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Introduction to Nonlinear Gyrokinetic Theory

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This Lecture

- Properties of Tokamak Micro-turbulence
- **Modern Nonlinear Gyrokinetics:**
  - Emphasis on Conservation Laws
  - Systematic Derivation
  - Single Particle Dynamics
    and Gyrokinetic Vlasov Equation
  - Gyrokinetic Maxwell's Equation
    and Pullback Transformation
- Further Extensions

Microinstabilities in Tokamaks

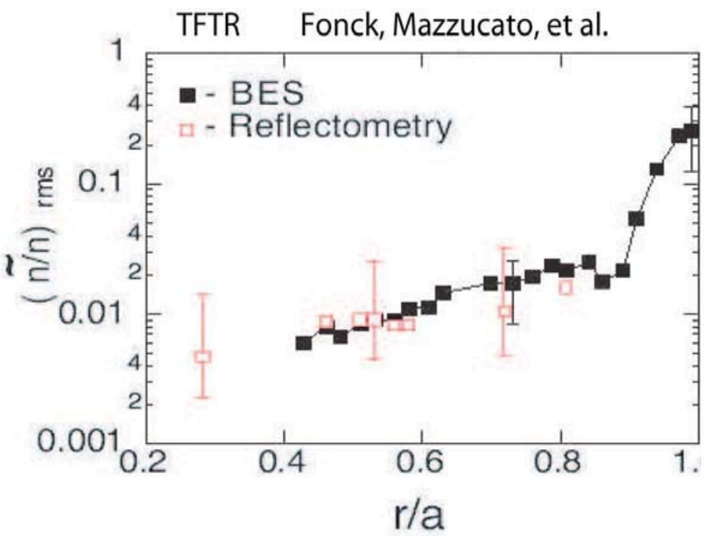
• Tokamak transport is usually anomalous, even in the absence of large-scale MHD

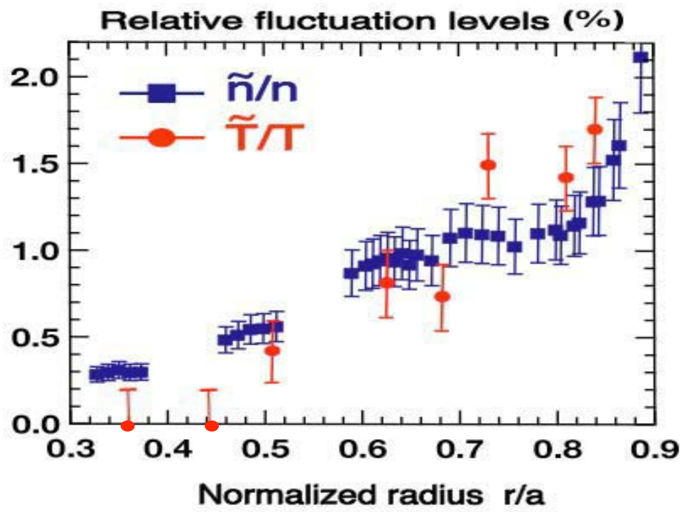
• Caused by small-scale collective instabilities driven by gradients in temperature, density, …

• Instabilities saturate at low amplitude due to nonlinear mechanisms

• Particles **E** x **B** drift radially due to fluctuating electric field

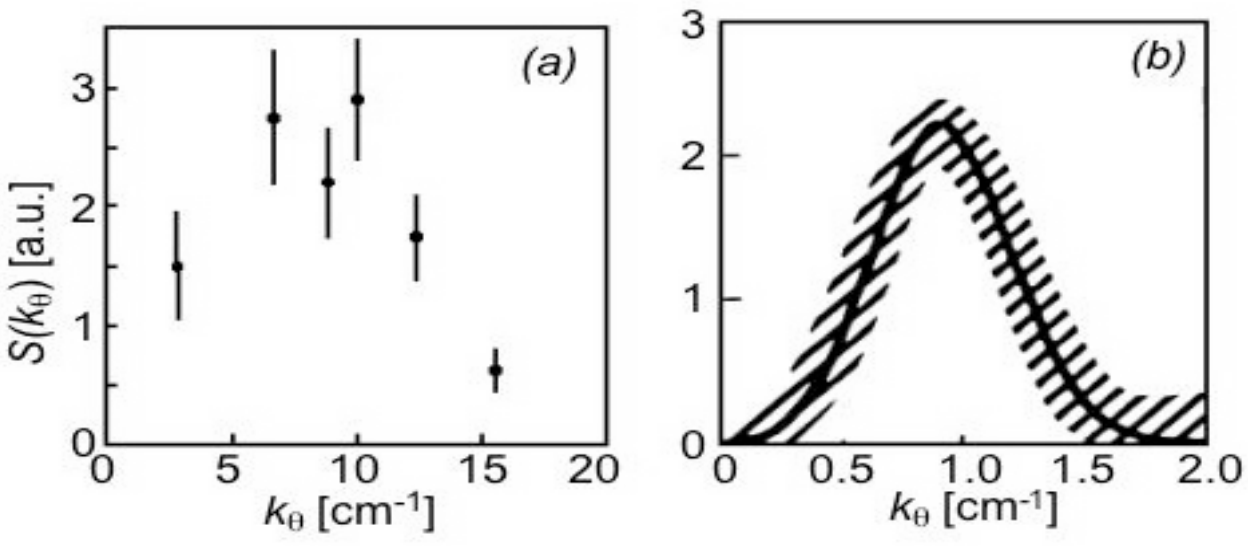
Amplitude of Tokamak Microturbulence





- Relative fluctuation amplitude $\delta n / n_0$ at core typically less than 1%
- At the edge, it can be greater than 10%
- Confirmed in different machines using different diagnostics

k-spectra of tokamak micro-turbulence



$k_\theta\ \rho_i \sim 0.1 - 0.2$

-from Mazzucato et al., PRL '82 (μ-wave scattering on ATC)
Fonck et al., PRL '93 (BES on TFTR)

