The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

2052-8

Summer College on Plasma Physics

10 - 28 August 2009

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(Talk at Trieste ICTP Summer College of Plasma Physics)

July 21, 2009

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1.1 Introduction

- Historically, the plasma equations tended to be referred to an **inertial frame of reference,** for sound reasons: **Maxwell's equations** take their simplest, classical forms in such frames, and there was little motivation to consider electrodynamics in non-inertial systems.
- However, there are a number of problems in astrophysics (pulsar/Kerr black-hole magnetospheres, for example) and in tokamak physics where rapidly rotating plasmas are encountered. The physics of these systems can perhaps be better understood in suitable **non-inertial** frames.
- In this talk, a **first-principles approach** to the formulation and application of the complete, un-averaged (non-relativistic) plasma equations - **Newton-Lorentz; Vlasov; Langevin; Landau-Fokker-Planck** -in such frames will be considered, with particular emphasis on rapidly rotating systems. It will be shown how **reduced** equations like, **gyrokinetic, drift kinetic and two-fluid/moment** equations are readily obtained from the exact equations.
- The applications of the theory to strongly rotating plasmas in tokamaks will be explored.

1.2 The problem and its solution

- The **Newton-Lorentz** equations of motion of a non-radiating charged particle of mass m and charge Ze subject only to electromagnetic forces take the following well-known, simple forms in **inertial frames of reference:**

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$
$$m\frac{d\mathbf{v}}{dt} = Ze[\mathbf{E} + \mathbf{v} \times \mathbf{B}]$$

where $\mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t)$ are **Maxwell fields**, satisfying Maxwell's equations in such frames.

- What forms do these equations (in particular, the momentum balance equation) take in an **arbitrarily accelerating, co-moving frame**? Remarkably simple, **Solution:** if $\mathbf{K}_f$ is an **arbitrary non-inertial frame** co- moving non-relativistically so that, $\mathbf{v} = \mathbf{u}_f + \mathbf{V}$, where $\mathbf{u}_f$ is the "frame velocity" and $\mathbf{V}(t)$ is the particle's velocity relative to the frame, then the **Newton-Lorentz** equation of motion in this frame is:

$$m\frac{d\mathbf{V}}{dt} = Ze[\mathbf{E}_* + \mathbf{V} \times \mathbf{B}_*] \quad \textbf{where,} \tag{1}$$
$$\mathbf{E}_* = [\mathbf{E} + \mathbf{u}_f \times \mathbf{B}] - \frac{m}{Ze}\frac{\partial \mathbf{u}_f}{\partial t} + \frac{m}{Ze}\nabla(\frac{\mathbf{u}_f . \mathbf{u}_f}{2}), \tag{2}$$
$$\mathbf{B}_* = \mathbf{B} + (\frac{m}{Ze})\nabla \times \mathbf{u}_f. \tag{3}$$

2.1 Steady, uniform frame rotation

- We start with a **simple, but important special case.**
- We will study in the first instance, **uniformly rotating** frames of reference with a fixed **axis of rotation** and constant angular velocity, $\mathbf{\Omega} = \Omega \mathbf{e}_z$, where, $\mathbf{K}_{\rm lab} : \mathbf{r} = (x, y, z); \mathbf{r}_\perp^2 = x^2 + y^2$ represents a standard, **Cartesian, inertial frame of reference** with a fixed origin.
- The **rotating frame** is denoted by $\mathbf{K}_{\rm rot} : \mathbf{R} = (X, Y, Z = z)$. Thus, the rotation axis is taken along the z-axis.
- We consider **non-relativistic** mechanics and therefore assume that all relevant lengths in the problem are $\ll \frac{c}{\Omega}$, (= the so-called "light cylinder" radius). If, at $t = 0$, the two frames were coincident, at any time t, a point P at $\mathbf{K}_{\rm lab}(x, y, z)$ has "rotating frame coordinates" $\mathbf{K}_{\rm rot}(X, Y, Z)$ according to the transformation law:

$$x = X \cos \Omega t - Y \sin \Omega t \tag{4}$$

$$y = X \sin \Omega t + Y \cos \Omega t \tag{5}$$

$$z = Z \tag{6}$$

2.2 Uniform rotation kinematics: contd.

- Differentiating Eqs.(1-3) with respect to t, we obtain the transformation rule for velocities in the two frames ($\mathbf{K}_{\rm lab}$ is assumed to be an inertial frame!):

$$[\frac{d\mathbf{r}}{dt}]_{\rm lab} = [\frac{d\mathbf{R}}{dt}]_{\rm rot} + \mathbf{\Omega} \times \mathbf{R} \tag{7}$$

Key point: The time derivatives are taken in each frame holding its **basis vectors** constant. To avoid cumbersome suffix notation we use $\frac{d}{dt}$ for the "lab" time derivative and $\frac{\partial}{\partial t}$ for the "rotational frame" time derivative. The arguments apply to time derivatives **of any vector!** Hence for the accelerations we have:

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= [\frac{\partial}{\partial t} + \mathbf{\Omega}\times][\frac{\partial \mathbf{R}}{\partial t} + \mathbf{\Omega} \times \mathbf{R}] \\ &= (\frac{\partial^2 \mathbf{R}}{\partial t^2}) + 2\mathbf{\Omega}\times(\frac{\partial \mathbf{R}}{\partial t}) + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R}) \\ &= (\frac{\partial^2 \mathbf{R}}{\partial t^2}) + 2\mathbf{\Omega}\times(\frac{\partial \mathbf{R}}{\partial t}) - \frac{1}{2}\nabla \Omega^2 R_\perp^2 \end{aligned} \tag{8}$$

where, $\mathbf{R}_\perp = \mathbf{R} - (\frac{\mathbf{R}.\mathbf{\Omega}}{\Omega^2})\mathbf{\Omega}$. We see that **Newton's second law** in the rotating frame must take the form:

$$m(\frac{\partial^2 \mathbf{R}}{\partial t^2}) = 2m(\frac{\partial \mathbf{R}}{\partial t})\times\mathbf{\Omega} + \frac{m}{2}\nabla \Omega^2 R_\perp^2 + \mathbf{F} \tag{9}$$

2.3 Uniform, steady rotation dynamics

- The **force** on the particle is frame-independent, but its **components** may depend on Ω if they are written in terms of rotating frame coordinates. We see that in the rotating frame, as is well-known, we obtain two **"pseudo-forces".**
- **The Coriolis force (velocity dependent but does no work!):** $2m(\frac{\partial \mathbf{R}}{\partial t}) \times \mathbf{\Omega}$, and the
- **The Centrifugal force (derivable from a potential, and thus a "conservative field"):** $\frac{m}{2} \nabla \Omega^2 R_\perp^2$.
- These forces are "inertial" in character. They depend only upon the **mass** of the particle (and the frame velocity) but not on its **charge.** In **Einstein's view**, they cannot be distinguished in principle from gravity in their **local** action. They clearly result from frame **accelerations.**
- Note carefully that the **Coriolis force** resembles **Lorentz'** $\mathbf{v} \times \mathbf{B}$ force, whilst the centrifugal force looks like an **electrostatic/gravitational** force-field. These formal resemblances will recur throughout the talk.
- Note also that the equivalent fields are functions **only of position** and not of the particle's velocity in the frame. This is a very fundamental property of electromagnetic fields in inertial frames: it remains invariant when we transform to noninertial frames, as we shall discover.

3.1 Electrodynamics in a rotating frame

- The general theory of relativity [cf. **Landau and Lifschitz** *Classical theory of fields*] tells us how the fields $\mathbf{E}, \mathbf{B}$ (aka $F_{\mu\nu}$) must transform for **completely general motions**. Fortunately, if the motion is **non-relativistic** (as in the present case), we have simpler rules:

$$\mathbf{B}_{\rm rot} = \mathbf{B}_{\rm lab} \quad (10)$$

$$\mathbf{E}_{\rm rot} = \mathbf{E}_{\rm lab} + (\mathbf{\Omega} \times \mathbf{R}) \times \mathbf{B}_{\rm lab} \quad (11)$$

  This is exactly what we would guess from special relativity for a "co-moving observer" with non-relativistic $\mathbf{v}_{\rm frame} = (\mathbf{\Omega} \times \mathbf{R})$ relative to the lab. frame.

- We can now write down the **Lorentz-Newton** equations in both frames:

$$m\frac{d\mathbf{v}}{dt} = Ze[\mathbf{E}_{\rm lab} + \mathbf{v} \times \mathbf{B}_{\rm lab}] \qquad [\mathbf{K}_{\rm lab}] \quad (12)$$

$$m(\frac{\partial \mathbf{V}}{\partial t}) = Ze[\mathbf{E}_{\rm rot} + \mathbf{V} \times \mathbf{B}_{\rm rot}] + 2m\mathbf{V} \times \mathbf{\Omega} + \frac{m}{2}\nabla \Omega^2 R_\perp^2 \qquad [\mathbf{K}_{\rm rot}] \quad (13)$$

  where $\mathbf{V} = (\frac{\partial \mathbf{R}}{\partial t}) = \mathbf{v} - \mathbf{\Omega} \times \mathbf{R}$ and $\mathbf{B}_{\rm rot} = \mathbf{B}_{\rm lab} = \mathbf{B}; \mathbf{E}_{\rm rot} = \mathbf{E}_{\rm lab} + (\mathbf{\Omega} \times \mathbf{R}) \times \mathbf{B}$.
  The magnetic field is the same (to $O(\frac{v^2}{c^2})$ accuracy) in both frames at the same point.

