Matrix Canonical Forms

Roger Horn

University of Utah

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SVD remarks

- $A = [a_1 \ \ldots \ a_n] = [a_{ij}]$
  - $\sigma_1 = \|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 \geq \|Ae_j\|_2 = \|a_j\|_2 \geq |a_{ij}|$
- U unitary $\Rightarrow U^*U = I \Rightarrow \Sigma = I$
- A given, $A = V\Sigma W^*$ and $\Sigma = I \Rightarrow A = VIW^*$ is unitary

Structured unitary equivalence: The CS decomposition

- Given: $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$ unitary, $U_{11} \in M_p$, $U_{22} \in M_q$, $p \leq q$, $p + q = n$
- Available to choose: $V, W \in M_n$ unitary, which we insist must be structured conformally to the partitioning of U: $V = V_1 \oplus V_2$, $W = W_1 \oplus W_2$, $V_1, W_1 \in M_p$, $V_2, W_2 \in M_q$
- Then $U \to Z = VUW = \begin{bmatrix} V_1 U_{11} W_1 & V_1 U_{12} W_2 \\ V_2 U_{21} W_1 & V_2 U_{22} W_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$ is unitary. We want to choose the small unitary matrices V_1, V_2, W_1, W_2 so that Z has a simple structure. We may then pre- or post-multiply Z by any unitary matrices of the forms $\hat{V} \oplus I_q$ or $I_p \oplus \hat{W}$ in which $\hat{V} \in M_p$ and $\hat{W} \in M_q$ are unitary.
- Use the SVD to choose V_1 and W_1 so that $V_1 U_{11} W_1 = \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_p)$, $1 \geq \sigma_1 \geq \cdots \geq \sigma_p \geq 0$. Now $Z = \begin{bmatrix} \Sigma & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$

Structured unitary equivalence: The CS decomposition

- Pre- and post-multiply by $K_p \oplus I_q$ (K is the reversal matrix). Now Z has the form

$$Z = \begin{bmatrix} K_p \Sigma K_p & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}, \quad K_p \Sigma K_p = \begin{bmatrix} 0 & & \\ & C & \\ & & I \end{bmatrix}$$

  with $C = \operatorname{diag}(c_1, \ldots, c_s)$ and $0 < c_1 \leq \cdots \leq c_s < 1$

- Z_{21} is q-by-p so there is a unitary $Q_1 \in M_q$ such that $Q_1 Z_{21} = R_{21} = \begin{bmatrix} R \\ 0 \end{bmatrix}$ is q-by-p and R is *upper* triangular
- Z_{12} is p-by-q so there is a unitary $Q_2 \in M_q$ such that $Z_{12} Q_2 = L_{12} = \begin{bmatrix} L & 0 \end{bmatrix}$ is p-by-q and L is *lower* triangular

Structured unitary equivalence: The CS decomposition

Pre-multiply by $I_p \oplus Q_1$ and post-multiply by $I_p \oplus Q_2$. Now Z has the form

$$\begin{bmatrix} \begin{bmatrix} 0 & & \\ & C & \\ & & I \end{bmatrix} & \begin{bmatrix} L & 0 \end{bmatrix} \\ \begin{bmatrix} R \\ 0 \end{bmatrix} & [Z_{22}] \end{bmatrix}$$

Partition L and R conformally to Z_{11} and keep in mind that each is triangular, so their diagonal blocks are triangular:

Structured unitary equivalence: The CS decomposition

$$\begin{bmatrix} \begin{bmatrix} 0 & & \\ & C & \\ & & I \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} ? & 0 & 0 \\ ? & ? & 0 \\ ? & ? & ? \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} ? & ? & ? \\ 0 & ? & ? \\ 0 & 0 & ? \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{bmatrix} & [Z_{22}] \end{bmatrix}$$

Let $S = \operatorname{diag}(\sqrt{1-c_1^2}, \ldots, \sqrt{1-c_s^2})$. Invoke orthonormality of the top p rows and the left p columns to get

$$\begin{bmatrix} \begin{bmatrix} 0 & & \\ & C & \\ & & I \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} & \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} \\ \begin{bmatrix} ? & ? & ? \end{bmatrix} & [?] \end{bmatrix} \end{bmatrix}$$

Now invoke orthonormality of rows $p+1, \ldots, 2p$ and of columns $p+1, \ldots, 2p$ to get

Structured unitary equivalence: The CS decomposition

$$\begin{bmatrix} \begin{bmatrix} 0 & & \\ & C & \\ & & I \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -C & 0 \\ 0 & 0 & ? \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ ? \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & ? \end{bmatrix} & \begin{bmatrix} ? \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

which is...

