Critical asymptotics for Toeplitz determinants

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Outline

- Toeplitz determinants
- Szegő and Fisher-Hartwig asymptotics
- Critical asymptotics and Painlevé V
- Toeplitz determinants and orthogonal polynomials
- 2d Ising model

Toeplitz determinants

- Toeplitz matrix = matrix which is constant along diagonals

$$\begin{pmatrix} c_0 & c_{-1} & c_{-2} & \cdots & c_{-n+1} \\ c_1 & c_0 & c_{-1} & \ddots & \vdots \\ c_2 & c_1 & \ddots & \ddots & c_{-2} \\ \vdots & \ddots & \ddots & c_0 & c_{-1} \\ c_{n-1} & \cdots & c_2 & c_1 & c_0 \end{pmatrix}$$

- Toeplitz determinant is the determinant of a Toeplitz matrix
- Asymptotics for Toeplitz determinants when the size of the matrices tends to infinity?

Toeplitz determinants

- Consider a weight $f(e^{i\theta})$ on the unit circle C_1
- Fourier coefficients

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ik\theta}d\theta$$

- Fourier series $f(e^{i\theta}) \sim \sum_{j=-\infty}^{+\infty} c_j e^{ij\theta}$
- Toeplitz determinant for weight/symbol f

$$D_n(f) = \det(c_{j-k})_{j,k=0}^{n-1}$$

Toeplitz determinants

- If the weight $f(\theta)$
  - is "smooth"
  - has no zeros
  - has a continuous logarithm (winding number 0 around the origin)
- Szegő's strong limit theorem: as $n \to \infty$,

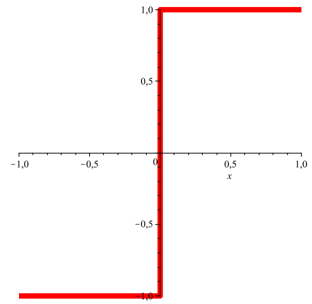
$$\ln D_n(f) = n(\ln f)_0 + \sum_{k=1}^{\infty} k(\ln f)_k(\ln f)_{-k} + o(1),$$

with

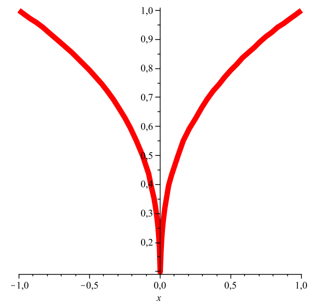
$$(\ln f)_k = \frac{1}{2\pi}\int_0^{2\pi} \ln f(e^{i\theta})e^{-ik\theta}d\theta.$$

Fisher-Hartwig singularities

- Two types of weights for which Szegő asymptotics are not valid
  - jump discontinuities



  - root type singularities



- Example

$$f(e^{i\theta}) = (2-2\cos\theta)^{\alpha} e^{i\beta(\theta-\pi)} e^{V(e^{i\theta})}, \qquad \text{for } 0 < \theta < 2\pi,$$

with Re $\alpha > -\frac{1}{2}$

  - Fisher-Hartwig singularity at 1

Fisher-Hartwig singularities

- For weights with one Fisher-Hartwig singularity with parameters α (root) and β (jump),

$$\ln D_n(f) = nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} - (\alpha-\beta)\sum_{k=1}^{\infty} V_k - (\alpha+\beta)\sum_{k=1}^{\infty} V_{-k}$$

$$+ (\alpha^2-\beta^2)\ln n + \ln \frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)} + o(1),$$

as $n \to \infty$, where G is Barnes' G-function, and

$$V_k = \frac{1}{2\pi}\int_0^{2\pi} V(e^{i\theta})e^{-ik\theta}d\theta.$$

(Fisher-Hartwig '68, Widom '73, Ehrhardt-Silbermann '97, Basor-Ehrhardt '01, Deift-Its-Krasovsky '08)

Toeplitz determinants

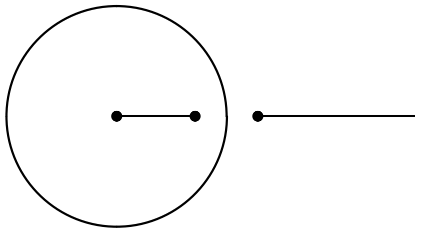
- Asymptotics are known for
  - Szegő weights (smooth, not winding around the origin)
  - Fisher-Hartwig weights (jump and root type singularities)
- asymptotic behavior is different in those two cases
- What happens if we deform the weight in such a way that a Szegő weight turns into a Fisher-Hartwig weight?
  - transition in asymptotic expansion where $\mathcal{O}(\ln n)$-term appears and $\mathcal{O}(n)$-term disappears