3.2 Charged particles in rotating frames

- Let us consider a situation in which $\mathbf{E}, \mathbf{B}, \mathbf{\Omega}$ are all independent of time in the lab. frame (also in the rotating frame!). A charged particle of mass m and charge Ze then satisfies the equations:

$$\frac{\partial \mathbf{R}}{\partial t} = \mathbf{V} \tag{14}$$

$$m\frac{\partial \mathbf{V}}{\partial t} = Ze[\mathbf{E}_{\mathrm{rot}} + \mathbf{V} \times \mathbf{B}] + 2m\mathbf{V} \times \mathbf{\Omega} + \frac{m}{2}\nabla \Omega^2 R_\perp^2$$

$$= Ze[\mathbf{E}_{\mathrm{rot}} + \frac{m}{Ze}\nabla(\frac{\Omega^2 R_\perp^2}{2}) + \mathbf{V} \times (\mathbf{B} + (\frac{2m}{Ze})\mathbf{\Omega})] \tag{15}$$

From the fact that the fields are time-independent in the rotating frame, we may write, $\mathbf{E}_{\mathrm{rot}} = -\nabla \Phi_{\mathrm{rot}} = \mathbf{E}_{\mathrm{lab}} + (\mathbf{\Omega} \times \mathbf{R}) \times \mathbf{B}_{\mathrm{lab}}$ and introduce the **equivalent fields and potential**:

$$\mathbf{E}_* = -\nabla \Phi_*$$

$$\Phi_* = \Phi_{\mathrm{rot}} - \frac{m}{Ze}\frac{\Omega^2 R_\perp^2}{2} \tag{16}$$

$$\mathbf{B}_* = \mathbf{B} + (\frac{2m}{Ze})\mathbf{\Omega}, \quad \textbf{consequently}, \tag{17}$$

$$m\frac{\partial \mathbf{V}}{\partial t} = Ze[\mathbf{E}_* + \mathbf{V} \times \mathbf{B}_*] \tag{18}$$

3.3 Uniform steady rotation: summary

- The results derived can be summarised in the following simple theorem which contains all the relevant physics of charged particles in uniformly rotating frames in non-relativistic conditions :

  1. **In a uniformly rotating frame with constant electric and magnetic fields, a charged particle moves, exactly as in an inertial frame but in "equivalent" magnetic and electric fields, $\mathbf{B}_*, \mathbf{E}_*$ [cf. Eqs.(16,17)].**
  2. **The centrifugal force adds to the electrostatic potential in the frame, a "centrifugal potential", and leads to a force which is always directed radially outward from the rotation axis.**
  3. **The Coriolis force adds to the stationary magnetic field in the frame a uniform component parallel to the rotation axis. The strength of this uniform field relative to the stationary field is measured by the non-dimensional "rotation parameter":** $\rho_\Omega^* = \frac{2\Omega}{\omega_c}$ where, $\omega_c = (\frac{Ze}{m})B_{\rm lab}$.

- **Deductions:** The theorem is quite precise about the **exact** electrodynamic effects of the **Coriolis and Centrifugal forces** in $\mathbf{K}_{\rm rot}$. The former simply adds a "vertical field" of order ρ_Ω^* to the field in the inertial system. It is **linear** in Ω and hence "knows" about the sign of the rotation ("directionality").

- The **Centrifugal potential** may be written as, $\Phi_{\rm cent} = -\frac{T}{Ze}\frac{\Omega^2 R_\perp^2}{V_{\rm th}^2} = -\frac{T}{Ze}M_\Omega^2; T = \frac{1}{2}mV_{\rm th}^2$. It is **independent of the sign** of the frame angular velocity and depends only upon the **rotation Mach number**, M_Ω.

3.4 Uniformly rotating frames: summary

- In $\mathbf{K}_{\rm rot}$ with $\boldsymbol{\Omega} = \Omega \mathbf{e}_z$, and we write (without confusion!) $\frac{\partial}{\partial t} = \frac{d}{dt}; \mathbf{B} = \mathbf{B}_{\rm lab}$. The particle executes **Larmor gyrations** about $\mathbf{B}_*$ with the **"effective gyro frequency"** ω_c^* and has an **"effective Larmor" radius** r_L^*:

$$\omega_c^* = \frac{Ze}{m} B_* \tag{19}$$

$$B_* = B[1 + 2\rho_\Omega^* \mathbf{b}.\mathbf{e}_Z + (\rho_\Omega^*)^2]^{1/2} \tag{20}$$

$$r_L^* = \frac{c_\perp}{\omega_c^*} \tag{21}$$

$$\rho_\Omega^* = \frac{2\Omega}{\omega_c} = \frac{2m\Omega}{ZeB} \tag{22}$$

Here, $\mathbf{b} = \frac{\mathbf{B}}{B}$ and $c_\perp$ is the "Larmor gyration velocity" about $\mathbf{B}_* = B_* \mathbf{b}_* \neq \mathbf{B}_{\rm lab}$!

- Subject to the usual drift-ordering requirements on the **equivalent fields**, the particle will perform all the classical drifts: $\mathbf{E}^* \times \mathbf{B}^*, \nabla B^*,$ curvature, and any external force drifts.

- Plainly then, **adiabatic invariance, energy conservation** (latter, if and only if the fields are time-independent) apply and the "effective magnetic moment" is given by, $\mathcal{E}_\perp = \frac{1}{2} m c_\perp^2 = \mu^* B_*$. Canonical **angular momentum conservation** also applies **if** $\mathbf{E}_*, \mathbf{B}_*$ **are axi-symmetric about the rotation axis.**

3.5 Uniformly rotating frames: examples

- **Ex. 1: (Larmor)** Let $\mathbf{B}_{\rm lab} = B\mathbf{e}_Z, \mathbf{E}_{\rm lab} = 0, B$ uniform and constant. In the lab. we have uniform "parallel" motion along the Z-axis and "Larmor gyration" with frequency and radius: $\omega_{cZ} = \frac{ZeB}{m}; r_{\rm L} = \frac{v_\perp}{\omega_{cZ}}; (e > 0)$. We can choose the frame rotation rate of $\mathbf{K}_f$ so that $\mathbf{B}_* = 0$! Then,

$$
\begin{aligned}
\Omega &= -\frac{\omega_{cZ}}{2} = -\frac{ZeB}{2m} \\
\mathbf{E}_{\rm rot} &= \Omega R B \mathbf{e}_R \\
\mathbf{E}_* &= \mathbf{E}_{\rm rot} + \frac{m}{Ze}\Omega^2 R \mathbf{e}_R = \frac{\Omega R B}{2}\mathbf{e}_R
\end{aligned}
$$

  This centripetal (ie pointing to the axis of rotation) radial field in the rotating frame can be "cancelled" by the particle rotating about the origin with angular velocity $\Omega = -\frac{\omega_{cZ}}{2}$, agreeing perfectly with the lab. frame trajectory.

- In this "Larmor frame" there is no "field" to gyrate about and the circular motion around the rotation axis takes place due to the **centripetal** force embodied in $\mathbf{E}_*$. This result was known to and used by **Larmor** to prove his famous theorem! [cf. **Landau and Lifschitz** *Classical theory of fields* sec. 45]

3.6 Uniformly rotating frames: contd.

- **Ex.2:** In the lab. suppose we have a purely toroidal field with no electric field:, $\mathbf{B}_{\rm lab} = R_0 B_0 \nabla\phi ; \mathbf{E}_{\rm lab} = 0$. Then the motion in the lab. frame, is “Larmor+uniform circular+vertical -(gradB+ curvature) drift” giving helical motion of guiding centres.
- Now consider a rotating frame with $\mathbf{u}_f = \Omega R^2 \nabla\phi$. Then,

$$
\begin{aligned}
\mathbf{B}_* &= R_0 B_0 \nabla\phi + \frac{2m\Omega}{Ze}\mathbf{e}_Z \neq \mathbf{B}_{\rm lab} \\
\mathbf{E}_* &= \frac{m\Omega^2 R}{Ze}\mathbf{e}_R
\end{aligned}
$$

  The equivalent magnetic field is purely helical and the electric field, purely radial.

- Analysis of the guiding centre motion is an extremely interesting exercise. A special choice of Ω lead to a purely vertical motion in the rotating frame which transforms back to the trajectory in the lab. frame.

4.1 Action principle in arbitrary frames

- The relativistically invariant "action integral" S for a charged particle of rest mass m and charge Ze is [cf. **Landau and Lifshitz** *The Classical theory of fields*]:

$$S = -\int_{\tau_{\rm in}}^{\tau_{\rm fin}} [m v_\mu v^\mu + Ze A_\mu v^\mu] d\tau \tag{23}$$

where, the "proper time" is related to the ordinary time (in Minkwoski space-time) by $\gamma d\tau = dt; \gamma = [1 - \frac{\mathbf{v}.\mathbf{v}}{c^2}]^{-1/2}$. The four-velocity is:
$v^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \mathbf{v}); v_\mu = \gamma(c, -\mathbf{v}); v_\mu v^\mu = c^2$. and the four-potential is:
$A_\mu = (\frac{\Phi}{c}, -\mathbf{A})$. This can be written in a more familiar form:

$$S = \int_{t_{\rm in}}^{t_{\rm fin}} \mathcal{L}_{\mathcal{I}} dt$$

$$\mathcal{L}_{\mathcal{I}} = [-mc^2(1 - \frac{\mathbf{v}.\mathbf{v}}{c^2})^{1/2} + Ze(\mathbf{v}.\mathbf{A} - \Phi)] \tag{24}$$

The first term in Eq.(23), represents the particle's inertia whilst the second is **Schwarzchild's Lorentz invariant** "electrokinetic potential". The formula is valid in **General relativity**, provided the co and contravariant components are related through the **Einsteinean metric tensor** $g_{\mu\nu}$. We can, in principle, obtain the correct electrodynamics of the particle moving in an **arbitrary** frame from this Lagrangian. For our purposes, in this talk; it suffices to use its non-relativistic limit, when $v^2 \ll c^2$ holds.