-similar results from

TS, ASDEX, JET, JT-60U and DIII-D

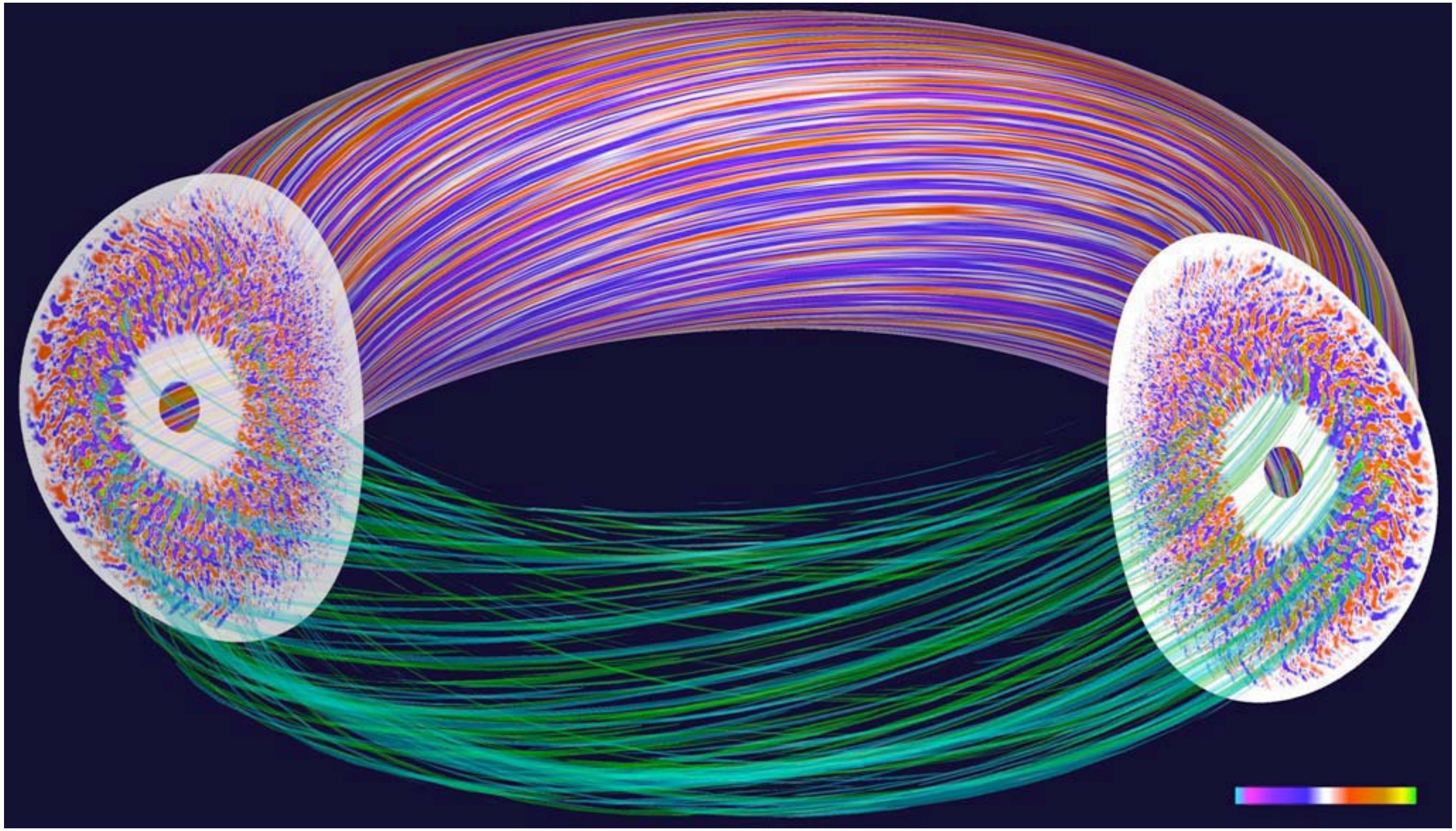
Properties of Tokamak Core Microturbulence

- $\delta n / n_0 \sim 1\%$
- $k_r \rho_i \sim k_\theta \rho_i \sim 0.1 - 0.2$
- $k_{||} < 1/qR << k_\perp$: Rarely measured
- $\omega - \mathbf{k} \cdot \mathbf{u}_E \sim \Delta\omega \sim \omega_{*pi}$:

Broad-band $\Rightarrow$ Strong Turbulence

Sometimes Doppler shift dominates in rotating plasmas

L'aspect Cinématique de la Théorie Gyrocinétique



GTS simulation of ITG Turbulence: S. Ethier, W. Wang et al.,

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Electrostatic Microinstabilities in Tokamaks

Classification: Free energy	Spatio-temporal Scales (wavelength, frequency direction, rough mag.)	Accessibility Mechanism for Instability
Trapped Ion Mode n, T_e, (ITG-TIM) T_i	$\sim \rho_\theta$ $\sim \omega^*_e$	Trapped ion precession resonance (coll-less) Collisions btwn trapped and passing ions (dissipative)
Ion Temp. Grad. Mode T_i	$> \rho_i$ $< \omega^*_{pi}$	Bad curvature or Negative compressibility
Trapped Electron Mode n or T_e	$\sim \rho_i$ $< \omega^*_e$	Trapped electron precession resonance (coll-less) Collisions btwn trapped and passing e^-s (dissipative)
Electron Temp. G Mode T_e	$> \rho_e$ $< \omega^*_{pe}$	Bad curvature or Negative compressibility

Standard Nonlinear Gyrokinetic Ordering I.

Frieman and Chen, Phys. Fluids 1982

Minimum number of ordering assumption

- $\omega/\Omega_i \sim k_{||}/k_\perp \sim \varepsilon_{k,\omega} << 1$; from spatio-temporal scales of fluctuations
- $k_\perp\rho_i \sim 1$ for generality:
  Short wavelength modes (with higher γ_{lin}) can affect
  the modes at NL peak ($k_\perp\rho_i \sim 0.1 \sim 0.2$) through NL coupling.
  $\rightarrow \omega \sim k_{||}v_{Ti}$ for wave-particle resonance

  i.e., Landau damping

- $\delta f/f_0 \sim e\delta\phi/T_e \sim 1/k_\perp L_p \sim \varepsilon_\phi << 1$; from small relative fluctuation amplitude

  - $k\, e\delta\phi/T_e \sim 1/L_p$: **ExB** Nonlinearity ~ Linear Drive

  - $\delta n/n_0 \sim \rho/L$ ~ roughly experimental values.

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Standard Nonlinear Gyrokinetic Ordering II.

- While the physics origins of $\varepsilon_{k,\omega}$ and ε_{ϕ} are different, the maximal ordering for NL GK corresponds to $\varepsilon_{k,\omega} \sim \varepsilon_{\phi}$

- $\varepsilon_{k,\omega} >> \varepsilon_{\phi}$ leads back to the Linear Gyrokinetics:

Taylor-Hastie, Plasma Phys. **10**, 419 '68
Rutherford-Frieman, Phys. Fluids **11**, 569 '68
Tang, Nuclear Fusion **18**, 1089 '78
Antonsen-Lane, Phys. Fluids **23**, 1205 '80
Horton, Rev. Mod. Phys **71**, 735 '99

- With $\varepsilon_{k,\omega} << \varepsilon_{\phi}$, one cannot recover the linear dispersion relation of instabilities:

Self-sustained Turbulence, BS from **BDS**

Scott, Phys. Rev. Lett. **65**, 3289 ‘90

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Conventional Nonlinear Gyrokinetic Equation

[eg., Frieman and Chen, Phys. Fluids 1982]

- Foundations of Tokamak Nonlinear Kinetic Theory
  for analytic applications, ballooning codes...
- Number of assumptions minimum
- Based on direct gyro-phase average of Vlasov equation
  Lots of algebra and book keeping
- Direct expansions in ε: Self-consistent up to $O(\varepsilon^2)$ →

Should be fine for linear phase and saturation due to **E**x**B** nonlinearity

- Velocity space nonlinearity: $\nabla_{||}\, \delta\phi\, \partial_{v||}\delta f \sim O(\varepsilon^3)$

Energy, phase space volume **not** conserved.