Transition from Szegő to FH

- weight

$$f(z) = (z-e^t)^{\alpha+\beta}(z-e^{-t})^{\alpha-\beta} z^{-\alpha+\beta} e^{-i\pi(\alpha+\beta)} e^{V(z)},$$

with V analytic and $t \geq 0$



  - f analytic on C with winding number zero around the origin for $t > 0$
  - f has a singularity at 1 for $t = 0$,

$$f(e^{i\theta}) = (2-2\cos\theta)^{\alpha} e^{i\beta(\theta-\pi)} e^{V(e^{i\theta})}, \quad \text{for } 0 < \theta < 2\pi,$$

Transition from Szegő to FH

- Asymptotics as $n \to \infty$ for Toeplitz determinant with weight

$$f(z) = (z - e^t)^{\alpha+\beta}(z - e^{-t})^{\alpha-\beta} z^{-\alpha+\beta} e^{-i\pi(\alpha+\beta)} e^{V(z)}$$

  - Szegő asymptotics for $t > 0$ fixed,

$$\ln D_n(t) = nV_0 + nt(\alpha+\beta) + \mathcal{O}(1), \quad \text{as } n \to \infty$$

  - Fisher-Hartwig asymptotics for $t = 0$,

$$\ln D_n(0) = nV_0 + (\alpha^2 - \beta^2)\ln n + \mathcal{O}(1), \quad \text{as } n \to \infty,$$

  with $V_0 = \frac{1}{2\pi}\int_C V(z)\frac{dz}{iz}$.

Transition from Szegő to FH

- what happens if $t \to 0$ simultaneously with $n \to \infty$ (double scaling limit)?

$$f(z) = (z - e^t)^{\alpha+\beta}(z - e^{-t})^{\alpha-\beta} z^{-\alpha+\beta} e^{-i\pi(\alpha+\beta)} e^{V(z)}$$

- Result *(TC-Its-Krasovsky)*:
  If $\alpha > -\frac{1}{2}$ and Re $\beta = 0$,

$$\ln D_n(t) = nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} - (\alpha - \beta) \sum_{k=1}^{\infty} V_k - (\alpha + \beta) \sum_{k=1}^{\infty} V_{-k}$$
$$+(\alpha+\beta)nt+(\alpha^2-\beta^2)\ln n+\ln \frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)}$$
$$+\int_0^{2nt} w(x)dx + \mathcal{O}(t) + o(1),$$

Transition from Szegő to FH

- w is a solution to the system

$$
\begin{aligned}
xu_x &= xu - 2v(u-1)^2 + (u-1)[(\alpha-\beta)u - \beta - \alpha],\\
xv_x &= uv[v-\alpha+\beta] - \frac{v}{u}(v-\beta-\alpha),\\
w &= -v - \frac{1}{x}\left[\alpha^2 - \beta^2 - 2\alpha v + 2v^2 - \frac{v}{u}(v-\alpha-\beta) - uv(v-\alpha+\beta)\right]
\end{aligned}
$$

- w is real analytic on $(0,+\infty)$, and it has the asymptotics

$$
\begin{aligned}
w(x) &= -\frac{\alpha^2-\beta^2}{x} + \mathcal{O}(e^{-cx}), && \text{as } x \to +\infty,\\
w(x) &= \mathcal{O}(1) + \mathcal{O}(x^{2\alpha}), && \text{as } x \searrow 0.
\end{aligned}
$$

Transition from Szegő to FH

- if u and v solve the differential system, then u solves the Painlevé V equation

$$u_{xx} = \left(\frac{1}{2u} + \frac{1}{u-1}\right) u_x^2 - \frac{1}{x} u_x + \frac{(u-1)^2}{x^2}\left(Au + \frac{B}{u}\right) + \frac{Cu}{x} + D\frac{u(u+1)}{u-1},$$

with

$$A = \frac{1}{2}(\alpha - \beta)^2, \quad B = -\frac{1}{2}(\alpha + \beta)^2, \quad C = 1 + 2\beta, \quad D = -\frac{1}{2}.$$