4.2 Non-relativistic, non-inertial frames

- It is essential to note that **both** the four-velocity of the particle and the electromagnetic "four-potential" **transform** when we "change frames".
- **What about more general accelerating frames? Hamilton's principle of Least Action** $\delta S = 0$ gets exact results more simply than elementary kinematics. Let $\mathbf{u}_f$ be a non-relativistic flow-field in $\mathbf{K}_{\rm lab}$, varying in space and time. The **Lagrangian** for a particle (mass m and charge Ze) with coordinates $\mathbf{X}(t)$ and velocity $\mathbf{V}(\mathbf{t})$ referred **locally to the flow** is sought . We assume in $\mathbf{K}_f : \mathbf{V} = \frac{d\mathbf{X}}{dt}$,$\mathbf{X} = (X, Y, Z)$;$R^2 = X^2 + Y^2$. Hence, $\mathbf{v}_{\rm lab} = \mathbf{V}(t) + \mathbf{u}_f(\mathbf{X}(t), t)$. Let the vector potential in $\mathbf{K}_{\rm lab}$ be $\mathbf{A}_{\rm lab}$ and the electrostatic potential, $\Phi_{\rm lab}$. These imply: $\mathbf{E}_{\rm lab} = -\frac{\partial \mathbf{A}_{\rm lab}}{\partial t} - \nabla\Phi_{\rm lab}$; $\mathbf{B}_{\rm lab} = \nabla \times \mathbf{A}_{\rm lab}$.
- Special relativity+ general covariance imply (when $\mathbf{u}_f^2 \ll c^2$) the relations: $dt_{\rm lab} = dt_f; \mathbf{A}_{\rm lab} = \mathbf{A}_f; d\mathbf{x} = d\mathbf{X} + \mathbf{u}_f dt$; ie, $\frac{\partial \mathbf{X}}{\partial t} + \mathbf{u}_f.\nabla\mathbf{X} = 0$.

$$\begin{aligned} \Phi_{\rm lab} - \mathbf{v}_{\rm lab}.\mathbf{A}_{\rm lab} &= \Phi_f - \mathbf{V}.\mathbf{A}_f \\ \Phi_{\rm lab} - \mathbf{A}_{\rm lab}.\mathbf{u}_f &= \Phi_f \end{aligned} \qquad (25)$$

Leading to the familiar transformation rules:

$$\mathbf{B}_f = \mathbf{B}_{\rm lab} \qquad (26)$$

$$\mathbf{E}_f = \mathbf{E}_{\rm lab} + \mathbf{u}_f \times \mathbf{B}_{\rm lab} \qquad (27)$$

4.3 Lagrangian dynamics in $\mathbf{K}_f$

- The Lagrangian for the particle in the frame $\mathbf{K}_f$ **co-moving** with the non-relativistic velocity, $\mathbf{u}_f$ relative to $\mathbf{K}_{\rm lab}$ is then given by:

$$L_f \quad = \quad \frac{1}{2}m\mathbf{V}^2 + Ze\mathbf{V}.[\mathbf{A}_f + (\frac{2m}{Ze})\frac{\mathbf{u}_f}{2}] - Ze[\Phi_f - \frac{m}{Ze}\frac{\mathbf{u}_f.\mathbf{u}_f}{2}] \tag{28}$$

This is obtained by substituting $\mathbf{v}_{\rm lab} = \mathbf{V} + \mathbf{u}_f$ in

$$L_{\rm lab} \quad = \quad \frac{1}{2}m\mathbf{v}_{\rm lab}^2 + Ze\mathbf{A}_{\rm lab}.\mathbf{v}_{\rm lab} - Ze\Phi_{\rm lab} \tag{29}$$

(this is the non-relativistic limit of $\mathcal{L}_\mathcal{I}$ in Eq.(24)) using the transformation rules: $\mathbf{A}_{\rm lab} = \mathbf{A}_f = \mathbf{A}; \Phi_f = \Phi_{\rm lab} - \mathbf{A}.\mathbf{u}_f$. Then, Hamilton's principle, $\delta S = 0$ leads to the **Euler-Lagrange equations of motion** in $\mathbf{K}_f$:

$$\frac{d}{dt}(\frac{\partial L_f}{\partial \dot{X}}) \quad = \quad \frac{\partial L_f}{\partial X} \tag{30}$$

$$\frac{d}{dt}(\frac{\partial L_f}{\partial \dot{Y}}) \quad = \quad \frac{\partial L_f}{\partial Y} \tag{31}$$

$$\frac{d}{dt}(\frac{\partial L_f}{\partial \dot{Z}}) \quad = \quad \frac{\partial L_f}{\partial Z} \tag{32}$$

4.4 Hamiltonian dynamics in K_f

- From now on, we will drop the suffix on L_f and take Eq.(28) to be the definition of the (non-relativistic) Lagrangian in K_f. Returning to Eqs. (30-32), the **canonical momenta** are given by setting, $(\mathbf{A}_f = \mathbf{A})$:

$$\mathbf{P} = \frac{\partial L}{\partial \dot{\mathbf{X}}} = m\mathbf{V} + Ze[\mathbf{A} + (\frac{2m}{Ze})\frac{\mathbf{u}_f}{2}] \quad (33)$$

Using the transformation, $\mathcal{H} = \mathbf{P}.\dot{\mathbf{X}} - L$, the **Hamiltonian** $\mathcal{H}$ of the system in K_f expressed in terms of the canonical momenta is found to be:

$$\mathcal{H} = \frac{1}{2m}[\mathbf{P} - Ze(\mathbf{A} + (\frac{2m}{Ze})\frac{\mathbf{u}_f}{2})]^2 + Ze[\Phi_f - \frac{m}{Ze}\frac{\mathbf{u}_f.\mathbf{u}_f}{2}] \quad (34)$$

- **Important observations:** The flow $\mathbf{u}_f$ contributes to both the "equivalent" vector and scalar potentials. The contribution to the "equivalent" **vector potential is linear** in the flow, whilst the contribution to the "equivalent" electrostatic potential is linear in the **specific kinetic energy**.

- Both contributions are proportional to the mass-to-charge ratio of the particle, exhibiting the fact that they arise from **inertial (ie non-electromagnetic) effects** of frame acceleration: thus they are non-zero even in the limit, $Ze/m \rightarrow 0$, (ie, in a neutral fluid).

4.5 The Euler-Lagrange equations

- The Euler-Lagrange equations can be transformed into a more familiar form when we introduce two **equivalent potentials**:

$$\mathbf{A}_* = [\mathbf{A}+(\frac{2m}{Ze})\frac{\mathbf{u}_f}{2}] \quad (35)$$

$$\Phi_* = \Phi_f - \frac{m}{Ze}\frac{\mathbf{u}_f.\mathbf{u}_f}{2} = [\Phi_{\rm lab} - \mathbf{A}.\mathbf{u}_f - \frac{m}{Ze}\frac{\mathbf{u}_f.\mathbf{u}_f}{2}] \quad (36)$$

The Euler-Lagrange equations for the system in terms of the equivalent potentials are derived using the fact that, $\nabla_{\mathbf{X}}$ applies holding $\mathbf{V}$ **fixed** and the vector identity: $\nabla_{\mathbf{X}}(\mathbf{V}.\mathbf{A}_*) = \mathbf{V}.\nabla_{\mathbf{X}}\mathbf{A}_* + \mathbf{V}\times(\nabla_{\mathbf{X}}\times\mathbf{A}_*)$ Then, we obtain the following key relations:

$$m\frac{d\mathbf{V}}{dt} = Ze[\mathbf{E}_* + \mathbf{V}\times\mathbf{B}_*] \quad \textbf{where,} \quad (37)$$

$$\mathbf{E}_* = \mathbf{E}_f - \frac{m}{Ze}\frac{\partial \mathbf{u}_f}{\partial t} + \frac{m}{Ze}\nabla\frac{\mathbf{u}_f.\mathbf{u}_f}{2}$$

$$= \mathbf{E}_{\rm lab} + \mathbf{u}_f\times\mathbf{B}_{\rm lab} - \frac{m}{Ze}\frac{\partial \mathbf{u}_f}{\partial t} + \frac{m}{Ze}\nabla\frac{\mathbf{u}_f.\mathbf{u}_f}{2} \quad (38)$$

$$\mathbf{B}_* = \nabla\times\mathbf{A}_*$$

$$= \mathbf{B}_f+(\frac{2m}{Ze})\frac{\mathbf{W}_f}{2} \quad (39)$$

4.6 Physical interpretation

- In these equations $\mathbf{W}_f = \nabla \times \mathbf{u}_f$ is the vorticity (W for "whorliness"!) in the flow, and $\mathbf{E}_f = \mathbf{E}_{\rm lab} + \mathbf{u}_f \times \mathbf{B}_{\rm lab}, \mathbf{B}_f = \mathbf{B}_{\rm lab} = \mathbf{B}$ are the usual electromagnetic fields (ie solutions of **Maxwell's equations)** in $\mathbf{K}_f$ at the location $\mathbf{X}(t)$ of the particle. Consider the simple example where, $\mathbf{u}_f = \mathbf{\Omega} \times \mathbf{X}; \mathbf{\Omega} = \Omega \nabla Z,$ and we assume the angular velocity to be uniform and constant. We obviously recover our earlier results. In this case, $\mathbf{W}_f = \nabla \times \mathbf{u}_f = 2\mathbf{\Omega}.$ In general, $\mathbf{W}_f$ is the vorticity in the "frame flow" $\mathbf{u}_f,$ and is of fundamental importance: $\mathbf{W}_f.\mathbf{W}_f$ represents the local flow **enstrophy.**
- Coriolis-like inertial effects are velocity-dependent but can do no work on the particle: they *must* be **perpendicular to the velocity vector of the particle in the frame!** Hence they combine with the magnetic field in the Lorentz-Newton equations to give $\mathbf{B}_*.$ The centrifugal-like inertial effects are non-dissipative: they *must* produce purely **conservative accelerations** along and perpendicular to the equivalent magnetic field. They result in modifications to Φ_f and give $\Phi_*.$ The constant $\frac{m}{Ze}$ appears in $\mathbf{A}_*, \mathbf{B}_*, \Phi_*$ since the frame acceleration depend upon the **mass and not the charge** of the particle. From nature of the Coriolis (does no work, always transverse to the velocity) and centrifugal forces (conservative field), we can **guess** the form of Eqs.(37), although the expressions for the equivalent fields [Eqs.(38,39)] are not intuitively "obvious".
- In inertial frames, $\mathbf{E}, \mathbf{B}$ **are functions only of particle position, not its velocity.** We have verified that $\mathbf{E}_*, \mathbf{B}_*$ have this same property in an arbitrary **non-inertial frame.**