- May not be able to describe long term behavior accurately

Topic of Current Research: [Villard, Hatzky, Sorge, Lee, Wang, Ku]

→ Physics responsible for the difference?

Conventional Nonlinear GK Derivation:Heuristic

• Transforming to guiding center variables, R = x + ρ, $\mu = v_\perp^2/2B$,
$\mathbf{v} = v_\parallel \mathbf{b} + (\mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta)$, one can write the Vlasov equation as

$$\frac{\partial}{\partial t} f + v_\parallel \mathbf{b} \cdot \frac{\partial}{\partial \mathbf{R}} f + \frac{\mathbf{E}\times\mathbf{b}}{\mathbf{B}} \cdot \frac{\partial}{\partial \mathbf{R}} f + (q/m) E_\parallel \frac{\partial}{\partial v_\parallel} f - \Omega \frac{\partial}{\partial \theta} f = 0$$

• Since $\Omega >> \omega$, to the lowest order $\Omega(\partial/\partial\theta)f=0$

• Writing $f = \langle f \rangle + \tilde{f}$, with $f = \langle f \rangle >> f$ in which $\langle \cdots \rangle$ indicates gyrophase average,

$$\frac{\partial}{\partial t}\langle f \rangle + v_\parallel \mathbf{b} \cdot \frac{\partial}{\partial \mathbf{R}}\langle f \rangle + \frac{\mathbf{E}\times\mathbf{b}}{\mathbf{B}} \cdot \frac{\partial}{\partial \mathbf{R}}\langle f \rangle + (q\tilde{/}m) E_\parallel \frac{\partial}{\partial v_\parallel}\langle f \rangle - \Omega \frac{\partial}{\partial \theta} f = 0$$

which is a solubility condition for $\langle f \rangle$.

• Gyro-phase averaging, one gets an electrostatic NL GK equation in a uniform B field:

$$\frac{\partial}{\partial t}\langle f \rangle + v_\parallel \mathbf{b} \cdot \frac{\partial}{\partial \mathbf{R}}\langle f \rangle + \frac{\langle \mathbf{E} \rangle\times\mathbf{b}}{\mathbf{B}} \cdot \frac{\partial}{\partial \mathbf{R}}\langle f \rangle + (q/m)\langle E_\parallel \rangle \frac{\partial}{\partial v_\parallel}\langle f \rangle = 0$$

• Frequency-wave number expansion and amplitude expansion,and geometric expansion (if it were included) are all lumped together in this procedure. If one modifies an ordering, needs to do the derivation all over again

Conventional (old-fashioned) Derivation of Non-linear Gyrokinetic Equation

- Closely follow Guiding Center transformation by
  P.J. Catto, Plasma Phys. **20**, 719 (1977)
- Resulting equation
  Frieman and Chen, PF **25**, 502 (1982)
  Lee, PF **26** 556 (1983)
- Purpose: illustrate basic physics and
  mathematical complexity
  involved in this conventional method.

Consider uniform $\mathbf{B} = B\hat{\mathbf{b}}$ to emphasize nonlinear effects

- Goal: from

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{q}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial \mathbf{v}}\right] f(\mathbf{x}, \mathbf{v}, t) = 0 \quad \text{6D Vlasov Eqn}$$

get

$$\left(\frac{\partial}{\partial t} + \frac{d\mathbf{R}}{dt}\cdot\frac{\partial}{\partial \mathbf{R}} + \frac{dv_\parallel}{dt}\frac{\partial}{\partial v_\parallel}\right)\langle f\rangle(\mathbf{R}, \mu, v_\parallel, t) = 0 \quad \text{5D GK Eqn}$$

with

$$\frac{d\mu}{dt} = 0 \text{ and } \frac{\partial}{\partial \theta}\langle f\rangle = 0$$

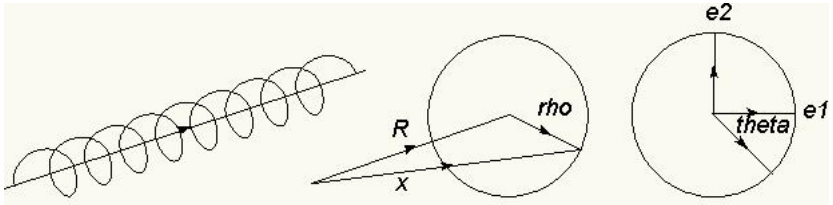
$\mu \simeq v_\perp^2/(2B)$: magnetic moment, an adiabatic invariant at lowest order

- Assumption:
  - $\omega \ll \Omega_{ci}$
  - $k_\parallel \ll k_\perp \sim \rho_i^{-1}$
  - $\delta f/f_0 \sim \delta n/n_0 \sim e\delta\phi/T_e \ll 1$

Guiding Center Transformation à la Catto

$(\mathbf{x}, \mathbf{v}) \to (\mathbf{R}, \mathbf{v}_\parallel, \mu, \theta)$, θ : gyrophase-angle
$\mathbf{R} = \mathbf{x} - \boldsymbol{\rho}$, $\boldsymbol{\rho} = \frac{\hat{\mathbf{b}} \times \mathbf{v}}{\Omega}$, $\Omega = \frac{eB}{mc}$
$v_\parallel = \hat{\mathbf{b}} \cdot \mathbf{v}$, $\mu = v_\perp^2/(2B)$
θ defined by

$$\begin{cases} \mathbf{v} = v_\parallel \hat{\mathbf{b}} + v_\perp \hat{\mathbf{e}}_\perp \\ \hat{e}_\perp = -\hat{\mathbf{e}}_2 \cos\theta - \hat{\mathbf{e}}_1 \sin\theta \\ \hat{\mathbf{e}}_\rho = \hat{\mathbf{e}}_1 \cos\theta - \hat{\mathbf{e}}_2 \sin\theta \end{cases}$$



Note that for uniform $\mathbf{B}$,

$$d^3\mathbf{x} d^3\mathbf{v} = \underbrace{B} d\mu d\theta dv_\parallel d^3\mathbf{R}$$

B : “phase-space volume”

Then, we would like to express $\dfrac{\partial}{\partial \mathbf{x}}$ and $\dfrac{\partial}{\partial \mathbf{v}}$ in G.C. space i.e., in terms of $\mu, v_{\|}, \mathbf{R},$ and θ;

$$\frac{\partial}{\partial \mathbf{x}} = \frac{\partial \mathbf{R}}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{R}} + \frac{\partial \mu}{\partial \mathbf{x}} \frac{\partial}{\partial \mu} + \frac{\partial v_{\|}}{\partial \mathbf{x}} \frac{\partial}{\partial v_{\|}} + \frac{\partial \theta}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial \mathbf{v}} = \frac{\partial \mathbf{R}}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{R}} + \frac{\partial \mu}{\partial \mathbf{v}} \frac{\partial}{\partial \mu} + \frac{\partial v_{\|}}{\partial \mathbf{v}} \frac{\partial}{\partial v_{\|}} + \frac{\partial \theta}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \theta}$$

