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Tutorial on the theory of plasma turbulence (Part I)

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References

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Vlasov-Poisson equation

$$\left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial x} + \frac{e_a E}{m_a}\frac{\partial}{\partial v}\right)\, f_a = 0,$$

$$\frac{\partial E}{\partial x} = 4\pi\hat{n}\sum_a e_a \int dv\; f_a.$$

Separation into Average and Fluctuation

$$f_a(x,v,t) = F_a(v,t) + \delta f_a(x,v,t),$$

$$E(x,t) = \delta E(x,t).$$

Rewrite the equations

$$\left(\frac{\partial}{\partial t} + \frac{e_a}{m_a}\,\delta E\,\frac{\partial}{\partial v}\right)\, F_a + \left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial x} + \frac{e_a}{m_a}\,\delta E\,\frac{\partial}{\partial v}\right)\, \delta f_a = 0,$$

$$\frac{\partial}{\partial x}\,\delta E = 4\pi\hat{n}\sum_a e_a \int dv\; \delta f_a.$$

Random phase approximation

$$< \delta f_a >= 0, \qquad < \delta E >= 0.$$

Upon averaging

$$\left(\frac{\partial}{\partial t} + \frac{e_a}{m_a}\,\delta E\,\frac{\partial}{\partial v}\right)\,F_a + \left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial x} + \frac{e_a}{m_a}\,\delta E\,\frac{\partial}{\partial v}\right)\,\delta f_a = 0,$$

we have

$$\boxed{\frac{\partial\,F_a}{\partial t} = -\frac{e_a}{m_a}\frac{\partial}{\partial v}\,\langle\delta f_a\,\delta E\rangle.}$$

Insert the above to original equation

$$\boxed{\left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial x}\right)\,\delta f_a = -\frac{e_a}{m_a}\,\delta E\,\frac{\partial F_a}{\partial v} - \frac{e_a}{m_a}\frac{\partial}{\partial v}\,\left(\delta f_a\,\delta E - < \delta f_a\,\delta E >\right).}$$

Fourier-Laplace transformation:

$$\delta f_a(x,v,t) = \int dk \int_L d\omega \, \delta f^a_{k\omega}(v,t) \, e^{ikx-i\omega t},$$
$$\delta f^a_{k\omega}(v,t) = \frac{1}{(2\pi)^2} \int dx \int_0^\infty dt \, \delta f_a(x,v,t) \, e^{-ikx+i\omega t},$$

$$\delta E(x,t) = \int dk \int_L d\omega \, \delta E_{k\omega}(t) \, e^{ikx-i\omega t},$$
$$\delta E_{k\omega}(t) = \frac{1}{(2\pi)^2} \int dx \int_0^\infty dt \, \delta E(x,t) \, e^{-ikx+i\omega t},$$

where the Landau integration path L has $i\sigma$ ($\sigma > 0$ and $\sigma \to 0$).

The slow time-dependence of spectral amplitudes, $\delta f^a_{k\omega}(v,t)$ and $\delta E_{k\omega}(t)$, is a short-cut approach (a sleazy way to avoid rigorous multiple-time step analysis – for a full-fledged multiple-time scale analysis, see Davidson, 1972).

Formal equations are now expressed as follows:

$$\delta E_{k\omega}(t) = -i \sum_a \frac{4\pi \hat{n} e_a}{k} \int dv \, \delta f^a_{k\omega}(v,t),$$

$$\frac{\partial \, F_a(v,t)}{\partial t} = -\frac{e_a}{m_a} \int dk \int d\omega \, \frac{\partial}{\partial v} < \delta E_{-k,-\omega}(t) \, \delta f^a_{k\omega}(v,t) >,$$

$$\left(\omega - kv + i \frac{\partial}{\partial t}\right) \delta f^a_{k\omega}(v,t) = -i \, \frac{e_a}{m_a} \, \delta E_{k\omega}(t) \, \frac{\partial F_a(v,t)}{\partial v}$$
$$- \; i \, \frac{e_a}{m_a} \int dk' \int d\omega' \, \frac{\partial}{\partial v} \left[\delta E_{k'\omega'}(t) \, \delta f^a_{k-k' \, \omega-\omega'}(v,t) \right.$$
$$\left. - < \delta E_{k'\omega'}(t) \, \delta f^a_{k-k',\omega-\omega'}(v,t) > \right].$$

New definition for ω,

$$\omega \rightarrow \omega + i \, \partial/\partial t.$$

This is another short-cut trick.

