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B Physics and CP Violation - I

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1 The flavour structure of the Standard Model

We start from the Lagrangian describing electroweak interactions of Standard Model (SM) fermions:

$$\mathcal{L}_{\rm int} = -\frac{g_2}{\sqrt{2}}\left(W^+_\mu J^\mu_{\rm ch} + W^-_\mu J^{\mu\dagger}_{\rm ch}\right) - g_1\cos\theta_W A_\mu J^\mu_{\rm em} - \frac{g_2}{\cos\theta_W} Z^\mu J^{\rm neutral}_\mu\,, \quad (1)$$

where $g_{1,2}$ denote the $SU(2)_L$ and $U(1)_Y$ gauge couplings respectively, θ_W is the electroweak mixing angle and the currents involving quarks are given by:

$$\begin{aligned} J^\mu_{\rm em} &= \frac{2}{3}\bar{u}'_{Li}\gamma^\mu u'_{Li} + \frac{2}{3}\bar{u}'_{Ri}\gamma^\mu u'_{Ri} - \frac{1}{3}\bar{d}'_{Li}\gamma^\mu d'_{Li} - \frac{1}{3}\bar{d}'_{Ri}\gamma^\mu d'_{Ri} \quad (2)\\ J^\mu_{\rm ch} &= \bar{u}'_{Li}\gamma^\mu d'_{Li} \\ J^\mu_{\rm neutral} &= g_L^{(u)}\bar{u}'_{Li}\gamma^\mu u'_{Li} + g_R^{(u)}\bar{u}'_{Ri}\gamma^\mu u'_{Ri} + g_L^{(d)}\bar{d}'_{Li}\gamma^\mu d'_{Li} + g_R^{(d)}\bar{d}'_{Ri}\gamma^\mu d'_{Ri}\,, \end{aligned}$$

where $g_L^{(u)} = 1/2 - 2/3\sin^2\theta_W$, $g_R^{(u)} = -2/3\sin^2\theta_W$, $g_L^{(d)} = -1/2 + 1/3\sin^2\theta_W$, $g_R^{(d)} = 1/3\sin^2\theta_W$, the electric charge is given by $e = g_1\cos\theta_W = g_2\sin\theta_W$ and

$$q_{L,R} = P_{L,R}\, q = \frac{(1\mp\gamma_5)}{2}\, q\,. \quad (3)$$

Let us now consider Yukawa interactions:

$$-\mathcal{L}_Y = Y^u_{ij}\bar{Q}'_{Li}\tilde{H}u'_{Rj} + Y^d_{ij}\bar{Q}'_{Li}H d'_{Rj} + H.c.\,, \quad (4)$$

where

$$Q_{Li} = \begin{pmatrix} u_{Li} \\ d_{Li}\end{pmatrix}, \qquad H = \begin{pmatrix} h^+ \\ h^0\end{pmatrix}, \qquad \tilde{H} = \begin{pmatrix} h^{0^*} \\ -h^{+^*}\end{pmatrix}. \quad (5)$$

These interactions break the $U(3)_{Q_L}\otimes U(3)_{d_R}\otimes U(3)_{u_R}$ flavour symmetries of the SM gauge Lagrangian. Once the neutral component of the Higgs field gets a vacuum expectation value $\langle h^0\rangle = v/\sqrt{2}$, breaking the electroweak symmetry, the Yukawa interactions generate masses for quarks:

$$-\mathcal{L}_m = \hat{m}^u_{ij}\bar{u}'_{Li}u'_{Rj} + \hat{m}^d_{ij}\bar{d}'_{Li}d'_{Rj} + H.c.\,. \quad (6)$$

To go to the mass eigenstate basis for quarks, we must diagonalize the complex matrices $\hat{m}^u$ and $\hat{m}^d$. This can be done via a bi-unitary transformation. Indeed, the Hermitian matrix $\hat{m}^u\hat{m}^{u\dagger}$ can be diagonalized with a unitary matrix U^{u_L}:

$$U^{u_L^\dagger}\hat{m}^u\hat{m}^{u\dagger}U^{u_L} = m^{u^2} = \mathrm{diag}(m_u^2, m_c^2, m_t^2)\,. \quad (7)$$

In the same way, we obtain

$$U^{u_R^\dagger}\hat{m}^{u\dagger}\hat{m}^u U^{u_R} = m^{u^2} = \mathrm{diag}(m_u^2, m_c^2, m_t^2)\,. \tag{8}$$

Finally, we have

$$\begin{aligned} U^{u_L^\dagger}\hat{m}^u U^{u_R} &= m^u = \mathrm{diag}(m_u, m_c, m_t)\,, \\ U^{d_L^\dagger}\hat{m}^d U^{d_R} &= m^d = \mathrm{diag}(m_d, m_s, m_b)\,. \end{aligned} \tag{9}$$