$\rightarrow$ important to check what quantities are held constant when taking partial derivatives

Since

$$\left.\frac{\partial}{\partial \mathbf{x}} \mu\right|_{\mathrm{v=const}} = 0, \left.\frac{\partial}{\partial \mathbf{x}} v_{\|}\right|_{\mathrm{v=const}} = 0, \left.\frac{\partial}{\partial \mathbf{x}}\right|_{\mathrm{v=const}} \theta = 0, \text{ and } \mathbf{R} = \mathbf{x} - \frac{\hat{\mathbf{b}} \times \mathbf{v}}{\Omega}$$

$\frac{\partial}{\partial \mathbf{x}} \rightarrow$ only the 1st term on the R.H.S. survives $\Rightarrow$

$$\frac{\partial}{\partial \mathbf{x}} = \mathbf{I} \cdot \frac{\partial}{\partial \mathbf{R}} = \frac{\partial}{\partial \mathbf{R}}$$

Also, noting that

$$\left.\frac{\partial}{\partial \mathbf{v}}\right|_{\mathrm{x=const}} v_\parallel = \left.\frac{\partial}{\partial \mathbf{v}}\right|_{\mathrm{x=const}} \mathbf{v}\cdot\hat{\mathbf{b}} = \hat{\mathbf{b}}, \quad \frac{\partial}{\partial \mathbf{v}}\mu = \mathbf{v}_\perp / B$$

$$\frac{\partial}{\partial \mathbf{v}}\mathbf{R} = \frac{\partial}{\partial \mathbf{v}}(\mathbf{x} - \frac{\hat{\mathbf{b}}\times\mathbf{v}}{\Omega}) \to -\frac{\partial}{\partial \mathbf{v}}(\frac{\hat{\mathbf{b}}\times\mathbf{v}}{\Omega}) = \frac{\mathbf{I}\times\hat{\mathbf{b}}}{\Omega}$$

$$\frac{\partial}{\partial \mathbf{v}} = \hat{\mathbf{b}}\frac{\partial}{\partial v_\parallel} + \frac{\mathbf{v}_\perp}{B}\frac{\partial}{\partial \mu} - \frac{\hat{\mathbf{b}}\times\hat{\mathbf{e}}_\perp}{v_\perp}\frac{\partial}{\partial \theta} + \frac{\mathbf{I}\times\hat{\mathbf{b}}}{\Omega}\frac{\partial}{\partial \mathbf{R}}$$

$\Rightarrow$

$$\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} = v_\parallel \hat{b}\cdot\frac{\partial}{\partial \mathbf{R}} + \mathbf{v}_\perp\cdot\frac{\partial}{\partial \mathbf{R}} \quad (1)$$

$$\frac{q}{m}\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{v}} = \frac{q}{m}\left(E_\parallel\cdot\frac{\partial}{\partial v_\parallel} + \frac{\mathbf{E}\cdot\mathbf{v}_\perp}{B}\frac{\partial}{\partial \mu} - \frac{\mathbf{E}\cdot\hat{\mathbf{b}}\times\mathbf{v}_\perp}{v_\perp^2}\frac{\partial}{\partial \theta}\right) + \underline{\frac{c\mathbf{E}\times\mathbf{B}}{B^2}\cdot\frac{\partial}{\partial \mathbf{R}}} \quad (2)$$

$$\begin{aligned}\frac{q\mathbf{v}\times\mathbf{B}}{mc}\cdot\frac{\partial}{\partial \mathbf{v}} &= 0 + 0 - \Omega\frac{\mathbf{v}\times\mathbf{B}\cdot\mathbf{B}\times\mathbf{v}_\perp}{B^2 v_\perp^2}\frac{\partial}{\partial \theta} + \Omega\frac{(\mathbf{v}\times\hat{\mathbf{b}})\times\hat{\mathbf{b}}}{\Omega}\cdot\frac{\partial}{\partial \mathbf{R}} \\ &= \Omega\frac{\partial}{\partial \theta} - \underline{\mathbf{v}_\perp\cdot\frac{\partial}{\partial \mathbf{R}}} \quad (3)\end{aligned}$$

We also want to express $\phi(\mathbf{x})$ and $\mathbf{E}(\mathbf{x})$ in terms of $(\mathbf{R}, \mu, \mathbf{v}_{\parallel}, \theta)$

$$\phi(\mathbf{x}) = \phi(\mathbf{R} + \boldsymbol{\rho}(\theta)) \Rightarrow$$

$$\frac{\partial \phi}{\partial \theta} = \left.\frac{\partial \mathbf{x}}{\partial \theta}\right|_{\mathbf{R}} \cdot \frac{\partial \phi}{\partial \mathbf{x}} = \frac{\partial \boldsymbol{\rho}}{\partial \theta} \cdot \frac{\partial \phi}{\partial \mathbf{x}} = \frac{\mathbf{v}_{\perp}}{\Omega} \cdot \frac{\partial \phi}{\partial \mathbf{x}} = -\frac{\mathbf{E} \cdot \mathbf{v}_{\perp}}{\Omega}$$

$\therefore$ the 2nd term of RHS of Eq. (4.2)

$$\frac{q}{m} \frac{\mathbf{E} \cdot \mathbf{v}_{\perp}}{B} \frac{\partial}{\partial \mu} = -\frac{1}{c} (\frac{q}{m})^2 \frac{\partial \phi}{\partial \theta} \frac{\partial}{\partial \mu}$$

Collecting all terms in Eqs. (1)-(3),

$$\left[\frac{\partial}{\partial t} + v_{\parallel} \hat{\mathbf{b}} \cdot \frac{\partial}{\partial \mathbf{R}} + c \frac{\mathbf{E} \times \mathbf{B}}{B^2} \cdot \frac{\partial}{\partial \mathbf{R}} - \frac{q}{m} \nabla_{\parallel} \phi \frac{\partial}{\partial v_{\parallel}} + \Omega \frac{\partial}{\partial \theta} - \frac{q\Omega}{mB} \frac{\partial \phi}{\partial \theta} \frac{\partial}{\partial \mu} - \Omega \frac{\mathbf{v}_E \cdot \mathbf{v}_{\perp}}{v_{\perp}^2} \frac{\partial}{\partial \theta}\right] f = 0 \tag{4}$$

$-i\omega$ $\quad ik_{\parallel} v_{\parallel}$ $\quad \mathbf{k}_{\perp} \cdot \mathbf{v}_E$ $\quad k_{\parallel} v_{\parallel} \left(\frac{e\phi}{T_e}\right)$ $\quad \Omega$ $\quad \underbrace{(i) \qquad\qquad (ii)}_{\text{ugly!}}$