4.7 Key results: 1

- The Lagrangian and Hamiltonian take the same form as in the lab. frame. The frame flow effectively “adds” to the vector potential. The relative size of this “effective magnetic field” scales like, $\rho^*_\Omega = \frac{2\Omega}{\omega_c}$ but more generally like, $\frac{|\mathbf{K}_f|}{\omega_c}$. The “effective electric field” acquires a purely potential "local centrifugal" term, $\frac{m}{Ze}\frac{u_f^2}{2}$.and a term proportional to $\frac{\partial \mathbf{u}_f}{\partial t}$.
- Since the frame electric potential is of order $\frac{T_f}{e}; T_f = \frac{1}{2} m_i v_{\rm th}^2$, the relative size of the centrifugal potential is $\frac{mM_f^2}{m_i Z}$, where $M_f = \frac{u_f}{v_{\rm th,f}}$ is the **Mach number of the bulk-ion** $[m_i, Z_i e, T_f, v_{\rm th,f}]$ flow.
- If all the fields appearing in $\mathcal{H}$ are explicitly time-independent, it is a **constant of the motion.**
- Consider cylindrical coordinates (R, ϕ, Z) in $\mathbf{K}_f$ with the Lagrangian:

$$L = \frac{1}{2} m[(\dot{R})^2 + (R\dot{\phi})^2 + (\dot{Z})^2] + Ze[\dot{R}A_{*R} + \dot{\phi}RA_{*\phi} + \dot{Z}A_{*Z}] - Ze\Phi_* \quad (40)$$

If the basic magnetic configuration and the frame flow, $\mathbf{u}_f$ possess **azimuthal symmetry** (say, about the Z-axis), the equivalent potential components are independent of ϕ.

4.8 Key results: 2

- The corresponding canonical momenta and Hamiltonian are:

$$P_R = m\dot{R} + ZeA_{*R} \quad (41)$$

$$P_\phi = mR^2\dot{\phi} + ZeRA_{*\phi} \quad (42)$$

$$P_Z = m\dot{Z} + ZeA_{*Z} \quad (43)$$

$$\mathcal{H} = \frac{1}{2m}[(P_R - ZeA_{*R})^2 + (P_\phi - ZeA_{*\phi})^2 + (P_Z - ZeA_{*Z})^2] + Ze\Phi_* \quad (44)$$

From Hamilton's equations and azimuthal symmetry, we see that P_ϕ, the **canonical angular momentum relative to the equivalent field** (not the **laboratory** field!) is a constant of the motion as can be expected from the most elementary kinematic considerations.

- If $\mathbf{u}_f$ is a purely axisymmetric toroidal flow it is divergence-free and **only influences** the **poloidal** equivalent magnetic field. If the rotation is in the **co-current** direction, it reduces the effective poloidal field **outboard of the magnetic axis** and increases it inboard. If the flow is toroidal and depends only upon R, the Coriolis contribution is a **purely vertical field**.
- A significant **toroidal vorticity** in the background **poloidal flow** would change the equivalent **toroidal magnetic field.** This may be relevant to ITBs/ETBs.

5.1 The Vlasov equation in $\mathbf{K}_f$

- **All charged particles with the same charge to mass ratio** $\frac{Ze}{m}$ have **identical dynamics in given fields** $\Phi_f, \mathbf{A}_f, \mathbf{u}_f$, for the same initial positions and velocities.
- The **Vlasov equation** satisfied by the distribution function, $f(\mathbf{X}, \mathbf{V}, t)$ in the frame is the one-particle Liouville equation:

$$\frac{\partial f}{\partial t} + \Sigma_{k=1}^{3}[\frac{\partial \mathcal{H}}{\partial P_k}\frac{\partial f}{\partial X_k} - \frac{\partial \mathcal{H}}{\partial X_k}\frac{\partial f}{\partial P_k}] \quad = \quad 0 \qquad (45)$$

Setting, $\mathbf{B}_* = \nabla \times \mathbf{A}_*; \mathbf{E}_* = -\frac{\partial \mathbf{A}_*}{\partial t} - \nabla \Phi_*,$ and taking account of Hamilton's equations, we see that the Vlasov equation assumes the standard form:

$$\frac{\partial f}{\partial t} + \mathbf{V}.\frac{\partial f}{\partial \mathbf{X}} + \frac{Ze}{m}[\mathbf{E}_* + \mathbf{V} \times \mathbf{B}_*].\frac{\partial f}{\partial \mathbf{V}} \quad = \quad 0 \qquad (46)$$

Note that $\mathbf{E}_*, \mathbf{B}_*$ contain the standard electromagnetic (**Maxwell**) fields as well as the "pseudo-fields" due to the frame velocity $\mathbf{u}_f$ and its vorticity $\mathbf{W}_f,$ and depend on $\frac{m}{eZ}$. but **not** on $\mathbf{V}$.

- The particles perform "parallel", "Larmor" and "drift motions" with respect to the **equivalent fields, not the Maxwell fields!** This distinction between inertial and non-inertial frames is **exact and should be respected** in charged particle dynamics in $\mathbf{K}_f$.

5.2 Drift orbit theory in $\mathbf{K}_f$

- The derivation of "drift orbit" theory (and associated gyrokinetics, subject to usual orderings) in $\mathbf{K}_f$:
  1. Start with the exact Lagrangian (cf. Eq.(28)): $L(\mathbf{R}(t), \mathbf{V}(t), \mathbf{A}_*, \Phi_*)$.
  2. Write down the "averaged Lagrangian" taking out the fast gyromotions about the equivalent field $\mathbf{B}_*$ (this is the so-called " **Whitham averaged Lagrangian approach**";cf. *Collisional transport in magnetized plasmas*, **Helander and Sigmar**, p. 100 *et seq.*).
  3. Obtain the adiabatically invariant "magnetic moment" $\mu^*; (\mathcal{E}_\perp^* = \mu^* B_*)$
  4. Set up the gyrokinetic Hamiltonian in its canonical variables. The collisionless gyrokinetic equation follows from the corresponding Liouville equation.

- The drift orbit equations can be written down “informally”: introduce the unit vector $\mathbf{b}_* = \frac{\mathbf{B}_*}{B_*}$ then,$V_\parallel^* = \mathbf{V}.\mathbf{b}_*.$ The **guiding centre** velocity $\mathbf{V}_{\rm gc}^* = \frac{d\mathbf{R}_{\rm gc}^*}{dt}$ satisfies:

$$m\frac{d\mathbf{V}_{\rm gc}^*}{dt} = ZeB_*\mathbf{V}_{\rm gc}^* \times \mathbf{b}_* + Ze\mathbf{E}_* - \mu^*\nabla B_* \quad (47)$$

$$\mathbf{V}_{\rm gc}^* = V_\parallel^*\mathbf{b}_* + [\frac{\mathbf{E}_*}{B_*} - (\frac{\mu^*\nabla B_*}{ZeB_*}) - (\frac{1}{\Omega_{cZ}}\frac{d\mathbf{V}_{\rm gc}^*}{dt})] \times \mathbf{b}_* \quad (48)$$

from which **all relevant information about gyrokinetics in** $\mathbf{K}_f$ follows! $\mathbf{E}_*, \mathbf{B}_*$ are “gyro-averages” since they must satisfy the “drift ordering” for Eq.(47) to be valid.

5.3 Collisionless gyro/drift kinetics in K_f

- We write, following **Hazeltine and Meiss** (cf. *Plasma confinement*)

$$\mathbf{V}^*_{\rm gc} = V^*_{\|}\mathbf{b}_* + \mathbf{v}_D \quad (49)$$

$$\mathcal{E} = \frac{1}{2}m(V^*_{\|})^2 + Ze\Phi_* + \mu^* B_* \quad (50)$$

Then the collisionless drift-kinetic equation (for example) takes the form:

$$\frac{\partial F}{\partial t} + (V^*_{\|}\mathbf{b}_* + \mathbf{v}_D).\nabla F + [Ze\frac{\partial \Phi_*}{\partial t} + \mu^*\frac{\partial B_*}{\partial t} - ZeV^*_{\|}\mathbf{b}_*.\frac{\partial \mathbf{A}_*}{\partial t}]\frac{\partial F}{\partial \mathcal{E}} = 0 \quad (51)$$

- The gyro-averaging of the exact equations is standard (cf. **Bernstein and Catto**, Phys. Fluids **28**, 1342 (1985).) The only (but essential!) difference is that we must use $\mathbf{E}_*, \mathbf{B}_*, \Phi_*$ and start from Eq.(47)!
- We obtain, the correct, collisionless, electromagnetic, non-linear gyrokinetic equation of Bernstein-Catto (cf. their Eq.(44) without collisions, using their notation for variables and averages):

$$\frac{\partial F}{\partial t} + << \dot{\mathbf{r}}' >> .\nabla' F + << \dot{u}' >> \frac{\partial F}{\partial u'} = 0 \quad (52)$$

5.4 Collisional theories in $\mathbf{K}_f$

- Setting up the **Landau-Fokker-Planck-Rosenbluth** collision terms is standard and no different to their derivation in the presence of gravitational fields in inertial systems! See **Lifschitz-Pitaevski** *Physical kinetics* and paper by **Bernstein and Catto** cited.