We thus write

$$\begin{aligned} -\mathcal{L}_m &= \bar{u}'_L U^{u_L} U^{u_L^\dagger}\hat{m}^u U^{u_R} U^{u_R^\dagger} u'_R + \bar{d}'_L U^{d_L} U^{d_L^\dagger}\hat{m}^d U^{d_R} U^{d_R^\dagger} d'_R \\ &= \bar{u}_L m^u u_R + \bar{d}_L m^d d_R\,, \end{aligned} \tag{10}$$

where $u_{L,R} = U^{u_{L,R}^\dagger} u'_{L,R}$ and $d_{L,R} = U^{d_{L,R}^\dagger} d'_{L,R}$. We now rewrite the interactions with gauge bosons in terms of mass eigenstates. The rotation matrices cancel in neutral currents:

$$\begin{aligned} J^\mu_{\mathrm{em}} &= \frac{2}{3}\bar{u}_L U^{u_L^\dagger}\gamma^\mu U^{u_L} u_L + \frac{2}{3}\bar{u}_R U^{u_R^\dagger}\gamma^\mu U^{u_R} u_R \\ &\quad -\frac{1}{3}\bar{d}_L U^{d_L^\dagger}\gamma^\mu U^{d_L} d_L - \frac{1}{3}\bar{d}_R U^{d_R^\dagger}\gamma^\mu U^{d_R} d_R \\ &= \frac{2}{3}\bar{u}_L\gamma^\mu u_L + \frac{2}{3}\bar{u}_R\gamma^\mu u_R - \frac{1}{3}\bar{d}_L\gamma^\mu d_L - \frac{1}{3}\bar{d}_R\gamma^\mu d_R \\ J^\mu_{\mathrm{neutral}} &= g_L^{(u)}\bar{u}_L U^{u_L^\dagger}\gamma^\mu U^{u_L} u_L + g_R^{(u)}\bar{u}_R U^{u_R^\dagger}\gamma^\mu U^{u_R} u_R \\ &\quad + g_L^{(d)}\bar{d}_L U^{d_L^\dagger}\gamma^\mu U^{d_L} d_L + g_R^{(d)}\bar{d}_R U^{d_R^\dagger}\gamma^\mu U^{d_R} d_R \\ &= g_L^{(u)}\bar{u}_L\gamma^\mu u_L + g_R^{(u)}\bar{u}_R\gamma^\mu u_R + g_L^{(d)}\bar{d}_L\gamma^\mu d_L + g_R^{(d)}\bar{d}_R\gamma^\mu d_R\,. \end{aligned} \tag{11}$$

The absence of tree-level Flavour Changing Neutral Currents (FCNC) is a fundamental property of the SM. It is interesting to note that it is an accidental symmetry, *i.e.* a property of renormalizable interactions due to the field content and gauge structure of the SM. Indeed, extensions of the SM generally do not preserve this accidental symmetry, since with additional fields it is generally possible to write down FCNC vertices involving new particles. Even within the SM, as we shall see in detail in the following, the symmetry is broken by non-renormalizable interactions generated at the loop level by the exchange of SM particles.

Turning to charged currents, we obtain

$$J^\mu_{\mathrm{ch}} = \bar{u}_L U^{u_L^\dagger}\gamma^\mu U^{d_L} d_L = \bar{u}_L V\gamma^\mu d_L\,, \tag{12}$$

where we have introduced the Cabibbo-Kobayashi-Maskawa (CKM) matrix

$$V = U^{u_L^\dagger} U^{d_L} \,. \tag{13}$$

Let us now discuss the CP invariance of the SM Lagrangian in the weak eigenstate basis. The gauge part can be shown to be invariant under CP transformations. Let us then concentrate on the Yukawa interactions in eq. (4). The action of CP on fermionic fields is given by:

$$\psi \to \gamma^0 C \bar{\psi}^T \qquad \bar{\psi} \to \psi^T C \gamma^0 \,, \tag{14}$$

where the matrix C has the property $C^T = C^{-1} = -C$. We then obtain for example

$$Y^d_{ij} \bar{Q}'_{Li} H d'_{Rj} \to Y^d_{ij} Q'^T_{Li} C \gamma^0 H^* \gamma^0 C \bar{d}'^T_{Rj} = -Y^d_{ij} Q'^T_{Li} H^* \bar{d}'^T_{Rj} = Y^d_{ij} \bar{d}'_{Rj} H^\dagger Q'_{Li} \,, \tag{15}$$

where in the last step we have taken into account the anticommutative nature of the quark fields. Comparing the result above with the Hermitean conjugate of $\mathcal{L}_Y$, we see that Yukawa interactions are CP invariant if and only if $Y^{u,d}_{ij}$ are real. In general, $Y^{u,d}_{ij}$ are arbitrary complex 3×3 matrices, corresponding to 18 real parameters and 18 complex phases. However, not all of these parameters are physical. We already noticed that SM gauge interactions have an $U(3)_{Q_L} \otimes U(3)_{d_R} \otimes U(3)_{u_R}$ flavour symmetry. Thus, we can perform a $U(3)_{Q_L} \otimes U(3)_{d_R} \otimes U(3)_{u_R}$ transformation on $\mathcal{L}_Y$ without affecting the gauge Lagrangian. Since an $n \times n$ unitary matrix has $n(n-1)/2$ real parameters (angles) and $n(n+1)/2$ phases, an $U(3)_{Q_L} \otimes U(3)_{d_R} \otimes U(3)_{u_R}$ transformation contains $3 \times 3(3-1)/2 = 9$ angles and $3 \times 3(3+1)/2 = 18$ phases. However, one combination of phase transformations corresponds to baryon number conservation $U(1)_B$ which is a property of the whole SM Lagrangian, so it leaves Yukawa couplings unchanged. We conclude that Yukawa couplings contain $18 - 9 = 9$ real parameters and $18 - (18 - 1) = 1$ phase. In the mass eigenstate basis, these parameters correspond to 6 quark masses and to 3 angles and one phase in the CKM matrix. It is interesting to notice that for a number of generations smaller than three, all phases can be reabsorbed so that no CP violation arises in weak interactions. Indeed, the 2008 Nobel prize was awarded to Kobayashi and Maskawa for introducing three-family quark mixing to accommodate CP violation in SM weak interactions.