- Term (i) can be shown to be the 1st order correction to μ

i.e.,

$$\frac{d\mu}{dt} = \frac{d\mu^{(0)}}{dt} + \frac{d\mu^{(1)}}{dt} \Rightarrow \frac{d}{dt}(\frac{v_\perp^2}{2B})^{(1)} = \frac{\mathbf{v}_\perp^{(0)}}{B} \cdot \frac{d}{dt}\mathbf{v}_\perp^{(1)}(\theta)$$

where

$$\frac{d}{dt}\mathbf{v}_\perp^{(1)} = \frac{q}{m}(\mathbf{v}_\perp^{(1)} \times \mathbf{B} + \mathbf{E}^{(1)}) \Rightarrow \mathbf{v}_\perp^{(0)} \cdot \frac{d}{dt}\mathbf{v}_\perp^{(1)} = \frac{q}{m}\mathbf{E}_\perp^{(1)} \cdot \mathbf{v}_\perp^{(0)}$$

- Term (ii) similarly, 1st order correction to the gyrophase θ, i.e., gyration speed is slightly nonuniform due to $\mathbf{E}_\perp^{(1)}$, $\rightarrow$Not of primary physical interest
- Now, we perform perturbation theory:

with

$$\Omega \gg \omega \sim k_\| v_\|,\ \frac{\omega}{\Omega} \sim \frac{e\delta\phi}{T} \ll 1,\ k_\| \ll k_\perp \sim \rho_i^{-1}$$

- Eq. (4)$\Rightarrow$

$$\underbrace{\Omega\frac{\partial f}{\partial\theta}}_{\text{Largest term}} + \left(\frac{\partial}{\partial t} + v_{\parallel}\hat{\mathbf{b}}\cdot\frac{\partial}{\partial\mathbf{R}} + c\frac{\mathbf{E}\times\mathbf{B}}{B^2}\cdot\frac{\partial}{\partial\mathbf{R}} - \frac{q}{m}\nabla_{\parallel}\phi\frac{\partial}{\partial v_{\parallel}} - \frac{q\Omega}{mB}\frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial\mu}\right) f = 0 \tag{5}$$

Let $f = f^{(0)} + f^{(1)} + \cdots$, with expansion parameter $\delta \sim \frac{\omega}{\Omega} \sim \frac{k_{\parallel}v_{\parallel}}{\Omega} \sim \frac{|e|\phi}{T_e}$

- 0-th order $\Rightarrow \Omega\frac{\partial}{\partial\theta}f^{(0)} = 0 \Rightarrow f^{(0)}$ is independent ofθ,
  $\therefore f = \langle f\rangle + f_{AC}, \langle\cdots\rangle = \frac{1}{2\pi}\oint d\theta\{\cdots\}$gyrophase average
  with $f^{(0)} = \langle f\rangle, f^{(1)} = f_{AC} \ll f^{(0)} = \langle f\rangle$
- 1-st order $\Rightarrow$

$$\underbrace{\Omega\frac{\partial}{\partial\theta}f^{(1)}}_{\text{(a)}} + \left(\frac{\partial}{\partial t} + v_{\parallel}\hat{\mathbf{b}}\cdot\frac{\partial}{\partial\mathbf{R}} + c\frac{\mathbf{E}\times\mathbf{B}}{B^2}\cdot\frac{\partial}{\partial\mathbf{R}} - \frac{q}{m}\nabla_{\parallel}\phi\frac{\partial}{\partial v_{\parallel}} - \underbrace{\frac{q\Omega}{mB}\frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial\mu}}_{\text{(b)}}\right) f^{(0)} = 0 \tag{6}$$

(a) and (b) can be combined into

$$\Omega\frac{\partial}{\partial\theta}\left[f_{AC} - \frac{q\phi}{mB}\frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial\mu}\langle f\rangle\right]$$

- Taking gyro-phase average of Eq. (6): $\langle \cdots \rangle = \frac{1}{2\pi} \oint d\theta \cdots$

$$\langle \Omega \frac{\partial}{\partial \theta} \{\cdots\} \rangle = 0 \Rightarrow$$

$$\left[\frac{\partial}{\partial t} + v_\parallel \hat{\mathbf{b}} \cdot \frac{\partial}{\partial \mathbf{R}} + \frac{c}{B} \hat{\mathbf{b}} \times \nabla \langle \phi \rangle - \frac{q}{m} \hat{\mathbf{b}} \cdot \frac{\partial}{\partial \mathbf{R}} \langle \phi \rangle \frac{\partial}{\partial v_\parallel} \right] \langle f \rangle = 0 \tag{7}$$

Finally, the electrostatic NLGK vlasov equation in uniform $\mathbf{B}$

- $\langle \phi \rangle$ contains the Finite Larmor Radius (FLR) effect!

although it's gyrophase-averaged

$$\phi(\mathbf{x}) = \phi(\mathbf{R} + \boldsymbol{\rho}) = \sum_{\mathbf{k}} \phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\mathbf{k}} \phi_{\mathbf{k}} e^{i\mathbf{k}_\perp \cdot \mathbf{R}} e^{ik_\perp \rho \sin\theta}$$

Fourier-Bessel Expansion:

$$e^{ik_\perp \rho \sin\theta} = \sum_n J_n(k_\perp \rho) e^{in\theta}$$

$$\langle e^{ik_\perp \rho \sin\theta} \rangle = \frac{1}{2\pi} \oint d\theta \sum_n J_n(k_\perp \rho) e^{in\theta} = J_0(k_\perp \rho)$$

$\therefore$

$$\langle \phi \rangle = \sum_{\mathbf{k}} J_0(k_\perp \rho) \phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}}$$

- Widespread Misconception: "Gyrokinetic Theory throws away the gyrophase-dependent information"

- Part of Reasons: Conventional (old-fashioned) derivation is rather opaque (much more complex in general geometry in nonuniform $\mathbf{B}$)

  Illustration in this note is a bit "modernized" version than the original papers up to mid 80's.

  - Hard to identify the role or necessity of $\theta-$dependent information
  - Also, most attention was paid to the nonlinear GK-"Vlasov" Equations.

Gyrokinetic Poisson Equation

- Maxwell's Eqns are still fine!
  but was NOT written in g.c. coordinates $(\mathbf{R})$
- So we need to express $n_i(\mathbf{x})$ in terms of $\langle f\rangle(\mathbf{R}, \mathbf{v}_\|, \mu)$

$$(\mathbf{R},\ \mathbf{v}_\|,\ \mu,\ \theta) \Rightarrow (\mathbf{x},\ \mathbf{v})$$

"Pull-Back" Transformation for GK Maxwell's Eqn
(ES $\Rightarrow$ Poisson)

$$(\mathbf{x},\ \mathbf{v}) \Rightarrow (\mathbf{R},\ \mathbf{v}_\|,\ \mu,\ \theta)$$

"Push-Forward" Transformation for GK-Vlasov

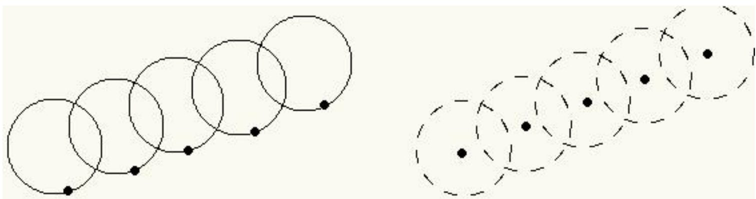
$$\nabla^2\phi = -4\pi e[n_i(\mathbf{x}) - n_e(\mathbf{x})]$$