- For a **test-particle/impurity species** $[m_Z, Ze]$ the corresponding **Langevin** equations are:

$$m_Z \frac{d\mathbf{V}_Z}{dt} = Ze[\mathbf{E}_* + \mathbf{V}_Z \times \mathbf{B}_*] + \frac{m}{\tau_{Zi}}(\mathbf{u}_i - \mathbf{V}_Z) + \mathbf{f}_{\text{Langevin}} \quad (53)$$

  where $\mathbf{u}_i$ is the "background ion" plasma flow in $\mathbf{K}_f$, **and not to be confused with** $\mathbf{u}_f$, the "frame flow"! τ_{Zi} is the momentum relaxation rate for the impurity species.

- The term involving τ_{Zi} is the **Einstein drag force** whilst the last term is the random **Langevin stochastic force**, characteristic of Brownian motion theory [cf. *Stochastic processes in physics and chemistry* by **N.G. van Kampen**].

- This equation has already been used **in the lab. frame** with great success by **Ken McClements and Robert McKay** (cf. R.J. McKay *et al*, Plasma Phys. Control. Fusion, **50**, 065017, (2008), also paper submitted (2009) to PPCF) to investigate trace impurity transport in MAST in static fields. Collisions with other species (eg. electrons) are easily included.

5.5 Applications: JET

- **In typical JET conditions:** $B \simeq 3$ T; $I_p \simeq 3$ MA; $T_D \simeq 10$ keV; $\Omega \leq 3 \times 10^5$ rads/s; $v_\phi^i \leq 1000$ km/s. Then for **background Deuterons**, $\rho_\Omega^* = \frac{2\Omega}{\omega_{cD}} \leq 4 \times 10^{-3}$. Hence, any **drift** effect of the **Coriolis force** (using our velocity space coordinates) on the background ions is negligible as the vertical field it creates is of order 12 mT (120 Gauss)$\ll B_{\rm pol} \simeq 0.5$ T. Since, $v_{\rm thD} \simeq 1000$ km/s, $M_f \simeq 1$. There should therefore be a strong centrifugal potential which is definitely not a flux function.
- Thus, we conclude that all effects on charged particles with $\frac{m}{eZ} \simeq \frac{m_D}{eZ_D}$ in JET due to strong rotation must arise largely from the frame electric fields and the centrifugal terms (Φ_* and equivalent drifts associated with them in $\mathbf{K}_f$).
- For partially stripped, massive impurities (eg. tungsten) with $\frac{m}{eZ} \simeq 10 \times \frac{m_D}{eZ_D}$, the "vertical field" is about 25-40 per cent of the poloidal field in JET and possibly comparable with typical vertical fields. The rotational effects on transport, especially asymmetries, are likely to be significant.

5.6 Applications: MAST, STs

- **In typical MAST conditions:** $B \simeq 0.5$ T; $I_p \simeq 1$ MA; $T_I \simeq 1$ keV; $\Omega \simeq 5 \times 10^5$ rads/s, the effect of the Coriolis "equivalent poloidal field" is likely to be of the order of 1-10 percent of the poloidal field even for deuterons. The background ions will see different flux surfaces (especially in the low field side) as compared with electrons. This implies that rather pronounced asymmetries in transport properties of the background plasma are likely with co and counter rotation under the conditions considered-see **McClements and McKay** paper submitted to PPCF.
- For impurities with $\frac{m}{eZ} \geq 10 \times \frac{m_D}{eZ_D}$ like tungsten (partially stripped under these conditions), we can indeed see large effects of fast rotation. Since the equivalent fields have a linear (as well as quadratic) dependence on Ω one should be able to observe significant asymmetries in impurity transport arising from co/counter rotation.
- The assessment of turbulent transport in these non-inertial frames can also be carried out rather effectively using the gyro-averaging procedure suggested in this paper, using appropriately defined equivalent fields in suitably chosen (possibly differentially rotating co-moving) frames involving “diagnostic” impurities (eg. argon).

6.1 Applications-tokamak frame flows

- **Theory of the frame flow and the equilibrium electrostatic potential: cf. Thyagaraja and McClements,** Phys. Plasmas, **13**, 062502 (2006). The equilibrium two-fluid momentum balance equations of the plasma in $\mathbf{K}_{\rm lab}$ (pure electron-deuteron, quasi-neutral, steady, dissipationless, negligible electron inertia) are:

$$\nabla p_i = -(\frac{m_i}{Z_i e}) Z_i n_i e \nabla(\frac{\mathbf{v}_i^2}{2}) - Z_i n_i e \nabla\Phi + Z_i n_i e \mathbf{v}_i \times (\mathbf{B} + \frac{m_i}{Z_i e}\mathbf{W}_i) \quad (54)$$

$$\nabla p_e - ne\nabla\Phi = -ne\mathbf{v}_e \times \mathbf{B} \quad (55)$$

whence,

$$\nabla p + ne\nabla[(\frac{m_i}{Z_i e})(\frac{\mathbf{v}_i)^2}{2}] = \mathbf{j}\times\mathbf{B} + ne(\frac{m_i}{Z_i e})[\mathbf{v}_i \times \mathbf{W}_i] \quad (56)$$

where, $\rho_{m_i} = m_i n_i; Z_i n_i = n_e = n; \mathbf{j} = ne(\mathbf{v}_i - \mathbf{v}_e) = \frac{1}{\mu_0}\nabla\times\mathbf{B}; \mathbf{W}_i = \nabla\times\mathbf{v}_i.$

- In the physically realistic cases (neglecting electron inertia, and when $v_\phi^i \gg v_{\rm pol}$), the following equations apply :

$$RB_\phi = F(\Psi) \quad (57)$$

$$\nabla p = \frac{m_i n (v_\phi^i)^2}{R}\nabla R + [\frac{R\mu_0 j_\phi - FF'}{\mu_0 R^2}]\nabla\Psi \quad (58)$$

6.2 Applications-tokamak flow equilibria

- We obtain the “solubility condition” on Eq.(58):

$$\frac{\partial p}{\partial l} = (\frac{\mathbf{B}_{\rm pol}}{B_{\rm pol}}).\nabla p = \frac{m_i n (v^i_\phi)^2}{R}\frac{\partial R}{\partial l} \quad (59)$$

A whole class of solutions have been identified, given the velocity profile. They determine p, n, Φ as functions of Ψ, R (given $T_{i,e}$) in the poloidal plane **self-consistently**. The above equation was solved exactly and two limiting cases were identified: rigid body rotation, such that Ω depends only on poloidal flux Ψ; and “Keplerian” rotation, in which the mechanical toroidal angular momentum per unit mass of the ion fluid is a flux function. In the rigid body case $\Phi_{\rm lab}$ is related to Ω by the expression

$$e\Phi_{\rm lab} = e\Phi_0(\Psi) + \frac{T_e}{2(T_e+T_i)} m_i \Omega(\Psi)^2 R^2, \quad (60)$$

where Φ_0 is a flux function and m_i is bulk ion mass. In the Keplerian case one obtains an expression of the form:

$$e\Phi_{\rm lab} = e\Phi_0(\Psi) - \frac{T_e}{2(T_e+T_i)} \frac{m_i \lambda(\Psi)^2}{R^2}, \quad (61)$$

where $\lambda(\Psi) = R^2\Omega$. Thus, in both cases $\Phi_{\rm lab}$ is not a pure flux function.

7.1 Relation to previous work-I

- **Artun and Tang**[Phys. Plasmas **1**, 2682 (1994)] consider frame toroidal flow velocities $\mathbf{u}_f = \omega_R(\Psi)R^2\nabla\phi$, and derive from a transformed Vlasov equation (Eq. (12) in their paper) a gyrokinetic equation. They retain laboratory frame position coordinates $\mathbf{x}_{\rm lab}$ while introducing a shift of origin in velocity space, $\mathbf{v}_{\rm rot} = \mathbf{v}_{\rm lab} - \mathbf{u}_f$ (note that our $\mathbf{u}_f = \mathbf{V}$ of Artun and Tang). unlike our $\mathbf{X}, \mathbf{P}$, these are not canonical coordinates.
- As we have shown, both our kinematic approach and Hamilton's principle deliver expressions of definitive simplicity for both the Vlasov equation and the Newton-Lorentz equations in $\mathbf{K}_f$. The Vlasov equation employed by Artun and Tang contains, in addition to the transformed Maxwell fields, velocity-dependent terms and $\partial/\partial\mathbf{x}$, the gradient operator in the laboratory frame rather than the rotating frame.
- In principle, the results derived by these authors should agree with any obtained from our equations, if carried to all orders, since both sets of equations are obtained from the inertial frame equations by purely mathematical manipulations. However, our equations exhibit a far simpler, canonical structure and reveal real physical characteristics of the motion.