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 & I & 0 \\ 0 & C & 0 & 0 & S \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & S & 0 & 0 & -C \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} & [Z_?] \end{bmatrix}$$

The remaining unknown block is a direct summand of a unitary matrix, so it is unitary and there are unitary matrices W' and W'' such that $W'Z_?W'' = I$.

Structured unitary equivalence: The CS decomposition

- Pre-multiply by $I \oplus W'$ and post-multiply by $I \oplus W''$ to obtain

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 & I & 0 \\ 0 & C & 0 & 0 & S \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & S & 0 & 0 & -C \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} & [I] \end{bmatrix}$$

- which we re-organize as

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & I \end{bmatrix} & \begin{bmatrix} I & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} I & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & -C & 0 \\ 0 & 0 & I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & [I_{q-p}] \end{bmatrix}$$

Structured unitary equivalence: The CS decomposition

- Finally, we adjust the signs and pre-/post-multiply by $K \oplus K \oplus I_{q-p}$ to permute the diagonal blocks (this also reverses the order of the diagonal entries in $C \to C'$ and $S \to S'$) to obtain

$$\begin{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & C' & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & S' & 0 \\ 0 & 0 & I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & -S' & 0 \\ 0 & 0 & -I \end{bmatrix} & \begin{bmatrix} I & 0 & 0 \\ 0 & C' & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} I_{q-p} \end{bmatrix} \end{bmatrix}$$

- For a more easily remembered form, let

$$\mathcal{C} = \begin{bmatrix} I & 0 & 0 \\ 0 & C' & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & S' & 0 \\ 0 & 0 & I \end{bmatrix}$$

so that $\mathcal{C}^2 + \mathcal{S}^2 = I_p$.

Structured unitary equivalence: The CS decomposition

- The *CS* decomposition of $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \in M_{p+q}$ is

$$\begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \begin{bmatrix} \mathcal{C} & \mathcal{S} & 0 \\ -\mathcal{S} & \mathcal{C} & 0 \\ 0 & 0 & I_{q-p} \end{bmatrix} \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}$$

  in which $U_{11}, \mathcal{C}, \mathcal{S}, V_1, W_1 \in M_p$; $U_{22}, V_2, W_2 \in M_q$; V_i, W_i are unitary; $\mathcal{C} = \operatorname{diag}(\sigma_1, \ldots, \sigma_p)$; $\sigma_1, \ldots, \sigma_p$ are the decreasingly ordered singular values of U_{11}; and $\mathcal{S} = \operatorname{diag}(\sqrt{1-\sigma_1^2}, \ldots, \sqrt{1-\sigma_p^2})$.
- This is a parametric representation for all unitary 2-by-2 block matrices with the given block sizes. The parameters are: p arbitrary numbers between zero and one (the diagonal entries of $\mathcal{C}$), and four arbitrary unitary matrices $V_1, W_1 \in M_p$, $V_2, W_2 \in M_q$.
- Applications: angles between subspaces, structured inverses, complementary nullities,...

Canonical forms for similarity: The Jordan canonical form

- The *Jordan block* of size ℓ with eigenvalue λ is

$$J_\ell(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}_{\ell \times \ell}$$

- A *Jordan matrix* is a direct sum of the form $J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_p}(\lambda_p)$
- (3.1.11) Each $A \in M_n$ is similar to a Jordan matrix.
- What about uniqueness?
- $J_\ell(\lambda) - \lambda I = J_\ell(0)$. Translation by λI permits us to reduce to the nilpotent case.
- $\operatorname{rank} J_\ell(0) = \ell - 1$, $\operatorname{rank} J_k(0)^2 = \ell - 2, \ldots, \operatorname{rank} J_\ell(0)^{\ell-1} = 1$, $\operatorname{rank} J_\ell(0)^\ell = 0$
- Convention: $\operatorname{rank} J_\ell(0)^0 := \ell$
- $\operatorname{rank} J_\ell(0)^k = \max\{\ell - k, 0\}$ for each $k = 1, 2, \ldots$