The CKM matrix is thus described by three angles and one phase. The position of the phase is arbitrary, but of course physical quantities do not depend on the phase convention. A measure of CP violation is given by the Jarlskog determinant

$$J_{\rm CP} \propto {\rm Im}(V_{ij}V_{kl}V_{il}^*V_{kj}^*)\,, \tag{16}$$

which is manifestly rephasing invariant.

The PDG advocates the use of the following parameterization for the CKM matrix:

$$\begin{aligned} V &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \end{aligned} \tag{17}$$

where we have used the notation $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$. Since experimentally there is a hierarchy between the three angles such that $s_{13} \ll s_{23} \ll s_{12} \sim 0.2$, it is possible to construct approximate parameterizations of the CKM matrix, such as the Wolfenstein parameterization and its generalizations to higher orders. As an example, we can identify

$$s_{12} = \lambda\,, \qquad s_{23} = A\lambda^2\,, \qquad s_{13}e^{-i\delta} = A\lambda^3(\rho - i\eta)\,. \tag{18}$$

Then, an approximate for for the CKM matrix can be obtained expanding in powers of λ the expressions for the cosines of the mixing angles $c_{ij} = \sqrt{1-s_{ij}}$.[1] At the lowest order we obtain

$$\begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho-i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}. \tag{19}$$

The unitarity of V implies the presence of triangular relations among CKM matrix elements. In particular, we have

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0 = 1 + \frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}\,, \tag{20}$$

[1] The three angles θ_{ij} can be chosen to lie in the first quadrant provided that δ varies between 0 and 2π.

where the last equality defines the so-called Unitarity Triangle (UT), as given in Fig. **??**. It is useful to define the following auxiliary quantities:

$$\begin{aligned}
\alpha &= \arg\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right)\,, \qquad \beta = \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right)\,, \qquad \gamma = \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right)\,, \\
R_b &= \left|\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right|\,, \qquad R_t = \left|\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}\right|\,. \qquad (21)
\end{aligned}$$

Employing the generalized Wolfenstein parameterization to $\mathcal{O}(\lambda^7)$, we obtain

$$\begin{aligned}
V_{ud}V_{ub}^* &= A\lambda^3(\bar\rho + i\bar\eta)\,, \qquad (22) \\
V_{cd}V_{cb}^* &= -A\lambda^3 \\
V_{td}V_{tb}^* &= A\lambda^3(1-\bar\rho - i\bar\eta)\,,
\end{aligned}$$

so that the apex of the UT has coordinates $(\bar\rho, \bar\eta)$, with $\bar\rho = \rho(1-\lambda^2/2)$ and $\bar\eta = \eta(1-\lambda^2/2)$.

Nuclear β decays and $K \to \pi$ semileptonic decays allow us to determine $|V_{ud}|$ and $|V_{us}|$, from which we can extract $\lambda = 0.2258 \pm 0.0014$. Exclusive and inclusive semileptonic B decays into charmed final states give us an estimate of $|V_{cb}|$:

$$|V_{cb}^{\mathrm{excl}}| = (39.2 \pm 1.1)10^{-3}\,, \qquad |V_{cb}^{\mathrm{incl}}| = (41.7 \pm 0.7)10^{-3}\,, \qquad (23)$$

while semileptonic charmless B decays give us access to $|V_{ub}|$:

$$|V_{ub}^{\mathrm{excl}}| = (35 \pm 4)10^{-4}\,, \qquad |V_{ub}^{\mathrm{incl}}| = (39.9 \pm 1.5 \pm 4.0)10^{-4}\,. \qquad (24)$$

These measurements allow us to determine the R_b side of the UT. Furthermore, we can measure the angle γ of the UT exploiting the interference between $b \to c\bar{u}s(d)$ and $b \to u\bar{c}s(d)$ transitions in $B \to DK(\pi)$ decays, up to a two-fold ambiguity. Experimental data give

$$\gamma = (78 \pm 12)^\circ \cup (-102 \pm 16)^\circ\,. \qquad (25)$$

Combining all this information gives us a determination of the UT entirely based on tree-level processes (see Fig. 1):

$$\bar\rho = \pm 0.06 \pm 0.08\,, \qquad \bar\eta = \pm 0.39 \pm 0.03\,. \qquad (26)$$