7.2 Relation to previous work-II

- **Brizard** [Phys. Plasmas **2**, 459, (1995)., **40**, 731 (1998)] considered the derivation of the nonlinear collisionless gyrokinetic equation for toroidally rotating axisymmetric tokamaks. **Notations:** Brizard lab. frame variables $(\mathbf{x}, \mathbf{v})$ with "frame flow" velocity $\mathbf{u}_s$. Brizard's moving frame [our $\mathbf{K}_f$] coordinates: $(\mathbf{r}, \mathbf{c})$ are equivalent to our $(\mathbf{X}, \mathbf{V})$.
- Derivation uses a phase-space Lagrangian Lie-transform perturbation method; no discussion of exact Lagrangian or transformation of the complete Newton-Lorentz equations or associated full Vlasov equation.
- In lieu of $\mathbf{u}_s$ Brizard uses a first-order approximation to it, $\mathbf{u}_0 = u_{0\|}\mathbf{b} + (1/B)\mathbf{b} \times \nabla\Phi_0$ where Φ_0 is a flux function, and introduces a **particle velocity** $\mathbf{u}_0^* \equiv \mathbf{u}_0 + W\mathbf{b}$, where W is the particle velocity parallel to $\mathbf{B}$, and an effective magnetic field

$$\mathbf{B}^* \quad \equiv \quad \nabla \times \mathbf{A}^* = \mathbf{B} + \frac{m}{Ze}\nabla \times \mathbf{u}_0^*. \qquad (62)$$

  This depends on **both the particle's position and its velocity** parallel to the magnetic field in $\mathbf{K}_{\mathrm{lab}}$.
- Brizard's "effective field" differs from our equivalent field $\mathbf{B}_*$, which, for given flow $\mathbf{u}_f$, depends only on the particle **position.** We have proved that (cf. Eq.(37)) in $\mathbf{K}_f$ the particle gyrates with respect to $\mathbf{B}_*$, **not** $\mathbf{B}$(or $\mathbf{B}^*$).

7.3 Relation to previous work-III

- The consequences of Brizard's analysis are exhibited in his Eqs. (18-19) for the drift orbit of the particle in the co-moving frame ($\mathbf{X}$ is the guiding center position):

$$\dot{\mathbf{X}} = \frac{\mathbf{b}}{eB_{\parallel}^{*}} \times \nabla H + \frac{\mathbf{B}^{*}}{mB_{\parallel}^{*}} \frac{\partial H}{\partial W}, \tag{63}$$

$$\dot{W} = -\frac{\mathbf{B}^{*}}{mB_{\parallel}^{*}} \cdot \nabla H, \tag{64}$$

$$H = e\Phi + \mu B + \frac{1}{2} m \mathbf{u}_0^{*} \cdot \mathbf{u}_0^{*}, \tag{65}$$

is Brizard's guiding-center Hamiltonian and $B_{\parallel}^{*}$ is the component of $\mathbf{B}^{*}$ parallel to $\mathbf{B}$

$$B_{\parallel}^{*} = B\left(1 + \frac{\mathbf{b}}{\Omega} \cdot \nabla \times \mathbf{u}_0^{*}\right). \tag{66}$$

Note two key differences between Eq. (71) our Eq. (48):

1. The "parallel" component of $\dot{\mathbf{X}} \equiv \mathbf{V}_{\mathrm{gc}}$ in Eq. (71) is taken along $\mathbf{B}^{*}$ rather than $\mathbf{B}_{*}$.
2. The perpendicular component is orthogonal to $\mathbf{B}$ rather than $\mathbf{B}_{*}$; the exact Lorentz-Newton equations [our Eq. (37)] suggest that $\mathbf{B}_{*}$ should be regarded as the most "natural" effective field in $\mathbf{K}_f$ from point of view of particle orbit theory.

7.4 Relation to previous work-IV

- As we have proved, **all particle drifts and adiabatic invariants** should be defined with respect to the **equivalent fields in** $\mathbf{K}_f : \mathbf{E}_*, \mathbf{B}_*$, defined by, Eqs.[35,36], and which are functions only of **particle position.**
- **The physical reason** for this is straightforward: in the laboratory frame the motion of a charged particle is determined solely by the $\mathbf{E}$ and $\mathbf{B}$ fields satisfying Maxwell's equations. In any **inertial frame** moving with constant velocity $\mathbf{u}$ relative to the laboratory, the motion is determined in the non-relativistic limit by the potentials, $\mathbf{A}_{\rm lab}$ and $\Phi_{\rm lab} - \mathbf{A} \cdot \mathbf{u}$.
- When the co-moving frame is an accelerating one, "inertial forces" contribute to **both** $\mathbf{A}_*$ and Φ_*. In these circumstances it is inadvisable to use $\mathbf{B}$ as the magnetic field in an orbit theory calculation, even if the differences between $\mathbf{B}, \mathbf{B}^*$ and $\mathbf{B}_*$ are small, since "secular terms" arise, which when suppressed properly to all orders lead to our results.
- We show that the canonical action conjugate to the gyrophase, denoted by μ^* which equals $\frac{\mathcal{E}_\perp}{B_*}$, in leading order, and **not** μ is conserved to all orders (demonstrated generally by **Kruskal**), if the drift ordering applies in $\mathbf{K}_f$ relative to the spatio-temporal variations of $\mathbf{E}_*, \mathbf{B}_*$. This can be explicitly verified with exact solutions in simple configurations.

7.5 Relation to previous work-V

- **Peeters** *et al* [Phys. Rev. Letts., **98**, 265003 (2007); Phys. Plasmas, **16**, 012505 (2009), Phys. Plasmas **16**, 042310 (2009).] base their work on Brizard's.
- Defining,

$$\frac{\mathbf{B}^*}{B_\parallel^*} = \mathbf{b} + \frac{mv_\parallel}{ZeB_\parallel^*}\mathbf{b}\times(\mathbf{b}.\nabla)\mathbf{b} + \frac{2m}{ZeB_\parallel^*}\mathbf{\Omega}_\perp \qquad (67)$$

The key gyro-orbit equations of Peeters *et al* (which they state is in the co-moving system-equivalent to our $\mathbf{K}_f$) for a charged particle of mass m and charge Ze with coordinates, $\mathbf{X}$, $\frac{d\mathbf{X}}{dt} \equiv \mathbf{V}_{\rm gc}$ is:

$$\frac{d\mathbf{X}}{dt} = v_\parallel\mathbf{b} + \frac{2mv_\parallel}{ZeB_\parallel^*}\mathbf{\Omega}_\perp - \frac{m\Omega^2 R}{ZeB_\parallel^*}\mathbf{b}\times\nabla R + \frac{mv_\parallel^2}{ZeB_\parallel^*}\mathbf{b}\times(\mathbf{b}.\nabla)\mathbf{b} + \frac{\mu}{Ze}\frac{\mathbf{b}\times\nabla B}{B_\parallel^*} + \frac{\mathbf{b}\times\nabla <\phi>}{B_\parallel^*} \qquad (68)$$

7.6 Relation to previous work-VI

- These equations are rather different in appearance from Eqs.(37,47), in our paper. As in Brizard, the material derivative $\mathbf{u}_0^*.\nabla\mathbf{u}_0^*, \mu, \mathbf{b}$ enter. Mathematically, they are presumably equivalent to our equations (both have the same starting point!) but have a more complicated structure due to the presence of $\mathbf{u}_0^*(\mathbf{X}, v_\parallel)$.
- The velocity vector is represented in terms of a component parallel to $\mathbf{B}_{\rm lab}$ and a perpendicular one which contains the “Coriolis drift” (the second term in Eq.(68)) in this reperesentation. Since $v_\parallel$ appears in $B_\parallel^*$ in a complicated non-linear manner, it is not immediately obvious whether this representation is mathematically equivalent to ours.
- Although one can indeed treat the Coriolis terms as “perturbations” to $\mathbf{V}\times\mathbf{B}_{\rm lab}$ in the orbit equations, as done by Brizard *et al*, this does not seem very natural, nor does it have a simple physical interpretation, as in our representation.

8.1 Discussion: general observations

- The **Lorentz-Newton** equations of motion in $\mathbf{K}_{\mathrm{lab}}$:

$$m\frac{d\mathbf{v}_{\mathrm{lab}}}{dt} = Ze[\mathbf{E}_{\mathrm{lab}}+\mathbf{v}_{\mathrm{lab}}\times\mathbf{B}_{\mathrm{lab}}]$$

In a **uniformly translating frame**, $\mathbf{v}_{\mathrm{lab}} = \mathbf{V} + \mathbf{u}_f$:

$$\begin{aligned} m\frac{d\mathbf{V}}{dt} &= Ze[\mathbf{E}_{\mathrm{f}}+\mathbf{V}\times\mathbf{B}_f] \\ &= Ze[\mathbf{E}_{\mathrm{lab}}+\mathbf{u}_f\times\mathbf{B}_{\mathrm{lab}}+\mathbf{V}\times\mathbf{B}_f] \end{aligned}$$

In an **arbitrarily accelerating frame**, $\mathbf{K}_f$,

$$\begin{aligned} m\frac{d\mathbf{V}}{dt} &= Ze[\mathbf{E}_{\mathrm{f}}+\mathbf{V}\times\mathbf{B}_f] + \mathbf{F}_{\mathrm{cent}} + \mathbf{F}_{\mathrm{Cor}} \\ &= Ze[\mathbf{E}_{\mathrm{lab}}+\mathbf{u}_f\times\mathbf{B}_{\mathrm{lab}}+\mathbf{V}\times\mathbf{B}_f] + Ze[\mathbf{E}_{\mathrm{cent}} + \mathbf{V}\times\mathbf{B}_{\mathrm{Cor}}] \\ \mathbf{E}_{\mathrm{cent}} &= \frac{m}{Ze}[-\frac{\partial\mathbf{u}_f}{\partial t} + \nabla\frac{\mathbf{u}_f.\mathbf{u}_f}{2}] \\ \mathbf{B}_{\mathrm{Cor}} &= \frac{m}{Ze}\nabla\times\mathbf{u}_f \\ \mathbf{E}_* &= \mathbf{E}_f + \mathbf{E}_{\mathrm{cent}} = \mathbf{E}_{\mathrm{lab}}+\mathbf{u}_f\times\mathbf{B}_{\mathrm{lab}} + \mathbf{E}_{\mathrm{cent}} \\ \mathbf{B}_* &= \mathbf{B}_f + \mathbf{B}_{\mathrm{Cor}} = \mathbf{B}_{\mathrm{lab}} + \mathbf{B}_{\mathrm{Cor}} \end{aligned}$$