Canonical forms for similarity: The Jordan canonical form

- $\operatorname{rank} J_\ell(0)^{k-1} - \operatorname{rank} J_\ell(0)^k = \begin{cases} 1 \text{ if } \ell \geq k \\ 0 \text{ if } \ell < k \end{cases}, \; k = 1, 2, \ldots$
- $J = J_{n_1}(\lambda) \oplus \cdots \oplus J_{n_p}(\lambda)$ and $J - \lambda I = J_{n_1}(0) \oplus \cdots \oplus J_{n_p}(0)$
- $\operatorname{rank}(J - \lambda I)^{k-1} - \operatorname{rank}(J - \lambda I)^k = (\operatorname{rank} J_{n_1}(0)^{k-1} - \operatorname{rank} J_{n_1}(0)^k) + \cdots + (\operatorname{rank} J_{n_p}(0)^{k-1} - \operatorname{rank} J_{n_p}(0)^k)$
- $= (1 \text{ if } n_1 \geq k) + \cdots + (1 \text{ if } n_p \geq k)$
- $=$ number of blocks with size k or larger
- Define $w_k(J, \lambda) = \operatorname{rank}(J - \lambda I)^{k-1} - \operatorname{rank}(J - \lambda I)^k$
- Then $w_k(J, \lambda) - w_{k+1}(J, \lambda) =$ (# blocks of J with size k or larger) $-$ (# blocks of J with size $k+1$ or larger) $=$ # blocks of J with size exactly k
- $w_k(SJS^{-1}, \lambda) = \operatorname{rank}(SJS^{-1} - \lambda I)^{k-1} - \operatorname{rank}(SJS^{-1} - \lambda I)^k$
- $= \operatorname{rank}(S(J - \lambda I)S^{-1})^{k-1} - \operatorname{rank}(S(J - \lambda I)S^{-1})^k$
- $= \operatorname{rank}(S(J - \lambda I)^{k-1}S^{-1}) - \operatorname{rank}(S(J - \lambda I)^k S^{-1})$
- $= \operatorname{rank}(J - \lambda I)^{k-1} - \operatorname{rank}(J - \lambda I)^k = w_k(J, \lambda)$
- Thus, $w_k(SJS^{-1}, \lambda) = w_k(J, \lambda)$ is a similarity invariant

- Each $A \in M_n$ is similar to a Jordan matrix, so the number of blocks $J_k(\lambda)$ in the Jordan canonical form of A is exactly $w_k(A,\lambda) - w_{k+1}(A,\lambda)$
- The sequence $w_1(A,\lambda), \ldots, w_n(A,\lambda)$ is the *Weyr characteristic* of A with respect to the eigenvalue λ. It is similarity invariant and is determined by the values of $\operatorname{rank}(A - \lambda I)^k$, $k = 1, \ldots, n$.
- The Jordan canonical form of A is unique (up to permutation of its direct summands): the number of blocks $J_k(\lambda)$ for each eigenvalue λ is determined by the Weyr characteristic of A.
- (3.1.18) A and B are similar if and only if they have the same eigenvalues, and the same Weyr characteristics with respect to each of those eigenvalues.

Some facts about the Weyr characteristic

- $w_1(A, \lambda) =$ total number of Jordan blocks $J_i(\lambda)$ of all sizes $=$ geometric multiplicity of λ as an eigenvalue of A
- $w_k(A, \lambda) =$ number of Jordan blocks $J_i(\lambda)$ with $i \geq k$
- $w_k(A, \lambda) = 0$ if $k > q_\lambda =$ *index* of $\lambda =$ size of largest $J_i(\lambda)$
- $w_1(A, \lambda) \geq w_2(A, \lambda) \geq \cdots \geq w_{q_\lambda}(A, \lambda) \geq 1 > w_{q_\lambda+1}(A, \lambda) = 0$

The Weyr Canonical Form

- Suppose that the distinct eigenvalues of $A \in M_n$ are $\lambda_1, \ldots, \lambda_d$. Choose one of them, call it λ, suppose the index of λ is q, and let $w_k := w_k(A, \lambda)$, $k = 1, \ldots, q$. The *Weyr block* of A associated with the eigenvalue λ is
- $W_A(\lambda) = \begin{bmatrix} \lambda I_{w_1} & G_{12} & & & \\ & \lambda I_{w_2} & G_{23} & & \\ & & \ddots & \ddots & \\ & & & \ddots & G_{w_{q-1}, w_q} \\ & & & & \lambda I_{w_q} \end{bmatrix}$, $G_{i,i+1} = \begin{bmatrix} I_{w_{i+1}} \\ 0 \end{bmatrix}$
- Only one Weyr block for each distinct eigenvalue.
- $W_A(\lambda) - \lambda I = W_A(0)$.
- rank $W_A(0) = w_2 + \cdots + w_q$, rank $W_A(0)^2 = w_3 + \cdots + w_q$, etc.
- rank $W_A(0) -$ rank $W_A(0)^2 = w_2$, rank $W_A(0)^2 -$ rank $W_A(0)^3 = w_3$, etc.
- Weyr characteristics of $W_A(\lambda)$ and A (with respect to λ) are the same!