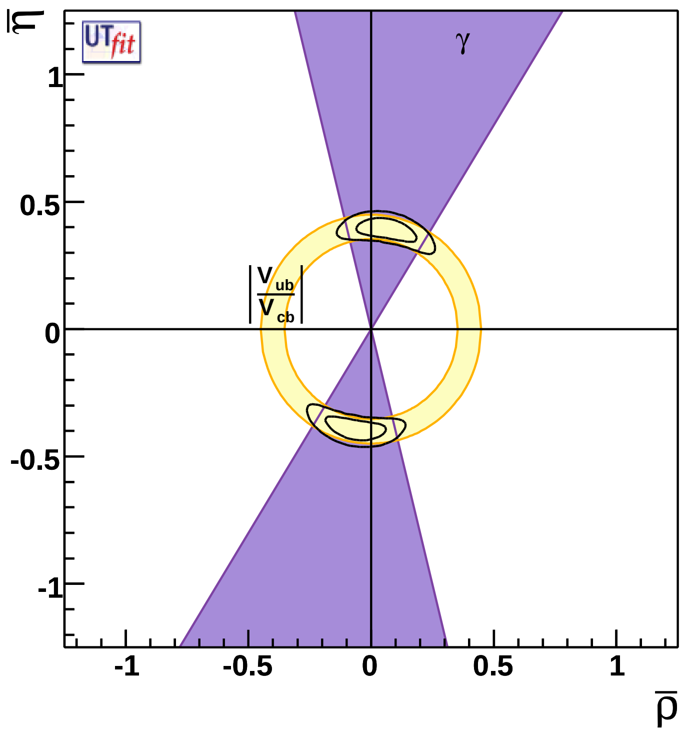

Figure 1: Determination of the UT using only tree-level processes.

The determination of the UT, and thus of the full CKM matrix, using only tree-level processes is a great experimental achievement. Indeed, New Physics (NP) contributions to tree-level processes are expected to be either absent (in all NP models with a discrete simmetry separating SM and NP particles) or strongly suppressed with respect to the SM. Therefore, the determination of the UT from tree-level processes can be considered to be valid also beyond the SM, and it is the starting point to study flavour and CP violation beyond the SM.

2 Effective Hamiltonian for $B-\bar{B}$ mixing

We have seen in the previous lecture that the SM has an accidental symmetry that forbids tree-level FCNC and confines flavour violation to charged current

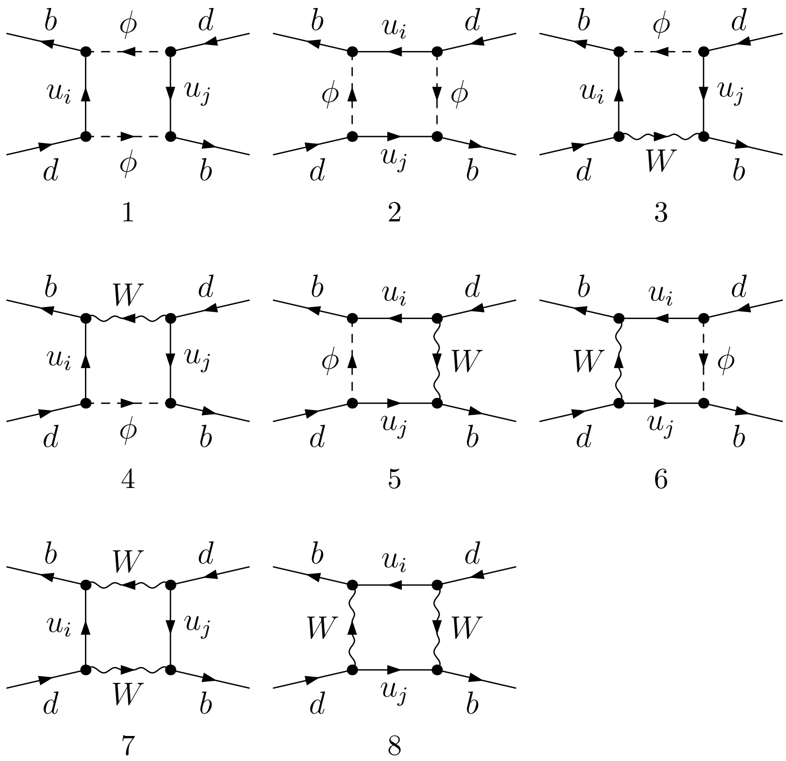

Figure 2: SM box diagrams for $\bar{b}d \to \bar{d}b$ transitions.

interactions. However, combining two or more charged current interactions we can generate FCNC transitions at the loop level. Let us now discuss two very important properties of FCNC processes in the SM. The first is the Glashow-Iliopoulos-Maiani (GIM) mechanism that suppresses loop-mediated FCNC processes. The second is the possibility to write down FCNC amplitudes using an effective Hamiltonian that contains only local operators.

2.1 $\bar{b}d \to \bar{d}b$ transitions in the full theory

As a first example, let us consider the $\Delta B = 2$ transition amplitude $\bar{b}d \to b\bar{d}$. In the 't-Hooft-Feynman gauge, the relevant diagrams are shown in Fig. 2. The relevant Feynman rules are reported in Fig. **??**. Let us concentrate first on diagram 7. We can write it down as follows:

$$\int \bar{u}_b \frac{ig}{\sqrt{2}}\gamma_\mu P_L V^*_{jb}\frac{i}{\not{p}-m_{u_j}}\frac{ig}{\sqrt{2}}\gamma_\nu P_L V_{jd} v_d \times \tag{27}$$
$$\bar{v}_b \frac{ig}{\sqrt{2}}\gamma^\nu P_L V^*_{ib}\frac{i}{\not{p}-m_{u_i}}\frac{ig}{\sqrt{2}}\gamma^\mu P_L V_{id} u_d \left(\frac{-i}{p^2-M_W^2}\right)^2 \frac{\mathrm{d}^4 p}{(2\pi)^4},$$

where we have neglected external momenta. Rationalizing propagators we see that the chiral projectors kill contributions from quark masses in the numerator. The relevant integral is thus

$$I^{ij}_{\alpha\beta} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{p_\alpha p_\beta}{(p^2-M_W^2)^2(p^2-m_{u_i})^2(p^2-m_{u_j}^2)}. \tag{28}$$