Transition from Szegő to FH

$$\ln D_n(t) = nV_0 + \sum_{k=1}^{\infty} kV_kV_{-k} - (\alpha-\beta)\sum_{k=1}^{\infty} V_k - (\alpha+\beta)\sum_{k=1}^{\infty} V_{-k}$$
$$+(\alpha+\beta)nt+(\alpha^2-\beta^2)\ln n+\ln\frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)}$$
$$+\int_0^{2nt} w(x)dx + \mathcal{O}(t) + o(1),$$

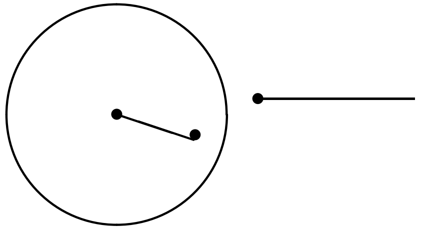
- special case 1: $n \to \infty$, $t \to 0$ in such a way that $nt \to 0$
  - terms $(\alpha+\beta)nt$ and $\int_0^{2nt} w(x)dx$ vanish $\longrightarrow$ FH asymptotics
- special case 2: $n \to \infty$, $t \to 0$ in such a way that $nt \to \infty$
  - divergent part of $\int_0^{2nt} w(x)dx$ kills $(\alpha^2-\beta^2)\ln n$ $\longrightarrow$ Szegő asymptotics

Transition from Szegő to FH

- Consistency of $\mathcal{O}(1)$-term with the Szegő asymptotics leads to the identity

$$\lim_{s\to+\infty}\left[\int_0^s w(x)dx\ +(\alpha^2-\beta^2)\ln s\right] = -\ln\frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)}.$$

- Extension to complex t?



Expansion is valid for $|\text{arg}t|<\frac{\pi}{2}$ if contour of integration does not contain poles of w

Transition from Szegő to FH

- what if Im $\alpha \neq 0$ and/or Re $\beta \neq 0$?
  - $w(x; \alpha, \beta)$ is not real for $x > 0$
  - w can have poles on $(0, +\infty)$
  - asymptotic expansion holds only if we integrate over a pole-free contour
    $\rightarrow$ expansion not valid if $2nt$ is a pole of $w(x; \alpha, \beta)$
  - poles correspond to Toeplitz determinants approaching 0
    $\rightarrow$ different choices of integration contour
    $\longrightarrow$ expansion picks up residue of w
    $\rightarrow$ residue of w at its poles is integer
    $\longrightarrow$ different branches of $\ln D_n$

Orthogonal polynomials

Relation between Toeplitz determinants and orthogonal polynomials

- let $f(e^{i\theta})$ be positive on the unit circle and in L^2
- OPs determined uniquely by conditions

$$\frac{1}{2\pi}\int_0^{2\pi} p_n(e^{i\theta})p_m(e^{-i\theta})f(\theta)d\theta = \delta_{nm},$$

or

$$\frac{1}{2\pi i}\int_C p_n(z)p_m(\bar{z})f(z)\frac{dz}{z} = 0,$$

Orthogonal polynomials

- Heine's formula: determinant formula for orthogonal polynomials

$$p_n(z) = \sqrt{\frac{1}{D_{n+1}(f)D_n(f)}} \begin{vmatrix} c_0 & c_{-1} & c_{-2} & \dots & c_{-n} \\ c_1 & c_0 & c_{-1} & \ddots & \vdots \\ c_2 & c_1 & \ddots & \ddots & c_{-2} \\ \vdots & \ddots & \ddots & c_0 & c_{-1} \\ 1 & z & \dots & z^{n-1} & z^n \end{vmatrix}$$

Orthogonal polynomials

Hint for Proof:

1. p_n is polynomial of degree n (expansion by last row),

$$p_n(z) = \sum_{j=0}^{n} a_j z^j$$

2. By residue argument, Fourier series and row expansion,

$$\frac{1}{2\pi i}\int_C p_n(z)\bar{z}^k f(z)\frac{dz}{z} = \sqrt{\frac{1}{D_{n+1}(f)D_n(f)}}\begin{vmatrix} c_0 & c_{-1} & c_{-2} & \cdots & c_{-n} \\ c_1 & c_0 & c_{-1} & \ddots & \vdots \\ c_2 & c_1 & \ddots & \ddots & c_{-2} \\ \vdots & \ddots & \ddots & c_0 & c_{-1} \\ c_k & c_{k-1} & \cdots & c_{k-n+1} & c_{k-n} \end{vmatrix}$$