- $n_i(\mathbf{x})$: typically obtained from GK Eqn
- $n_e(\mathbf{x})$: from adiabatic response for pure - ITG
  or from drift-kinetic or bounce-kinetic
  or from some other fluid eqns for more realistic case
  "GK" required for ETG

$$\begin{aligned} n_i(\mathbf{x}) &= \int d^3\mathbf{v} f_i(\mathbf{x},\ \mathbf{v}, t) \\ &= \int d^3\mathbf{x}' d^3\mathbf{v} f_i(\mathbf{x}',\ \mathbf{v})\delta(\mathbf{x}' - \mathbf{x}) \\ &= \int d^3\mathbf{R} d\mu dv_\parallel d\theta B f_i(\mathbf{R},\ \mu,\ v_\parallel, \theta)\delta(\mathbf{R} + \boldsymbol{\rho} - \mathbf{x}) \end{aligned} \tag{8}$$

not quite the same

$$\int d^3\mathbf{R} d\mu dv_\parallel B\langle f\rangle(\mathbf{R},\ \mu,\ v_\parallel)$$



Since

$$f_i(\mathbf{R},\ \mu,\ v_\parallel,\theta) = \langle f\rangle + f_{AC}(\mathbf{R},\ \mu,\ v_\parallel,\theta),$$

we need to know "f_{AC}" as well.
Back to Eq. (6):

$$\Omega\frac{\partial}{\partial\theta}\left[f_{AC} - \frac{q\phi}{mB}\frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial\mu}\langle f\rangle\right] + \text{``}\ \frac{d}{dt}\langle f\rangle\ \text{''} = 0$$

and Eq. (7)

$$\left.\frac{d}{dt}\right|^{(0)}\langle f\rangle = 0$$

$\Rightarrow$

$$\Omega\frac{\partial}{\partial\theta}\left[f_{AC} - \frac{q\phi}{mB}\frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial\mu}\langle f\rangle\right] + \left(\frac{d}{dt} - \left.\frac{d}{dt}\right|^{(0)}\right)\langle f\rangle = 0 \tag{9}$$

$$\frac{d}{dt} - \frac{d}{dt}^{(0)} \propto \text{``}\phi - \langle\phi\rangle\text{''}$$

integrating Eq. (9)

$$f_{AC}(\theta) \simeq \frac{q}{mB}(\phi - \langle\phi\rangle)\frac{\partial}{\partial\mu}\langle f\rangle \tag{10}$$

Polarization Density

Eq. (8) $\Rightarrow$

$$n_i(\mathbf{x}) = \underbrace{\int d^3\mathbf{R} d\mu dv_\parallel d\theta B \langle f \rangle \delta(\mathbf{R} + \boldsymbol{\rho} - \mathbf{x})}_{n_{i,gc}(\mathbf{x})}$$
$$+ \underbrace{\int d^3\mathbf{R} d\mu dv_\parallel d\theta B \frac{q}{mB} \left(\phi - \langle \phi \rangle\right) \frac{\partial \langle f \rangle}{\partial \mu} \delta(\mathbf{R} + \boldsymbol{\rho} - \mathbf{x})}_{n_{pol}(\mathbf{x})}$$

- $n_{i,gc}(\mathbf{x})$: G.C. density at particle position
- $n_{pol}(\mathbf{x})$: Polarization Density, one can dvaluate exactly for $\langle f \rangle \propto e^{-\mu B/T}$, i.e.,"Maxwellian in $\mu \propto v_\perp^2$"

Nonlinear Gyrokinetics for Large Scale Computation

• Direct simulation of actual size fusion plasmas in realistic geometry using the primitive nonlinear plasma equations (Vlasov-Maxwell), is far beyond the computational capability of foreseeable future.

• For turbulence problems in fusion plasmas, the temporal scales fluctuations much longer than the period of a charged particle's cyclotron motion, while the spatial scales and gyro-orbits are much smaller than the macroscopic length scales: → details of the charged particle's gyration motion are not of physical interest → Develop reduced dynamical equations which capture the essential features

• After decoupling of gyro-motion, gyrokinetic equation describes evolution of gyro-center distribution function, independent of the gyro-phase, θ, defined over a five-dimensional phase space ($\mathbf{R}$, $v_{||}$, μ). →
save enormous amounts of computing time by having a time step greater than the gyro-period, and by reducing the number of dynamical variables.

• In gyrokinetic approach, gyro-phase is an ignorable coordinate, magnitude of the perpendicular velocity enters as a parameter in terms of an adiabatic invariant μ

• Nonlinear gyrokinetic equations are now widely used in turbulence simulations.

Modern Nonlinear Gyrokinetics

• Starting from the original Vlasov-Maxwell system (6D), pursue **"Reduction of dimensionality"** for both computational and analytic feasibility.

• Keep intact the underlying symmetry/conservation of the original system.

• Perturbation analysis consists of near-identity coordinate transformation which **"decouples"** the gyration from the slower dynamics of interest in the single particle Lagrangian, rather than a direct "gyro-phase average" of Vlasov equation.

• This procedure is reversible:
The gyro-phase dependent information can be recovered when it is needed.

Phase Space Lagrangian Derivation of Nonlinear Gyrokinetics

[since Hahm, PF **31**, 2670 '88, followed by Brizard, Sugama,…]

- **Conservations Laws are Satisfied.**
- Various expansion parameters appear at different stages

$\rightarrow$Flexibility in variations of ordering for specific application

- Guiding center drift calculations in equilibrium field **B**:

Expansion in $\delta_B = \rho_i / L_B \sim \rho_i / R$.

- Perturbative analysis consists of near-identity transformations to new variables which remove the gyro-phase dependence in perturbed fields $\delta\mathbf{A}(\mathbf{x})$, $\delta\phi(\mathbf{x})$ where $\mathbf{x} = \mathbf{R} + \rho$:

Expansion in $\varepsilon_\phi = e[\delta\phi - (v_{||}/c)\delta A_{||}]/T_e \sim \delta B_{||}/B_0$.

- Derivation more transparent, less amount of algebra

Single Particle Phase Space Lagrangian

[Littlejohn, Cary '83,...]