Now,

$$\frac{1}{p^2-m_{u_i}^2} - \frac{1}{p^2-m_{u_j}^2} = \frac{m_{u_i}^2-m_{u_j}^2}{(p^2-m_{u_i}^2)(p^2-m_{u_j}^2)}, \tag{29}$$

so that

$$I^{ij}_{\alpha\beta} = \frac{I^i_{\alpha\beta}-I^j_{\alpha\beta}}{m_{u_i}^2-m_{u_j}^2}, \tag{30}$$

where

$$\begin{aligned} I^i_{\alpha\beta} &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{p_\alpha p_\beta}{(p^2-M_W^2)^2(p^2-m_{u_i})} = \frac{g_{\alpha\beta}}{4}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{p^2}{(p^2-M_W^2)^2(p^2-m_{u_i})} \\ &= \frac{g_{\alpha\beta}}{4}m_{u_i}^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{1}{(p^2-M_W^2)^2(p^2-m_{u_i})} + \text{terms indep. on } m_{u_i}^2. \end{aligned} \tag{31}$$

Now, using the Feynman parameterization

$$\frac{1}{a^n b} = n\int_0^1 \mathrm{d}x \frac{x^{n-1}}{[(1-x)b+xa]^{n+1}}, \tag{32}$$

we obtain

$$\begin{aligned} &\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{1}{(p^2-M_W^2)^2(p^2-m_{u_i})} \\ &= 2\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{x}{\left[(1-x)(p^2-m_{u_i}^2)+x(p^2-M_w^2)\right]^3} \end{aligned} \tag{33}$$

$$
\begin{aligned}
&= 2\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{x}{\left[p^2 - M_w^2 x - m_{u_i}^2(1-x)\right]^3} \\
&= 2i\frac{2\pi^2}{16\pi^4}\int_0^1 \mathrm{d}x \int_0^\infty \frac{p^3\mathrm{d}p}{\left[-p^2 - M_w^2 x - m_{u_i}^2(1-x)\right]^3} ,
\end{aligned}
$$

where in the last step we have performed a Wick rotation to go to Euclidean spacetime. Writing $C = M_W^2 x + m_{u_i}^2(1-x)$ we have

$$
\begin{aligned}
&\int_0^\infty \frac{p^3\mathrm{d}p}{(p^2+C)^3} = -\frac{1}{4}\frac{p^2}{(p^2+C)^2}\bigg|_0^\infty + \frac{1}{2}\int_0^\infty \frac{p\mathrm{d}p}{(p^2+C)^2} = \frac{1}{4}\int_0^\infty \frac{\mathrm{d}y}{(y+C)^2} \\
&= -\frac{1}{4}\frac{1}{y+C}\bigg|_0^\infty = \frac{1}{4C} . \qquad (34)
\end{aligned}
$$

Substituting the result (34) in eq. (33) we are led to

$$
\begin{aligned}
&-\frac{i}{16\pi^2}\int_0^1 \mathrm{d}x \frac{x}{xM_W^2 + m_{u_i}^2(1-x)} = -\frac{i}{16\pi^2 M_W^2}\int_0^1 \mathrm{d}x \frac{x}{x + x_i(1-x)} \\
&= -\frac{i}{16\pi^2 M_W^2}\int_0^1 \mathrm{d}x \frac{x}{x_i + x(1-x_i)} , \qquad (35)
\end{aligned}
$$

where

$$
x_i = m_{u_i}^2 / M_W^2 . \qquad (36)
$$

The integration on x can be carried out as follows:

$$
\begin{aligned}
&\int_0^1 \mathrm{d}x \frac{x}{x_i + x(1-x_i)} = \frac{1}{1-x_i}\int_0^1 \mathrm{d}x \frac{(x(1-x_i)+x_i) - x_i}{x_i + x(1-x_i)} \\
&= \frac{1}{1-x_i} - \frac{x_i}{1-x_i}\int_0^1 \mathrm{d}x \frac{1}{x_i + x(1-x_i)} \\
&= \frac{1}{1-x_i} - \frac{x_i}{1-x_i}\frac{1}{1-x_i}\log\left((1-x_i)x + x_i\right)\bigg|_0^1 \\
&= \frac{1}{1-x_i} + \frac{x_i \log x_i}{(1-x_i)^2} . \qquad (37)
\end{aligned}
$$

Thus, up to terms independent on quark masses, we have

$$
I^i_{\alpha\beta} = -\frac{g_{\alpha\beta}}{4}\frac{i}{16\pi^2} J(x_i) \qquad (38)
$$

with

$$J(x_i) = \frac{x_i}{1-x_i} + \frac{x_i^2 \log x_i}{(1-x_i)^2} \,. \tag{39}$$

Using eq. (30) we finally obtain

$$I^{ij}_{\alpha\beta} = -\frac{g_{\alpha\beta}}{4M_W^2}\frac{i}{16\pi^2}A(x_i,x_j) \tag{40}$$

with

$$A(x_i,x_j) = \frac{J(x_i)-J(x_j)}{x_i-x_j} \,. \tag{41}$$

We now turn to the Dirac structure. From eq. (27) we extract

$$\bar{u}_b\gamma_\mu P_L\gamma_\alpha\gamma_\nu P_L v_d \times \bar{v}_b\gamma^\nu P_L\gamma^\alpha\gamma^\mu P_L u_d \,. \tag{42}$$