The Weyr Canonical Form

- The *Weyr matrix* of A is $W_A = W_A(\lambda_1) \oplus \cdots \oplus W_A(\lambda_d)$ (d blocks)
- W_A is similar to J_A: same eigenvalues and same Weyr characteristics!
- (3.4.2.3) Weyr matrices are a canonical form for similarity.
- In fact, W_A and J_A are *permutation similar*. So why bother?

Jordan vs. Weyr: commutativity

- $J = \begin{bmatrix} J_2(\lambda) & 0 \\ 0 & J_2(\lambda) \end{bmatrix}$,
- $w_1(J,\lambda) = 2, w_2(J,\lambda) = 2 \Rightarrow W_J = \begin{bmatrix} \lambda I_2 & I_2 \\ 0 & \lambda I_2 \end{bmatrix}$
- $AJ = JA \Leftrightarrow A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$, each block is upper triangular Toeplitz
- $AW_J = W_J A \Leftrightarrow A = \begin{bmatrix} F & G \\ 0 & F \end{bmatrix}$, which is block upper triangular.
- Construct a Schur triangularization: $F = U\Delta U^*$, Δ upper triangular
- $V = U \oplus U$: $\quad V^* W_J V = W_J$, $\quad V^* A V = \begin{bmatrix} \Delta & D \\ 0 & \Delta \end{bmatrix}$. Thus, there is a block unitary matrix conformal to the block structure of W_J that leaves W_J invariant and reduces A to upper triangular form.

Jordan vs. Weyr: commutativity

- (3.4.2.10) $\mathcal{F} = \{A, A_1, A_2, \ldots\}$ a commuting family $\Rightarrow$ there is a simultaneous similarity $\mathcal{F} \to S\mathcal{F}S^{-1} = \{W_A, SA_1S^{-1}, SA_2S^{-1}, \ldots\}$ that puts A into Weyr canonical form and upper triangularizes each A_i (moreover, there are certain identities between blocks on the same superdiagonal of each SA_jS^{-1}).
- *There is no analog* of this simultaneous reduction for the Jordan canonical form!
- Many applications, e.g., sub-algebras of M_n generated by a commuting family (Gerstenhaber (1961), Neubauer/Sethuraman (1999), O'Meara/Visonhaler (2006))

The unitary Weyr form

- The *Weyr canonical form theorem* says that for each $A \in M_n$ there is a nonsingular $S \in M_n$ such that $A = SW_AS^{-1}$. Let $S = QR$, in which Q is unitary and R is nonsingular and upper triangular, with positive diagonal entries. Then $A = SW_AS^{-1} = Q(RW_AR^{-1})Q^*$, so A is unitarily similar to $F = RW_AR^{-1}$, which has the form

$$F = \begin{bmatrix} \mu_1 I_{n_1} & F_{12} & F_{13} & \cdots & F_{1p} \\ & \mu_2 I_{n_2} & F_{23} & \cdots & F_{2p} \\ & & \mu_3 I_{n_3} & \ddots & \vdots \\ & & & \ddots & F_{p-1,p} \\ & & & & \mu_p I_{n_p} \end{bmatrix}$$

- The block sizes of the μ_i are determined by the Weyr characteristics of A; if $\mu_i = \mu_{i+1}$ then $n_i \geq n_{i+1}$, $F_{i,i+1} \in M_{n_i,n_{i+1}}$ is upper triangular and has positive diagonal entries, so it has full rank.

The unitary Weyr form

- (3.4.3.1) The upper triangular form

$$F = \begin{bmatrix} \mu_1 I_{n_1} & F_{12} & F_{13} & \cdots & F_{1p} \\ & \mu_2 I_{n_2} & F_{23} & \cdots & F_{2p} \\ & & \mu_3 I_{n_3} & \ddots & \vdots \\ & & & \ddots & F_{p-1,p} \\ & & & & \mu_p I_{n_p} \end{bmatrix}$$

is a substantial refinement of the Schur upper triangular form because of the special structure of the superdiagonal blocks $F_{i,i+1}$. It has many applications to problems involving unitary similarity. (3.4.3.3), Problem 5 in (3.4)

References

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C. Paige and M. Wei, History and generality of the CS decomposition, *Linear Algebra Appl.* 208/209 (1994) 303-326.