- Fundamental 1-form (phase space Lagrangian in non-canonical variables)

$$\gamma \equiv (e\mathbf{A}(\mathbf{x}) + m\mathbf{v}) \cdot d\mathbf{x} - (m/2)v^2 dt$$

- Transformation to guiding center variables: $\mathbf{x} \equiv \mathbf{R} + \rho,\ \mu \equiv v_\perp^2/2\Omega,\ \theta \equiv tan^{-1}(\frac{\mathbf{v}\cdot\mathbf{e_1}}{\mathbf{v}\cdot\mathbf{e_2}}), \ldots$
- The zero-th order phase space Lagrangian for guiding center:

$$\gamma_0 = (e\mathbf{A}(\mathbf{R}) + mv_{||}\mathbf{b}(\mathbf{R})) \cdot d\mathbf{R} + \frac{\mu B}{\Omega} d\theta - H_0 dt$$

angle variable θ is ignorable
action is an adiabatic invariant μ

$$H_0 = \mu B + (m/2)v_{||}^2$$

Euler-Lagrange Equation

- From variation of phase space Lagrangian:

$$\frac{d\theta}{dt} = \Omega, \quad \frac{d\mu}{dt} = 0$$

  Decoupling of gyromotion, adiabatic invariant

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$$-e\mathbf{B}^* \times \frac{d\mathbf{R}}{dt} - m\mathbf{b}\frac{dv_\parallel}{dt} = \mu \nabla B$$

  where $\mathbf{B}^* \equiv \nabla \times (\mathbf{A} + \frac{m}{e} v_\parallel \mathbf{b}) = \mathbf{B} + \frac{m}{e} v_\parallel \nabla \times \mathbf{b}$

- Decompose via $\mathbf{b}\times$ and $\mathbf{B}^*$, to get

$$\frac{d\mathbf{R}}{dt} = v_\parallel \frac{\mathbf{B}^*}{B^*} + \frac{\mu}{e} \frac{\mathbf{b}}{B^*} \times \nabla B,$$

  and

$$\frac{dv_\parallel}{dt} = -\frac{\mu}{m} \frac{\mathbf{B}^*}{B^*} \cdot \nabla B$$

More on Guiding Center Drift

- Frequently asked question:
  "**Where is the curvature drift?**"
  Using an identity $\mathbf{B}^* = B^*\mathbf{b} + \frac{m}{e}v_\| \mathbf{b} \times (\mathbf{b}\cdot\nabla)\mathbf{b}$:

$$\frac{d\mathbf{R}}{dt} = v_\| \frac{B^*\mathbf{b} + \frac{m}{e}v_\| \mathbf{b} \times (\mathbf{b}\cdot\nabla)\mathbf{b}}{B^*} + \frac{\mu}{e}\frac{\mathbf{b}}{B^*}\times\nabla B$$

- Infrequently asked question: "**Do conventional guiding center drifts conserve energy?**"

$$\frac{d\mathbf{R}}{dt} = v_\|\mathbf{b} + \mathbf{v}_{curv} + \mathbf{v}_{gradB}, \quad \frac{dv_\|}{dt} = -\frac{\mu}{m}\mathbf{b}\cdot\nabla B$$

  do *not* conserve energy exactly, while our E-L eqns do.
- $\mathbf{B}^*$ is a manifestation of Hamiltonian structure
- B^* is the density of phase-volume, $d^6\mathbf{Z} = B^* d\mu d\theta dv_\| d^3\mathbf{R}$

Lie Perturbative Analysis I.

[from Hahm, PF **31**, 2670 '88]

- Consider electrostatic fluctuation only (for illustration): $\delta\phi(\mathbf{x}) = \delta\phi(\mathbf{R} + \rho)$
- While gyromotion has been decouple in the zero-th order phase space Lagrangian, it appears again in the perturbation. Since it is $O(\epsilon_\phi)$, we can remove it via *near-identity, phase-space preserving* Lie transform.
- In addition to zero-th order γ_0, $\gamma_1 = -e\delta\phi(\mathbf{R} + \rho)dt$
- Perform Lie-perturbation:

$$\Gamma_1 = \gamma_1 - L_1\gamma_0 + dS_1$$

where $(L_1\gamma)_\mu = g_1^\nu(\frac{\partial\gamma_\mu}{\partial z^\nu} - \frac{\partial\gamma_\nu}{\partial z^\mu})$, transformation of 1 form

Lie Perturbative Analysis II.

- One can choose the gauge function S_1 and the *generator* $\mathbf{g}_1$ such that the gyrophase is removed from Γ_1
- $\Omega\frac{d}{d\theta}S_1 - \frac{\partial S_1}{\partial t} - \frac{d\mathbf{R}}{dt}\cdot\nabla S_1 - \frac{dv_{||}}{dt}\frac{\partial}{\partial v_{||}}S_1 = \frac{e}{\Omega}(\delta\phi - <\delta\phi>)$
- Using $\epsilon_{k,\omega} << 1$, we obtain $dS_1 = \frac{e}{\Omega}(\delta\phi - <\delta\phi>)d\theta$
- $$\Gamma_1 = -e < \delta\phi > dt$$

  where $< ... >$ is the gyrophase average $\frac{1}{2\pi}\int(...)$
- Note that *decoupled gyrophase information* is kept in S_1 and $\mathbf{g}_1$ to be used later when necessary.

Lie Perturbative Analysis III.

- Now, $\Gamma = \Gamma_0 - e < \delta\phi > dt$,
  $H = H_0 + H_1 = \mu B + (m/2)v_\parallel^2 + e < \delta\phi >$
- Euler-Lagrange Equation

$$\frac{d\mathbf{R}}{dt} = v_\parallel \frac{\mathbf{B}^*}{B^*} + \frac{\mathbf{b}}{B^*} \times (\frac{\mu}{e}\nabla B + \nabla < \delta\phi >),$$

  and

$$\frac{dv_\parallel}{dt} = -\frac{1}{m}\frac{\mathbf{B}^*}{B^*} \cdot (\mu\nabla B + e\nabla < \delta\phi >)$$

- $\mathbf{B}^*$ correction in the last term crucial for momentum pinch
- The second order perturbation in $\epsilon_\phi \sim \rho/L_p$ is necessary for energy conservation.

Gyrokinetic Vlasov-Poisson System

• With Euler-Lagrange Eqns, Gyrokinetic Vlasov equation for gyrocenter distribution function $F(\overline{R},\overline{\mu},\overline{\mathrm{v}}_{\|})$ is:

$$\frac{\partial F}{\partial t}+\frac{d\overline{R}}{dt}\cdot\overline{\nabla}F+\frac{d\overline{\mathrm{v}}_{\|}}{\mathrm{dt}}\frac{\partial F}{\partial\overline{\mathrm{v}}_{\|}}=0$$

Note reduction of dimensionality achieved by $(\partial F/\partial\theta)=0$, $d\overline{\mu}/dt=0$

• Self-consistency is enforced by the Poisson's equation. Debye shielding is typically irrelevant, one must express the ion particle density $n_i(\mathbf{x})$ in terms of the gyrocenter distribution function $F(\overline{R},\overline{\mu},\overline{\mathrm{v}}_{\|})$

• Lee [PF **26**, 556 '83] has identified the *polarization density* (in addition to the guiding center density). It was a key breakthrough in advances in GK particle simulations.

$$\delta n_i(\mathbf{x}) = \delta n_{gc} + \rho_i^2\, \nabla_\perp\cdot N_0\nabla_\perp(e\delta\phi/T_i)$$

Pullback Transformation

• Widespread Misconception: *"Gyrokinetic theory throws away the gyrophase dependent part of F."*

• The gyrophase dependent information is kept in the gauge function S_1 or a generator g_1.