This can be simplified using a Fierz identity. From the basic Fierz identity

$$\begin{aligned}\delta_{\alpha\beta}\delta_{\gamma\delta} \;=\; & \frac{1}{2}(P_L)_{\alpha\delta}(P_L)_{\gamma\beta} + \frac{1}{2}(P_R)_{\alpha\delta}(P_R)_{\gamma\beta} \\ & +\frac{1}{2}(\gamma^\mu P_L)_{\alpha\delta}(\gamma_\mu P_R)_{\gamma\beta} + \frac{1}{2}(\gamma^\mu P_R)_{\alpha\delta}(\gamma_\mu P_L)_{\gamma\beta} + \frac{1}{8}(\sigma^{\mu\nu})_{\alpha\delta}(\sigma_{\mu\nu})_{\alpha\delta} \,,\end{aligned} \tag{43}$$

we obtain

$$\begin{aligned}&(P_L v_d\bar{v}_b P_R)_{\alpha\delta} = (P_L v_d)_\beta(\bar{v}_b P_R)_\gamma\delta_{\alpha\beta}\delta_{\gamma\delta} \\ &= (P_L v_d)_\beta(\bar{v}_b P_R)_\gamma(\frac{1}{2}(P_L)_{\alpha\delta}(P_L)_{\gamma\beta} + \frac{1}{2}(P_R)_{\alpha\delta}(P_R)_{\gamma\beta} \\ &+\frac{1}{2}(\gamma^\mu P_L)_{\alpha\delta}(\gamma_\mu P_R)_{\gamma\beta} + \frac{1}{2}(\gamma^\mu P_R)_{\alpha\delta}(\gamma_\mu P_L)_{\gamma\beta} + \frac{1}{8}(\sigma^{\mu\nu})_{\alpha\delta}(\sigma_{\mu\nu})_{\alpha\delta}) \\ &= (P_L v_d)_\beta(\bar{v}_b P_R)_\gamma\frac{1}{2}(\gamma^\mu P_R)_{\alpha\delta}(\gamma_\mu P_L)_{\gamma\beta} = \frac{1}{2}(\bar{v}_b\gamma^\mu P_L v_d)(\gamma_\mu P_R)_{\alpha\beta} \,,\end{aligned} \tag{44}$$

Plugging eq. (44) into eq. (42) we obtain

$$\frac{1}{2}(\bar{v}_b\gamma^\rho P_L v_d)(\bar{u}_b\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\rho P_R\gamma^\nu P_L\gamma^\alpha\gamma^\mu P_L u_d) \,, \tag{45}$$

which can be further simplified using

$$\gamma_\mu\gamma_\alpha\gamma^\mu = -2\gamma_\alpha \tag{46}$$

to obtain

$$-4(\bar{v}_b\gamma^\mu P_L v_d)(\bar{u}_b\gamma_\mu P_L u_d)\,. \tag{47}$$

The Feynman amplitude generated by this diagram is thus

$$\begin{aligned} i\mathcal{M}^{(7)} &= \frac{i}{16\pi^2}4\frac{g^4}{16M_W^2}\sum_{i,j=u,c,t}V_{ib}^*V_{id}V_{jb}^*V_{jd}A(x_i,x_j)(\bar{v}_b\gamma^\mu P_L v_d)(\bar{u}_b\gamma_\mu P_L u_d) \\ &= i\frac{G_F^2M_W^2}{2\pi^2}(\bar{v}_b\gamma^\mu P_L v_d)(\bar{u}_b\gamma_\mu P_L u_d)\sum_{i,j=u,c,t}\lambda_i\lambda_j A(x_i,x_j)\,, \end{aligned} \tag{48}$$

where $G_F/\sqrt{2} = g^2/(8M_W^2)$ is the Fermi constant and $\lambda_i = V_{ib}^*V_{id}$.

Diagram 8 in Fig. 2 is identical to the one we just computed, up to a Fierz identity $((\gamma^\mu P_L)_{\alpha\beta}(\gamma_\mu P_L)_{\gamma\delta} = -(\gamma^\mu P_L)_{\alpha\delta}(\gamma_\mu P_L)_{\gamma\beta}$:

$$i\mathcal{M}^{(8)} = -i\frac{G_F^2M_W^2}{2\pi^2}(\bar{v}_b\gamma^\mu P_L u_d)(\bar{u}_b\gamma_\mu P_L v_d)\sum_{i,j=u,c,t}\lambda_i\lambda_j A(x_i,x_j)\,. \tag{49}$$