Short-hand notations:

$$
\begin{aligned}
K &= (k,\omega), \qquad E_K = \delta E_{k\omega}, \qquad f_K = \delta f^a_{k\omega}, \\
F &= F_a, \qquad \int dK = \int dk \int d\omega, \\
g_K &= -i\frac{e_a}{m_a}\frac{1}{\omega - kv + i0}\frac{\partial}{\partial v}.
\end{aligned}
$$

Equation for the perturbed particle distribution:

$$
f_K = g_K\, F\, E_K + \int dK'\, g_K\,(E_{K'}\, f_{K-K'} - < E_{K'}\, f_{K-K'} >).
$$

Iteration:

$$
f_K = f_K^{(1)} + f_K^{(2)} + \cdots, \qquad (f_K^{(n)} \propto E_K^n).
$$

$$
\begin{aligned}
f_K^{(1)} &= g_K\, F\, E_K, \\
f_K^{(2)} &= \int dK'\, g_K\,(E_{K'}\, f^{(1)}_{K-K'} - < E_{K'}\, f^{(1)}_{K-K'} >) \\
&= \int dK'\, g_K\, g_{K-K'}\, F\,(E_{K'}\, E_{K-K'} - < E_{K'}\, E_{K-K'} >).
\end{aligned}
$$

Early monographs calculate up to $f_K^{(3)}$, but $f_K^{(3)}$ is unnecessary!

Simplified notations:

$$\sum_{1+2=K} = \int dK_1 \int dK_2 \, \delta(K_1 + K_2 - K),$$

$$E_1 = E_{K_1}, \quad E_2 = E_{K_2}, \qquad g_1 = g_{K_1}, \quad g_2 = g_{K_2}.$$

Iterative solution for f_K

$$f_K = g_K \, F \, E_K + \sum_{1+2=K} g_K \, g_2 \, F \, (E_1 E_2 - < E_1 E_2 >)$$

Symmetrized version

$$\boxed{f_K = g_K \, F \, E_K + \sum_{1+2=K} \frac{1}{2} g_{1+2} \, (g_1 + g_2) \, F \, (E_1 \, E_2 - < E_1 \, E_2 >).}$$

Insert f_K to the perturbed Poisson equation,

$$E_K = -i \sum_a \frac{4\pi \widehat{n} e_a}{k} \int dv \, f_K.$$

$$
\begin{aligned}
E_K &= -i\sum_a \frac{4\pi\hat{n}e_a}{k}\int dv\; f_K, \\
f_K &= g_K\, F\, E_K + \sum_{1+2=K} \frac{1}{2}\, g_{1+2}\,(g_1+g_2)\, F\,(E_1\, E_2 - < E_1\, E_2 >).
\end{aligned}
$$

The above can be re-written as

$$
\begin{aligned}
0 &= \left(1 + i\sum_a \frac{4\pi e_a\hat{n}}{k}\int dv\; g_K\, F\right) E_K \\
&\quad + i \sum_{1+2=K} \frac{1}{2}\sum_a \frac{4\pi e_a\hat{n}}{k}\int dv\; g_{1+2}\,(g_1+g_2)\, F\;(E_1 E_2 - < E_1 E_2 >)\,.
\end{aligned}
$$

Linear and nonlinear susceptibility response functions,

$$
\boxed{
\begin{aligned}
\epsilon(K) &= 1 + \sum_a i\,\frac{4\pi e_a\hat{n}}{k}\int dv\; g_K\, F, \\
\chi_a^{(2)}(1|2) &= \frac{i}{2}\frac{4\pi e_a\hat{n}}{k_1+k_2}\int dv\; g_{1+2}\,(g_1+g_2)\, F.
\end{aligned}
}
$$

