) is divided into

the compressible part \( E_{\text{kin}}^c = \frac{1}{\int dx} \int dx \left[ (|\Phi| \nabla \theta)^c \right]^2 \) with \( \text{rot} (|\Phi| \nabla \theta)^c = 0 \)

and

the incompressible part \( E_{\text{kin}}^i = \frac{1}{\int dx} \int dx \left[ (|\Phi| \nabla \theta)^i \right]^2 \) with \( \text{div} (|\Phi| \nabla \theta)^i = 0 \)

This incompressible kinetic energy \( E_{\text{kin}}^i \) should obey the Kolmogorov spectrum.

\[ \Delta: 2 < k < 12 \]
\[ \bigcirc: 2 < k < 14 \]
\[ \square: 2 < k < 16 \]

The right figure shows the energy spectrum at a moment. The left figure shows the development of the exponent \( n(t) \). The exponent \( n(t) \) goes through 5/3 on the way of the dynamics. In the late stage, however, the exponent deviates from 5/3, because the sound waves resulting from vortex reconnections disturb the cascade process of the inertial range.
They obtained a statistical steady state by introducing large-scale excitation and small-scale dissipation. The state showed the Kolmogorov spectra in the inertial range.
2. Quantum turbulence in trapped Bose gas


Two principal cooperative phenomena of quantized vortices are vortex arrays and vortex tangles.

<table>
<thead>
<tr>
<th></th>
<th>Vortex array</th>
<th>Vortex tangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Superfluid He</td>
<td><img src="image1" alt="Vortex array" /></td>
<td><img src="image2" alt="Vortex tangle" /></td>
</tr>
<tr>
<td>Packard (Berkeley)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Atomic BEC</td>
<td><img src="image3" alt="Vortex array" /></td>
<td>None</td>
</tr>
<tr>
<td>Ketterle (MIT)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Usual setup of atomic BECs


Optical spoon (oscillating laser beam)

Rotation frequency $\Omega$

“cigar-shape”

100 $\mu$m

5 $\mu$m

20 $\mu$m

16 $\mu$m

Laser cooling
Observation of quantized vortices in atomic BECs

ENS  K.W. Madison, et. al PRL 84, 806

MIT  J.R. Abo-Shaeer, et. al Science 292, 476

JILA  P. Engels, et. al PRL 87, 210403 (2001)

Dynamics of vortex lattice formation by the GP model

\[ \Omega = 0.7 \omega \]

\[ V_{\text{trap}}(r) = \frac{1}{2} m \omega^2 r^2 \]

\[ \Psi(r) = \sqrt{n_0(r)} e^{i \theta(r)} \]

Time development of condensate density \( n_0 \)

Experiment
K.W. Madison et.al., PRL 86, 4443 (2001)
Is it possible to produce turbulence in a trapped BEC?


(1) We cannot apply any flow to this system.

(2) This is a finite-size system. Can we make turbulence with a sufficiently wide inertial range?

The coherence length is not much smaller than the system size. However, we could confirm Kolmogorov’s law.
How to produce turbulence in a trapped BEC

1. Trap the BEC in a weak elliptical potential.

\[ U(x) = \frac{m \omega^2}{2} \left[ (1 - \varepsilon_1)(1 - \varepsilon_2)x^2 + (1 + \varepsilon_1)(1 - \varepsilon_2)y^2 + (1 + \varepsilon_2)z^2 \right] \]

2. Rotate the system first around the \(x\)-axis, then around the \(z\)-axis.

\[ \Omega(t) = (\Omega_x, \Omega_z \sin \omega_x t, \Omega_z \cos \omega_x t) \]
Actually, this idea has been already used in CT.

**Precession**

Spin axis itself rotates around another axis.

Are these two rotations represented by their sum? *No!*

Precessing motion of a gyroscope

We consider the case where the spinning and precessing rotational axes are perpendicular to each other. Hence, the two rotations do not commute, and thus cannot be represented by their sum.
Two precessions \((\omega_x \times \omega_z)\)  
Three precessions \((\omega_y \times \omega_x \times \omega_z)\)

Condensate density vortices  
Quantized  
Condensate density vortices  
Quantized
Energy spectra for two cases

Two precessions

Three precessions

Three precessions produce more isotropic QT, whose $\eta$ is closer to $5/3$. 

$E_{\text{kin}}^i \propto k^{-\eta}$

$\eta \approx 1.78 \pm 0.194$

$E_{\text{kin}}^i \propto k^{-\eta}$

$\eta \approx 1.69 \pm 0.037$
What can we learn from QT of atomic BEC?

- Controlling the transition to turbulence by changing the rotational frequency or interaction parameters, etc.
- We can visualize quantized vortices. We can consider the relation of real space cascade of vortices and the wavenumber space cascade (Kolmogorov’s law).
- Changing the trapping potential or the rotational frequency leads to dimensional crossover (2D↔3D) in turbulence.
Controlling the transition to turbulence

\[ \Omega_z \]

Vortex tangle

\[ \omega_x \times \omega_z \]

Vortex lattice

Vortex lattice

0

\[ \Omega_x \]
What can we learn from quantum turbulence of atomic BEC?

• Controlling the transition to turbulence by changing the rotational frequency or interaction parameters, etc.

• We can visualize quantized vortices. We can consider the relation of real space cascade of vortices and the wavenumber space cascade (Kolmogorov’s law).

• Changing the trapping potential or the rotational frequency leads to dimensional crossover (2D↔3D) in turbulence.
3. A bigger simulation of the GP model was recently made by M. Machida using the earth simulator (private communication).

256\(^3\) (Kobayashi and Tsubota) \(\rightarrow\) 2048\(^3\) (Machida)

![Diagram showing log-log plot with lines labeled 256\(^3\), 512\(^3\), 1024\(^3\), and 2048\(^3\). Lines are labeled as Normal turbulence(2048\(^3\)) and intermittent(2048\(^3\)). The question marks indicate areas of interest: Bottleneck effect? and Intermittency?}
Bundle-like structure seems to be present.
4. Two-dimensional quantum turbulence

Two-dimensional classical turbulence has different properties from three-dimensional one. The most important difference is the presence of the inverse cascade.

Our interests are

• Can we obtain the inverse cascade even in quantum case?
• If so, the behavior should be reduced to the motion of quantized vortices. What occurs?

Two-dimensional quantum turbulence can be realized in superfluid helium and cold atoms.
Two-dimensional turbulence is very common in our daily life.
Inverse cascade is one of the most remarkable phenomenon of two-dimansional classical turbulence!

R. H. Kraichnan, Phys. Fluids 10, 1417 (1967)
Two-dimensional classical turbulence (the NS equ.)

Vorticity distribution

Large-scale motion occurs with inverse cascade!
Two-dimensional quantum turbulence (the GP model)
Simulation by the vortex-point model
Summary

1. Introduction -Brief history-
2. Quantum turbulence in trapped Bose gas
3. Big simulation towards the classical-quantum crossover
4. Two-dimensional quantum turbulence