2.2 Introducing the effective Hamiltonian

We notice that these Feynman amplitudes can be written as matrix elements of an effective Hamiltonian involving a local operator. Let us consider the following effective interaction:

$$\mathcal{H}_{\text{eff}}^{\Delta B=2} = C\bar{b}_L\gamma^\mu d_L\bar{b}_L\gamma_\mu d_L\,, \tag{50}$$

built up of a numerical coefficient C, called Wilson coefficient, times a local operator of mass dimension six. The Wilson coefficient must then have dimensions mass^{-2}. The matrix element of $\mathcal{H}_{\text{eff}}^{\Delta B=2}$ is given by

$$\begin{aligned} i\mathcal{M}^{\mathcal{H}} &= -iC\langle\bar{d}b|\bar{b}_L\gamma^\mu d_L\bar{b}_L\gamma_\mu d_L|\bar{b}d\rangle \\ &= -2iC\left(\bar{u}_b\gamma^\mu P_L v_d\bar{v}_b\gamma_\mu P_L u_d - \bar{u}_b\gamma^\mu P_L u_d\bar{v}_b\gamma_\mu P_L v_d\right)\,. \end{aligned} \tag{51}$$

Comparing eq. (51) with eqs. (48) and (49) we obtain

$$C^{(7+8)} = \frac{G_F^2M_W^2}{4\pi^2}\sum_{i,j}\lambda_i\lambda_j A(x_i,x_j)\,. \tag{52}$$

Proceeding along the same lines we obtain

$$\begin{aligned} C^{(3+5)} = C^{(4+6)} &= -\frac{G_F^2M_W^2}{4\pi^2}\sum_{i,j}\lambda_i\lambda_j x_i x_j A'(x_i,x_j)\,, \\ C^{(1+2)} &= \frac{1}{4}\frac{G_F^2M_W^2}{4\pi^2}\sum_{i,j}\lambda_i\lambda_j x_i x_j A(x_i,x_j)\,, \end{aligned} \tag{53}$$

with

$$A'(x_i,x_j) = \frac{J'(x_i)-J'(x_j)}{x_i-x_j}\,, \qquad J'(x) = \frac{1}{1-x} + \frac{x\log x}{(1-x)^2}\,. \tag{54}$$

Summing the four contributions we finally obtain

$$C = \frac{G_F^2 M_W^2}{4\pi^2} \sum_{i,j} \lambda_i\lambda_j \bar{A}(x_i,x_j)\,, \tag{55}$$

with

$$\bar{A}(x_i,x_j) = A(x_i,x_j) - x_i x_j A'(x_i,x_j) + \frac{1}{4} x_i x_j A(x_i,x_j)\,. \tag{56}$$

The unitarity of the CKM matrix implies that

$$\lambda_u + \lambda_c + \lambda_t = 0 \tag{57}$$

so that we can replace $\lambda_u \to -\lambda_t - \lambda_c$ to obtain

$$\begin{aligned}\sum_{i,j} \lambda_i\lambda_j \bar{A}(x_i,x_j) &= (\lambda_c+\lambda_t)^2 \bar{A}(x_u,x_u) + 2\lambda_c\lambda_t \bar{A}(x_c,x_t) + \lambda_c^2 \bar{A}(x_c,x_c) \\ &+\lambda_t^2 \bar{A}(x_t,x_t) - 2\lambda_t(\lambda_c+\lambda_t)\bar{A}(x_u,x_t) - 2\lambda_c(\lambda_c+\lambda_t)\bar{A}(x_u,x_c) \\ &= \lambda_t^2 S_0(x_t) + \lambda_c^2 S_0(x_c) + 2\lambda_c\lambda_t S_0(x_c,x_t)\,, \end{aligned} \tag{58}$$

where

$$\begin{aligned} S_0(x_t) &= \bar{A}(x_t,x_t) + \bar{A}(x_u,x_u) - 2\bar{A}(x_u,x_t)\,, \\ S_0(x_c) &= \bar{A}(x_c,x_c) + \bar{A}(x_u,x_u) - 2\bar{A}(x_u,x_c)\,, \\ S_0(x_c,x_t) &= \bar{A}(x_c,x_t) + \bar{A}(x_u,x_u) - \bar{A}(x_u,x_t) - \bar{A}(x_u,x_c)\,. \end{aligned} \tag{59}$$

We note that S_0 contains only differences of $\bar{A}$ functions with different arguments. Thus, for massless or degenerate quarks no FCNC vertex can be generated, and the coefficients are suppressed by the GIM mechanism. Indeed, we have

$$S_0(x) \overset{x\to 0}{\sim} x\,. \tag{60}$$

Thus, the contribution of light quarks is suppressed by $m_{u_i}^2/M_W^2$, and it vanishes as M_W^{-4}.

We also notice that heavy quarks do not decouple from FCNC processes. Indeed, in the large x limit we have

$$S_0(x) \overset{x>>1}{\sim} x\,, \tag{61}$$

so that the top quark plays a dominant role in flavour physics. This non-decoupling effect can be immediately understood recalling that the would-be goldstone bosons couple to fermions with a strength proportional to quark masses.