Orthogonal polynomials

- As a consequence, we have

$$\kappa_n(f) = \sqrt{\frac{D_n(f)}{D_{n+1}(f)}}, \qquad D_n(f) = \prod_{j=0}^{n-1} \kappa_j(f)^{-2},$$

  where $\kappa_j > 0$ is leading coefficient of orthonormal polynomial p_j
- asymptotics as $n \to \infty$ for p_n, κ_n are known in many cases
- unfortunately $\kappa_0, \kappa_1, \ldots$ are also needed

Asymptotics for Toeplitz determinants

General approach to obtain asymptotics for Toeplitz determinants for weight f

- Step 1: deform weight f smoothly to a weight for which Toeplitz determinant is known (e.g. uniform weight),

$$f_t(z), \qquad f_1(z) = f, \qquad f_0(z) = 1$$

- Step 2: try to find differential identity for $\frac{d}{dt} \ln D_n(f_t)$ in terms of $p_n, p_{n-1}, \ldots, p_{n-k}$ and $\kappa_n, \kappa_{n-1}, \ldots, \kappa_{n-j}$
- Step 3: find asymptotics for orthogonal polynomials as $n \to \infty$, uniform in t
- Step 4: integrate differential identity from 0 to 1

Asymptotics for Toeplitz determinants

Step 2: differential identity

- Since $D_n(f_t) = \prod_{j=0}^{n-1} \kappa_j(f_t)^{-2}$, we have

$$\frac{d}{dt} \ln D_n(f_t) = -2 \sum_{j=0}^{n-1} \frac{\kappa_j'(f_t)}{\kappa_j(f_t)}$$

  - Write $\frac{\kappa_j'(f_t)}{\kappa_j(f_t)} = \frac{1}{4\pi i} \int_0^{2\pi} \frac{\partial}{\partial t} \left(p_j(z)p_j(\bar{z})\right) f_t(z) \frac{dz}{z}$
  - Use Christoffel-Darboux formula

$$\sum_{j=0}^{n-1} p_j(z)p_j(\bar{z}) = -np_n(z)p_n(\bar{z}) + z\left(p_n(\bar{z})p_n'(z) - p_n(z)p_n'(\bar{z})\right)$$

Step 2: differential identity

- We obtain

$$\frac{d}{dt}\ln D_n(f_t) = 2n\frac{\kappa_n'}{\kappa_n} + \frac{1}{2\pi}\int_C \frac{\partial}{\partial t}\left(p_n(\bar{z})p_n'(z) - p_n(z)p_n'(\bar{z})\right) f_t(z)dz$$

  - 'local' formula in n
  - asymptotics for OPs are now (in principle) sufficient to obtain asymptotics for Toeplitz determinants
  - depending on the deformation, it might be possible to simplify the differential identity

Asymptotics for Toeplitz determinants

Step 3: asymptotics for OPs

- this is the main difficulty
- for large classes of weights, asymptotics can be found using the Riemann-Hilbert approach

Step 4:

- Insert the asymptotic expansions in the differential identity and integrate

Transition from Szegő to FH

Applied to our transition between Szegő and FH

- Step 1: deformation of weight:

$$f_t(z) = (z - e^t)^{\alpha+\beta}(z - e^{-t})^{\alpha-\beta} z^{-\alpha+\beta} e^{-i\pi(\alpha+\beta)} e^{V(z)}$$

  - weight is complex, so orthogonal polynomials do not necessarily exist

- we know asymptotics for $\ln D_n(0)$ (Fisher-Hartwig) and for $\ln D_n(t_0)$ (Szegő)
  - we can integrate from 0 or from t_0

Transition from Szegő to FH

- Step 2: differential identity in terms of orthogonal polynomials
  - Let $\{p_k, \hat{p}_k, k = 0, 1, \ldots\}$ be an orthonormal system of polynomials (both with leading coefficient χ_k) satisfying

$$\frac{1}{2\pi i}\int_{C_1} p_k(z)\hat{p}_m(z^{-1})f(z)\frac{dz}{z} = \delta_{km}, \qquad k, m = 0, 1, \ldots$$