8.2 Discussion-II

- We have used use the **exact** Lagrangian/Hamiltonian, Eqs.(28-34) and the Newton-Lorentz equations, Eqs.(38-40) as a starting point. Our equations are valid for arbitrary ρ^*_Ω. Gyro-averaging can be done, a la **Bernstein-Catto-Helander-Sigmar** starting with these equations to obtaine reduced equations..
- We have derived the exact Newton-Lorentz equations **kinematically, for uniform rotation, and exactly from Hamilton's principle for arbitrary non-inertial, non-relativistic frame motions.** The results are remarkably simple and intuitively obvious, not to mention physically correct.
- Our derivations are **simpler, more general, and more natural** than previous ones of drift orbit theory. All reduced models follow from Whitham theory and rigorous asymptotics. The gyro-averaged drift orbit equations are **characteristics** of the gyrokinetic equation. Hence the latter can be correctly obtained if the former are deduced by uniformly valid perturbation expansions based on the exact Newton-Lorentz equations.
- The adiabatic invariance (to all orders) of μ^* properly defined as canonical conjugate action to gyrophase and the correct $P_\phi, \mathcal{E}$ invariances are **automatically guranteed in appropriate conditions.** Trapping, drifts, orbits, mirror points etc in $\mathbf{K}_f$ are all sensitive to both **magnitude and direction** of the equivalent fields $\mathbf{E}_*, \mathbf{B}_*$.
- In an ITB of an ST, $\frac{W_f}{\Omega_{ci}} \simeq \frac{10^6}{\rho_i \Omega_{ci}} \simeq 1,$ or possibly a sizable fraction thereof. Our derivations provide a useful starting point for analysing such situations.

9.1 Conclusions: I

- We have demonstrated, using **Hamilton's Principle** that:

1. The motion of the charged particle is exactly as in an inertial frame except that the Lorentz-Newton equations of motion involve the "equivalent frame fields" $\mathbf{E}_*, \mathbf{B}_*$ which depend both on the frame Maxwell fields (derived from $\Phi_f, \mathbf{A}$) and the effective fields due to the centrifugal and Coriolis terms generated by the frame flow, $\mathbf{u}_f$.
2. The effect of the Coriolis force in the equations of motion in a non-inertial frame is to introduce an additional vector potential, $\frac{2m}{Ze}\mathbf{u}_f$. If $\mathbf{u}_f$ is a pure toroidal rotation, depending only on R,it adds an equivalent vertical field to the magnetic field in a tokamak. If the angular velocity of rotation $\Omega = \frac{v_\phi}{R}$ is **uniform** in space, the effective vertical "Coriolis" field is also uniform and $B_{\rm Cor} \simeq \rho^*_\Omega B = \frac{2\Omega}{\Omega_{cI}} B$, relative to the total tokamak field B.
3. The flow (if steady) modifies the electrostatic potential. The frame potential $\Phi_f = \Phi_{\rm lab} - \mathbf{A}.\mathbf{u}_f$ is seen by electrons to a good approximation. The equivalent potential Φ_* differs from the frame potential due to centrifugal potential, $\frac{m}{Ze}\frac{\mathbf{u}_f.\mathbf{u}_f}{2}$.
4. Only the vorticity of the frame flow $\mathbf{W}_f = \nabla \times \mathbf{u}_f$ contributes to the equivalent $\mathbf{B}_*$ field, whilst the compressibility and flow kinetic energy both contribute to the equivalent potential Φ_*.

9.2 Conclusions: II

- The frame **Vlasov, Fokker-Planck/Langevin, drift-orbit, drift-kinetic and gyrokinetic equations** are formally derived exactly as they would be in an inertial frame, except that the equivalent fields must be used, as noted.
- A two-fluid, dissipationless, azimuthally symmetric equilibrium model first investigated by **Thyagaraja and McClements** is used to obtain the frame (equivalent) potentials $\Phi_*, \mathbf{A}_*$.
- Estimates and predictions are made for JET and MAST conditions.
- Our theory is valid (except for the drift orbit equations) for any value of the rotation parameter ρ^*_Ω and arbitrary non-relativistic frame flows, including strongly non-neoclassical poloidal flows with significant shear and provide the frame work for the study of electrodynamics of charged particles in such non-inertial frames.
- Our analysis is, in principle, both experimentally and computationally testable under tokamak plasma conditions, and can be readily extended to include relativistic flows and spacetime curvature, thereby making it applicable to extreme astrophysical plasma environments, such as the magnetospheres of rapidly-rotating pulsars and Kerr blackholes.

10.1 Suggested further reading

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Appendix A.1: electron physics relations

- $T_{e,i}$ are assumed "flux functions" (fast parallel transport).[Nb. T_i may not be, when there are large flows present].
- Two-fluid mass conservation imply: $\Theta_{i,e}(R,Z)$ such that,

$$
\begin{aligned}
n\mathbf{v}_{i,e} &= (-\frac{1}{R}\frac{\partial\Theta_{i,e}}{\partial Z})\mathbf{e}_R + nv_\phi^{i,e}\mathbf{e}_\phi + (\frac{1}{R}\frac{\partial\Theta_{i,e}}{\partial R})\mathbf{e}_Z, \\
&= \nabla\Theta_{i,e} \times \nabla\phi + nRv_\phi^{i,e}\nabla\phi,
\end{aligned}
$$

Electron momentum balance:

$$
\begin{aligned}
0 &= -\nabla nT_e + en\nabla\Phi - en\mathbf{v}_e \times \mathbf{B} \qquad \text{implies} \\
T_e \ln n &= e\Phi + h_e(\Psi) \\
T_e' \ln n &= (T_e + h_e)' + \frac{ev_\phi^e}{R} - \left[\frac{eB_\phi\Theta_e'}{nR}\right]
\end{aligned}
$$

A.2: Two-fluid equilibria

- Electron physics yields the relation upon setting $2T = T_e + T_i$:

$$\frac{T_e'}{2T}\ln p_{\rm tot} - \frac{T_e'}{2T}\ln 2T - \frac{(T_e + h_e)'}{2T} + \frac{\partial \ln p_{\rm tot}}{\partial \Psi} \quad = \quad \frac{e\Omega_\phi^i}{2T}.$$

- The system can be closed in the important special case when $T_{i,e}$ are **both** flux functions. Differentiate w.r.t R^2 and eliminate $p_{\rm tot}$:

$$\frac{T_e'}{T}\frac{m_i(\Omega_\phi^i)^2}{8T} + \frac{\partial}{\partial \Psi}\left(\frac{m_i(\Omega_\phi^i)^2}{4T}\right) \quad = \quad \frac{e}{2T}\frac{\partial \Omega_\phi^i}{\partial R^2}$$

- New variables: thus let, $M_\phi^2 = \frac{m_i(R_*\Omega_\phi^i)^2}{4T}; y = (\frac{R}{R_*})^2$, where R_* is a “typical”, fixed major radius, and define $\xi(\Psi)$:

$$\xi(\Psi) \quad = \quad \exp\left[\int_{\Psi_0}^{\Psi}\frac{T_e' d\psi}{2T}\right]$$

A.3: Two-fluid consistency equation

- Reduce the equation to,

$$\frac{\partial}{\partial\Psi}\left[\xi M_\phi^2\right] \quad = \quad \frac{\partial}{\partial y}\left[\frac{e\xi M_\phi}{\sqrt{m_i T}}\right].$$

- Setting $z = \xi^{1/2}M_\phi$ and $x = \int_{\Psi_0}^{\Psi}\frac{e}{R^*\sqrt{m_i T}}\xi^{1/2}(\psi)d\psi$ we obtain the non-dimensional, nonlinear p.d.e:

$$\frac{\partial z^2}{\partial x} \quad = \quad \frac{\partial z}{\partial y}$$

- This equation can be solved **exactly by Charpit's method**! The remarkably simple "complete integral" depends upon two arbitrary constants:

$$z \quad = \quad \frac{c-x}{2(y+d)}$$

A.4: Two-fluid solutions

- The solubility condition can be satisfied in an infinite number of ways. Consider only physically transparent cases: regard Ψ, R^2 as independent variables. Introducing $V(\Psi, R^2)$ such that:

$$\frac{m_i(\Omega_\phi^i)^2}{4T} = \frac{\partial V}{\partial R^2}$$

Then we obtain:

$$\begin{aligned} p &= P^*(\Psi)\exp\left[V(\Psi, R^2)\right] \\ \nabla p &= \frac{\partial p}{\partial R^2}\nabla R^2 + \frac{\partial p}{\partial \Psi}\nabla\Psi \\ \frac{\partial p}{\partial \Psi} &= \frac{j_\phi}{R} - \frac{FF'}{\mu_0 R^2} \end{aligned}$$

Observing that p, n, Φ are not flux functions, we obtain the **Two-fluid Grad-Shafranov Equation:**

$$\left[\frac{\partial^2\Psi}{\partial Z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right)\right] = -R^2\mu_0\frac{\partial p}{\partial \Psi} - FF'$$