For $B_d - \bar{B}_d$ mixing, we have $\lambda_c \sim \lambda_t$, so that we can safely neglect $S_0(x_c)$ and $S_0(x_c, x_t)$ in eq. (58), leading to

$$\mathcal{H}_{\rm eff}^{\Delta B=2} = \frac{G_F^2 M_W^2}{4\pi^2} \lambda_t^2 S_0(x_t) \bar{b}_L \gamma^\mu d_L \bar{b}_L \gamma_\mu d_L \,. \tag{62}$$

We have obtained the result above neglecting external momenta. We now show that the terms we neglected are suppressed by an additional factor of q^2/M_W^2, where q is the external momentum. To this aim, let us write down the full expression for $I^i_{\alpha\beta}$:

$$I^i_{\alpha\beta}(q) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{p_\alpha p_\beta}{((p-q)^2 - M_W^2)^2 (p^2 - m_{u_i})} \tag{63}$$

and Taylor expand it around $q = 0$:

$$I^i_{\alpha\beta}(q) = I^i_{\alpha\beta} + \frac{1}{2} \frac{\partial}{\partial q_\mu} \frac{\partial}{\partial q_\nu} I^i_{\alpha\beta}(q) \bigg|_{q=0} q^\mu q^\nu + \mathcal{O}(q^4) \,. \tag{64}$$

We have

$$\frac{\partial}{\partial q_\mu} \frac{\partial}{\partial q_\nu} \frac{1}{[(p-q)^2 - M_w^2]^2} \bigg|_{q=0} = 4 \left(- \frac{\delta_{\mu\nu}}{(p^2 - M_W)^3} + 6 \frac{p_\mu p_\nu}{(p^2 - M_W^2)^4} \right) \,. \tag{65}$$

For the sake of simplicity, let us focus on the term proportional to $\delta_{\mu\nu}$ in eq. (65); similar considerations apply to the other term. We obtain

$$I^i_{\alpha\beta}(q) = I^i_{\alpha\beta} - q^2 \frac{\partial}{\partial M_W^2} I^i_{\alpha\beta} + \ldots = I^i_{\alpha\beta} + \frac{q^2}{M_W^2} x \frac{\partial}{\partial x} I^i_{\alpha\beta} + \ldots \,, \tag{66}$$

showing explicitly that the effects of external momenta are suppressed by powers of q^2/M_W^2.

To summarize, we have seen how the transition $\bar{b}d \to \bar{d}b$ is generated at the loop level in the SM, how the GIM mechanism implies that this amplitude is proportional to $G_F^2 m_{u_i}^2$ for light quarks, and how this amplitude can be written as the matrix element of an effective Hamiltonian containing a local operator.

2.3 QCD corrections

Let us now very briefly comment on the inclusion of QCD effects in the calculation above. Let us consider for example the exchange of a gluon between external quark lines as in Fig. **??**. In the full theory, this diagram is convergent in the ultraviolet. However, the same diagram in the effective theory has a logarithmic divergence. Indeed, the W propagator acts as a regulator of the ultraviolet divergence. Thus, the result in the full theory contains terms proportional to

$$\alpha_S \log \frac{M_W^2}{m_b^2}\,, \tag{67}$$

so that the large log spoils the perturbative expansion. This is due to the presence of two widely different scales: the weak scale and the hadronic scale. The effective Hamiltonian and the renormalization group equations give us a very efficient tool to resum these large logs. Indeed, since the W boson acts as a regulator of the effective theory, the coefficient of the log is given by the anomalous dimension γ_0 of the operator $\bar{b}_L\gamma^\mu d_L \bar{b}_L\gamma_\mu d_L$, which can be easily computed using dimensional regularization. Then, large logs are resummed by computing the Wilson coefficient at a scale $M \sim M_W$ and then by evolving it to the hadronic scale $\mu \sim m_b$ using renormalization group equations. In the leading logarithmic approximation, we have

$$C(\mu) = \left[\frac{\alpha_S(M_W)}{\alpha_S(\mu)}\right]^{\frac{\gamma_0}{2\beta_0}} C = \eta(\mu) C\,, \tag{68}$$

where β_0 is the first coefficient of the QCD beta function, and C is the Wilson coefficient we computed above.

2.4 Hadronic matrix elements

We have obtained the effective Hamiltonian

$$\mathcal{H}_{\text{eff}}^{\Delta B=2} = \frac{G_F^2 M_W^2}{4\pi^2}\lambda_t^2 S_0(x_t)\eta(\mu)\bar{b}_L\gamma^\mu d_L \bar{b}_L\gamma_\mu d_L(\mu)\,, \tag{69}$$

where μ is the renormalization scale. We have seen that this Hamiltonian correctly describes the SM $\bar{b}d \to \bar{d}b$ transition amplitude, including the resummation of large logs. However, we are ultimately interested in computing a transition amplitude for B mesons: can we use $\mathcal{H}_{\text{eff}}^{\Delta B=2}$ for this purpose?

The answer is positive, since the Wilson coefficient $C(\mu)$ does not depend on the choice of the external states. Indeed, the dependence on external states drops in the matching between the full and the effective theories, since the two theories only differ in the ultraviolet. All the dependence on external states is thus encoded in the matrix elements of the effective Hamiltonian. Therefore, to compute the $B_d \to \bar{B}_d$ transition amplitude, we must compute the matrix element

$$\langle B | \bar{b}_L \gamma^\mu d_L \bar{b}_L \gamma_\mu d_L(\mu) | \bar{B} \rangle \,. \tag{70}$$

This matrix element contains all the low-energy hadronic dynamics, and it must be computed using non-perturbative methods such as lattice QCD or QCD sum rules.

It is customary to express the matrix element in eq. (70) in terms of the Vacuum Insertion Approximation (VIA) result times a B-parameter. We then have

$$\langle B | \bar{b}_L \gamma^\mu d_L \bar{b}_L \gamma_\mu d_L(\mu) | \bar{B} \rangle = \frac{1}{3} F_B^2 m_B B(\mu) \,. \tag{71}$$