  - if f is real, $p_k = \hat{p}_k$ are the usual orthogonal polynomials on the unit circle
  - for complex f, existence of polynomials is not guaranteed

Transition from Szegő to FH

- Step 2: differential identity

$$
\begin{aligned}
\frac{d}{dt}\ln D_n(t) = \beta n - \frac{\cosh t}{\sinh t}\frac{\alpha^2-\beta^2}{2} + \alpha\beta - \beta^2 - \frac{\alpha+\beta}{2}V'(e^t) + \frac{\alpha-\beta}{2}V'(e^{-t}) \\
- \frac{\alpha+\beta}{2}\left(\frac{n}{2}+\frac{\alpha-\beta}{2}\right)F(0,e^t) + \frac{\alpha-\beta}{2}\left(\frac{n}{2}+\frac{\alpha-\beta}{2}\right)F(0,e^{-t}) \\
+ \frac{\cosh t}{\sinh t}\frac{\alpha^2-\beta^2}{4}F(e^t,e^{-t}),
\end{aligned}
$$

where

$$
F(x,y) = \mathrm{Tr}\left[Y(x)\sigma_3 Y(x)^{-1}Y(y)\sigma_3 Y(y)^{-1}\right],
$$

$$
Y(z) = \begin{pmatrix} \chi_n^{-1}p_n(z) & p_n^{-1}\int_{C_1}\frac{p_n(\xi)}{\xi-z}\frac{f(\xi)d\xi}{2\pi i\xi^n} \\ -\chi_{n-1}z^{n-1}\hat{p}_{n-1}(z^{-1}) & -\chi_{n-1}\int_{C_1}\frac{\hat{p}_{n-1}(\xi^{-1})}{\xi-z}\frac{f(\xi)d\xi}{2\pi i\xi} \end{pmatrix}
$$

  - Y is solution of the Riemann-Hilbert problem for orthogonal polynomials

Transition from Szegő to FH

- Step 3: asymptotics for orthogonal polynomials
  - Riemann-Hilbert problem for orthogonal polynomials
  - asymptotic analysis of the RH problem
  - this is the step where the Painlevé V equation appears

Transition from Szegő to FH

Final step:

- integrate (asymptotics for) differential identity from 0 to t
- insert known FH asymptotics for $\ln D_n(0)$
- this leads to asymptotics for $\ln D_n(t)$

Remark:

- if Painlevé V function w has a pole at x
  - OPs do not necessarily exist (not even for large n) if $2nt \to x$
  - Toeplitz determinants approach 0 if $n \to \infty$ and $t \to 0$ such that $2nt \to x$

2d Ising model

- rectangular lattice of size $m \times n$
- assign a spin (± 1) to each point of the lattice
  - spin configuration
- probability measure on configurations:

$$\mathbb{P}(\sigma \in A) = Z(T)^{-1} \sum_{\sigma \in A} e^{-E(\sigma)/T},$$

$$E(\{\sigma\}) = -\sum_{j=-\mathcal{M}}^{\mathcal{M}-1} \sum_{k=-\mathcal{N}}^{\mathcal{N}-1} \left(\gamma_1 \sigma_{jk} \sigma_{j\,k+1} + \gamma_2 \sigma_{jk} \sigma_{j+1\,k}\right), \; 0 < \gamma_1 < \gamma_2,$$

$$Z(T) = \sum_{\{\sigma\}} e^{-E(\{\sigma\})/T}$$

2d Ising model

- 2-point correlation function is given as Toeplitz determinant for a certain weight (depending on T, γ_1, γ_2)
- phase transition for critical value of $T = T_c$ *(Onsager, McCoy-Wu)*
  - long range order $\longrightarrow$ long range disorder
  - magnetization $\neq 0 \longrightarrow$ magnetization 0
  - corresponding Toeplitz determinant has symbol of the form

$$f(z) = (z - e^t)^{\alpha+\beta}(z - e^{-t})^{\alpha-\beta} z^{-\alpha+\beta} e^{-i\pi(\alpha+\beta)} e^{V(z)}$$

    with $\alpha = 0$ and $\beta = -\frac{1}{2}$
  - $t \to 0$ corresponds to $T \nearrow T_c$