A.5: Two-fluid equilibria: general solution

- In the general case the variation with Ψ is complicated and depends upon the temperature profiles. However the $(\frac{R}{R^*})^2 = y$ dependence of p, n is **uniquely determined.**

$$
\begin{aligned}
A(\Psi) &= \frac{c - x(\Psi)}{2\xi^{1/2}(\Psi)} \\
p &= P^*(\Psi) \exp\left[-\frac{A^2(\Psi)}{(y+d)}\right] \\
n &= N^*(\Psi) \exp\left[-\frac{A^2(\Psi)}{(y+d)}\right] \\
\frac{e\Phi}{T_e} &= \frac{e\Phi^*(\Psi)}{T_e} - \frac{A^2(\Psi)}{(y+d)} \\
\Omega_\phi^i &= \left(\frac{4T(\Psi)}{m_i}\right)^{1/2} \frac{A(\Psi)}{y+d}
\end{aligned}
$$

A.6: Special solutions: rigid rotation

- For Thyagaraja-McClements general solution reduces in two limiting cases controlled by a parameter d. If $d \to \infty$ with Ω_ϕ^i is a **given flux-function** and putting $P^*(\Psi) = 2N^*(\Psi)T; m \simeq m_i; m_e \ll m_i$,

$$\frac{m_i(\Omega_\phi^i)^2}{4T} = \frac{m_i(\Omega_{\phi 0}^i)^2}{4T_0} \exp\left[-\int_{\Psi_0}^{\Psi} \frac{T_e' d\psi}{2T}\right]$$

$$p = P^*(\Psi) \exp\left(\frac{m_i(\Omega_\phi^i R)^2}{4T}\right)$$

$$n = N^*(\Psi) \exp\left(\frac{m_i(\Omega_\phi^i R)^2}{4T}\right)$$

$$\frac{e\Phi}{T_e} = \frac{e\Phi^*(\Psi)}{T_e} + \frac{m_i(\Omega_\phi^i R)^2}{4T}$$

Principal features:

1. The angular velocity is flux-function related to **temperature**
2. Φ is **not** a flux function, but "feels" the centrifugal force
3. Density, pressure, potential **larger on the outboard side.**

A.7: Special solutions: Keplerian rotation

- A second special case $d = 0$: mechanical toroidal angular momentum per ion is a **flux-function, i.e. the ions rotate toroidally in a Keplerian manner.** Density, pressure are higher on the **outboard side** just as in rigid rotation! $\Omega_\phi^i = \lambda(\Psi)/R^2; P_\phi^i = m_i \Omega_\phi^i R^2 + e\Psi$ **flux-function!** Then: $\lambda(\Psi) = \frac{1}{\mu(\Psi)}\left[\lambda_0 - \frac{e}{m_i}\int_{\Psi_0}^{\Psi}\mu(\psi)d\psi\right]; \mu(\Psi) = \exp\left[-\int_{\Psi_0}^{\Psi}\frac{T_i' d\psi}{4T}\right]$

$$
\begin{aligned}
p &= P^*(\Psi)\exp\left(-\frac{m_i\lambda^2}{4TR^2}\right) \\
n &= N^*(\Psi)\exp\left(-\frac{m_i\lambda^2}{4TR^2}\right) \\
\frac{e\Phi}{T_e} &= \frac{e\Phi^*(\Psi)}{T_e} - \frac{m_i\lambda^2}{4TR^2}
\end{aligned}
$$

Comparison with (single-fluid) ideal MHD:

1. The angular velocity is a flux-function, but explicitly related to temperature
2. The electrostatic potential, Φ is **not** a flux function, but "feels" the centrifugal force; **Hinton-Wong, Wesson** obtained special cases.

- We have thus obtained, $\Phi_{\text{lab}} = \Phi$ and related it to $\mathbf{v}_i = \mathbf{u}_f = \Omega_\phi^i(R, \Psi)R^2\nabla\phi$. It is now straight forward to work out Φ_f and proceed with the orbit/Vlasov/gyrokinetics in $\mathbf{K}_f$. If $\lambda(\Psi)$ is constant, flow is **irrotational**, hence $\mathbf{B}_* = \mathbf{B}$!

A.8: Does $\mathbf{u}_f$ have to be incompressible?

- In general, $\mathbf{u}_f$ may not be divergence-free. Quite generally, by the "Helmholtz" decomposition of a vector field: $\mathbf{u}_f = \mathbf{u}_f^* + \nabla\lambda_f$,

$$\nabla.\mathbf{u}_f^* = 0 \quad (69)$$

$$\nabla\times\mathbf{u}_f^* = \mathbf{W}_f \quad (70)$$

$$\nabla^2\lambda_f = \nabla.\mathbf{u}_f \quad (71)$$

Then,

$$L^* = \frac{1}{2}m\mathbf{V}^2 + Ze\mathbf{V}.[\mathbf{A}+(\frac{2m}{Ze})\frac{\mathbf{u}_f^*}{2}] - Ze[\Phi_f - \frac{m}{Ze}\frac{[(u_f^*)^2 + (\nabla\lambda_f)^2]}{2} - \frac{m}{Ze}\nabla.(\lambda_f\mathbf{u}_f^*)] \quad (72)$$

The **canonical momenta** depend only upon $\mathbf{u}_f^*$ (also implied by **gauge-invariance!)** Now, if $\mathbf{A}_{\rm lab}$ satisfies the Coulomb gauge, then $\mathbf{A}_*$ also satisfies it. The equivalent electrostatic potential is then,

$$\Phi_* = \Phi_f - \frac{m}{Ze}\frac{[(u_f^*)^2 + (\nabla\lambda_f)^2]}{2} - \frac{m}{Ze}\nabla.(\lambda_f\mathbf{u}_f^*) \quad (73)$$

Only solenoidal fields contribute to $\mathbf{B}_*$; **both irrotational and solenoidal fields** contribute to Φ_*

A.9: A possible novel application

- **An application of potential interest in both tokamak and astrophysics:** $\mathbf{u}_f(\mathbf{x},t)$ can be *arbitrary* (subject only to non-relativistic values)! Therefore, **define**, for given, $\mathbf{E}_{\rm lab}(\mathbf{x},t), \mathbf{B}_{\rm lab}(\mathbf{x},t)$, the frame velocity $\mathbf{u}_f(\mathbf{x},t)$ as the solution of:

$$m[\frac{\partial \mathbf{u}_f}{\partial t} + \mathbf{u}_f.\nabla \mathbf{u}_f] \quad = \quad Ze[\mathbf{E}_{\rm lab} + \mathbf{u}_f \times \mathbf{B}_{\rm lab}] \tag{74}$$

This represents the fluid velocity of a species m, Ze which is cold and dissipationless. Set, $\mathbf{v} = \mathbf{u}_f + \mathbf{V}$, and apply our results using Eqs.(37-39) to **co-evolve** $\mathbf{u}_f, V(t)$ obtaining the trajectory of the particle in the "lab. space" using,

$$\frac{d\mathbf{x}}{dt} \quad = \quad \mathbf{u}_f(\mathbf{x},t) + \mathbf{V}$$

The velocity $\mathbf{u}_f$ "feels" the $\mathbf{E}_{\rm lab} \times \mathbf{B}_{\rm lab}$ drift and inertial drifts associated with it, precisely as in ideal MHD! The "$\mathbf{E} \times \mathbf{B}$"-frame thus defined might provide a better physical understanding and ease of performing gyrokinetic averages than the lab. frame. If $\mathbf{E}_{\rm lab}, \mathbf{B}_{\rm lab}$ have large spatio-temporal gradients locally, this would strongly influence (for the ions) the equivalent fields, $\mathbf{E}_*, \mathbf{B}_*$. cf. Eqs.(38-39).

- Consequently, it might be easier to gain a better and more intuitive understanding of charged particle dynamics in such conditions: shape of things to come?!

A.10 Fluid equations in non-inertial frames

- Starting with the **Fokker-Planck-Landau** equation in $\mathbf{K}_f$, we may construct a fluid model for any species, m, Ze. If we set $\mathbf{v}_Z$ to be the fluid velocity of the species and restrict ourselves to time-independent frame flows $\mathbf{u}_f$ and source-free dissipationless steady equations of motion, we obtain:

$$\begin{aligned}
\nabla . n_Z \mathbf{v}_Z &= 0 \\
m n_Z \mathbf{W} \times \mathbf{v}_Z &= -\nabla p_Z - m n_Z \nabla [\frac{\mathbf{v}_Z . \mathbf{v}_Z}{2}] + Zen_Z[-\nabla \Phi_* + \mathbf{v}_Z \times \mathbf{B}_*] \\
\frac{3}{2} n_Z \mathbf{v}_Z . \nabla T_Z + n_Z T_Z \nabla . \mathbf{v}_Z &= 0
\end{aligned}$$

where, $\mathbf{W} = \nabla \times \mathbf{v}_Z; p_Z = n_Z T_Z, \mathbf{E}_*, \mathbf{B}_*$ are defined as usual by Eqs.(35-36) and Eqs.(39-40). If the collision operators and sources are included we can carry out classical/neo-classical analysis and determine the stress tensor and the heat-flux vector to complete the transport equations with source terms in the non-inertial frame. $\hat{\Phi} = \Phi + (\frac{m}{Ze})\frac{v_Z^2}{2}; \hat{\mathbf{B}} = \mathbf{B}_* + (\frac{m}{Ze})\mathbf{W}$ implying the pressure balance relation:

$$\frac{1}{Zen_Z} \nabla p_Z = -\nabla \hat{\Phi} + \mathbf{v}_Z \times \hat{\mathbf{B}} \qquad (75)$$