• This can be used reversibly whenever one wants to calculate a quantity in the particle frame from the gyrocenter distribution function.

$$\int d^6\overline{Z}(T_G^* F(Z))K(\overline{R})\delta^3(\overline{R}-\mathbf{x}+\overline{\rho}) \rightarrow K(x)$$

• Examples include the polarization density, diamagnetic current, and other quantities related to finite Larmor radius effects.

Polarization Density

- More systematic derivation of GK Poisson's eqn started since Dubin *et al.*,[PF **26**, 3524 '83] via *pullback* transformation:

$$\nabla^2 \delta\phi = -4\pi e [\int d^6\overline{Z}\ (T_G^* \delta f)\, \delta^3(\overline{R} - \mathbf{x} + \overline{\rho}) - \delta n_e(\mathbf{x}, t)],$$

where

$$T_G^* \delta f \equiv \delta f + \left(\frac{\partial S_1}{\partial \overline{\theta}}\right) \frac{\partial F_0}{\partial \overline{\mu}} + [\frac{1}{\Omega}\left(\overline{\nabla} S_1\right) \times \mathbf{b}] \cdot \overline{\nabla} F_0$$

- Contribution to the ion particle density which involves S_1 is the general form of polarization density. After linearization,

$$\{k^2 \lambda_{Di}^2\} \frac{e\delta\phi_{\mathbf{k}}}{T_{i\perp}} n_0 + \{1 - \Gamma_0(b)\} \frac{e\delta\phi_{\mathbf{k}}}{T_{i\perp}} n_0 = \delta \overline{N}_{i\mathbf{k}} - \delta n_{e\mathbf{k}}$$

- It is obvious that the *polarization density* statisfies

$$\frac{\partial}{\partial t} \delta n^{pol} + \frac{\partial}{\partial \mathbf{x}} \cdot n_0 \mathbf{v}^{pol} = 0$$

Conservation of Energy and Phase-Space Volume

- It is straight-forward to show the Liouville's theorem:

$$\overline{\nabla}\cdot\left(B_\parallel^*\frac{d\overline{R}}{dt}\right)+\frac{\partial}{\partial\overline{v}_\parallel}\left(B_\parallel^*\frac{d\overline{v}_\parallel}{dt}\right)=0$$

- The invariant energy for GK Vlasov-Poisson system is obtained by transforming the energy constant of the original Vlasov-Poisson system [Dubin *et al.*,'83]

$$E=\int d^6\mathbf{Z}\,F_i(\mu B+\frac{M}{2}v_\parallel^2)+\int d^6\mathbf{z}\,f_e(\mathbf{z})\frac{1}{2}m_e v^2$$

$$+\frac{1}{8\pi}\int d^3\mathbf{x}\,|\mathbf{E}|^2+\frac{e^2}{2\Omega}\int d^6\mathbf{Z}\,F_i\left(\frac{\partial}{\partial\mu}\langle\delta\tilde{\phi}^2\rangle+\frac{1}{\Omega}\langle\nabla\delta\tilde{\mathbf{\Phi}}\times\mathbf{b}\cdot\nabla\delta\tilde{\phi}\rangle\right)$$

Note that the sloshing energy (last term) can be obtained from perturbation up to $O(\epsilon_\phi^2)$.

Extensions to Edge

[for core transport barriers $\rightarrow$ Hahm, Phys. Plasmas **3**, 4658, '96]

Expansion in $\epsilon_B \sim \rho_i/L_E \sim \frac{B_\theta}{B}$:

- From $\rho_{ip} \sim L_P \sim L_E$,

  $u_E \sim u_{*i} \sim \frac{\rho_i}{L_p} v_{Ti}$, $\frac{e\Phi^{(0)}}{T_e} \sim 1$.
- $|S-1| \sim 1$ (banana orbit distortion), $\frac{\omega_E}{\Omega_i} \sim \epsilon_B^2$ (circular gyro-orbit)

  where $\omega_E \equiv \frac{(RB_\theta)^2}{B}\frac{\partial}{\partial\psi}(\frac{E_r}{RB_\theta})$ [Hahm-Burrell, PoP '95]

  $S \simeq 1 + (\frac{B}{B_\theta})^2 \frac{\omega_E}{\Omega_i}$ [Hinton-Kim, Furth-Rosenbluth, Shaing,...]

- The zero-th order phase space Lagrangian

$$\gamma_0 \equiv (e\mathbf{A} + m\mathbf{u}_E + mv_{\|}\mathbf{b}) \cdot d\mathbf{R} + \frac{\mu B}{\Omega} d\theta - H_0 dt$$

  with a guiding-center Hamiltonian

$$H_0 = e\Phi + \mu B + (m/2)(v_{\|}^2 + u_E^2) + \frac{\mu B}{2\Omega}\mathbf{b} \cdot \nabla \times \mathbf{u}_E.$$

Summary

- Modern Nonlinear Gyrokinetic Theory has provided a firm theoretical foundation for recent remarkable advances in gyrokinetic simulations and associated theories.
- Its elegance and relative simplicity have contributed to deeper understanding of the gyrokinetic system, not only improving treatment of familiar ones, but also indentification of novel physics effect.
- Significant example: Turbulent Convective Pinch of Toroidal Momentum
- It should be useful for even more complicated systems where several expansion parameters exist.

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References on Nonlinear Gyrokinetic Theory I.

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  Brizard and Hahm, Rev. Mod. Phys. **79**, 421 '07
- Pioneering paper on conventional NL GK

  Frieman and Chen, PF **25**, 502 '82
- NL GK for particle simulation:

  Lee, PF **26**, 556 '83
- Proto-type Modern NL GK using Hamiltonian method:

  Dubin, Krommes, Oberman, and Lee, PF **26**, 3524 '83

  Hagan and Frieman, PF **28**, 2641 '85

  Yang and Choi, Phys. Lett. A **108**, 25 '85 (Electrostatic)

  Hahm, Lee, and Brizard, PF **31**, 1940 '88

References on Nonlinear Gyrokinetic Theory II.

- Modern NL GK using phase-space Lagrangian Lie perturbation method:

  Hahm, PF **31**, 2670 '88 (General geometry, electrostatic)

  Brizard, J. Plasma Phys. **41**, 541 '89

  (General geometry, electromagnetic)
- NL GK for strongly rotating plasmas:

  Hahm, PF-B **4**, 2801 '92 (in slab)

  Brizard, PoP **2**, 459 '95 (in terms of toroidal rotation)

  Hahm, PoP **3**, 4658 '96 (in terms of E_r)
- Energy conservation theorem:

  Brizard, PoP **7**, 4816 '00

  Sugama, PoP **7**, 466 '00 (introduction of field theory)

References on Topics related to Modern NL GK using phase-space Lagrangian Method

- Bounce-averaged Nonlinear Kinetic equation

  Fong and Hahm, PoP **6**, 188 ’99 (electrostatic)

  Brizard, PoP **7**, 3238 ’00 (electromagnetic)

- High frequency linear gyrokinetic theory:

  Qin and Tang, PoP **11**, 1052 ’04 (recovery of compressional Alfven wave, elucidation of differential geometrical meaning of pullback transformation)