$$
\begin{aligned}
\epsilon(K) &= 1 + \sum_a \frac{\omega_{pa}^2}{k} \int dv \, \frac{\partial F_a / \partial v}{\omega - kv + i0}, \\
\chi_a^{(2)}(1|2) &= -\frac{i}{2} \frac{e_a}{m_a} \frac{\omega_{pa}^2}{k_1 + k_2} \int dv \, \frac{1}{\omega_1 + \omega_2 - (k_1 + k_2)\, v + i0} \\
&\quad \times \frac{\partial}{\partial v} \left[\left(\frac{1}{\omega_1 - k_1 v + i0} + \frac{1}{\omega_2 - k_2 v + i0} \right) \frac{\partial F_a}{\partial v} \right].
\end{aligned}
$$

$$
0 = \epsilon(K)\, E_K + \sum_{1+2=K} \chi^{(2)}(1|2) \; (E_1 E_2 - < E_1 E_2 >) \, .
$$

Multiply $E_{K'}$ and take ensemble average

$$
0 = \epsilon(K) < E_K E_{K'} > + \sum_{1+2=K} \chi^{(2)}(1|2) < E_1 E_2 E_{K'} > .
$$

Homogeneous turbulence,

$$< E(x,t)\, E(x',t') >= D(|x-x'|; t, t'),$$

Stationary turbulence,

$$< E(x,t)\, E(x',t') >= D(x, x'; |t-t'|).$$

Homogeneous and stationary turbulence,

$$< E(x,t)\, E(x',t') >=< E^2 >_{x-x',t-t'} .$$

The spectral representation

$$< E_{k\omega}\, E_{k'\omega'} >= \delta(k+k')\, \delta(\omega+\omega') < E^2 >_{k\omega} .$$

$$0 = \epsilon(K) < E_K E_{K'} > + \sum_{1+2=K} \chi^{(2)}(1|2) < E_1 E_2 E_{K'} >,$$

$\Rightarrow$

$$0 = \epsilon(K) < E^2 >_K + \sum_{1+2=K} \chi^{(2)}(1|2) < E_1 E_2 E_{-K} > .$$

Three-Body Cumulant and Closure of Hierarchy: If E_K is linear eigenmode,

$$\epsilon(K)\,E_K = 0.$$

then by definition $< E_1 E_2 E_{-K} >= 0$. But we are dealing with nonlinear theory, where

$$0 = \epsilon(K)\,E_K + \sum_{1+2=K} \chi^{(2)}(1|2)\,\left(E_1 E_2 - < E_1 E_2 >\right).$$

Let us write $E_K = E_K^{(0)} + E_K^{(1)}$, where $E_K^{(0)}$ satisfies $\epsilon(K)\,E_K^{(0)} = 0$. Then

$$\epsilon(K)\,(E_K^{(0)} + E_K^{(1)})$$

$$= -\int dK' \chi^{(2)}(K'|K-K')\,(E_{K'}^{(0)} E_{K-K'}^{(0)} - < E_{K'}^{(0)} E_{K-K'}^{(0)} >).$$

By definition $\epsilon(K)\,E_K^{(0)} = 0$. Thus,

$$E_K^{(1)} = -\frac{1}{\epsilon(K)} \int dK' \chi^{(2)}(K'|K-K')\left(E_{K'}^{(0)} E_{K-K'}^{(0)} - < E_{K'}^{(0)} E_{K-K'}^{(0)} >\right).$$

Three-body correlation,

$$
\begin{aligned}
< E_{K'}E_{K-K'}E_{-K} >=& < (E_{K'}^{(0)} + E_{K'}^{(1)}) \\
&\times(E_{K-K'}^{(0)} + E_{K-K'}^{(1)})(E_{-K}^{(0)} + E_{-K}^{(1)}) > \\
\\
&= \underbrace{< E_{K'}^{(0)} E_{K-K'}^{(0)} E_{-K}^{(0)} >}_{\begin{array}{c}\| \\ 0\end{array}} + < E_{K'}^{(1)} E_{K-K'}^{(0)} E_{-K}^{(0)} > \\
&+ < E_{K'}^{(0)} E_{K-K'}^{(1)} E_{-K}^{(0)} > + < E_{K'}^{(0)} E_{K-K'}^{(0)} E_{-K}^{(1)} >,
\end{aligned}
$$

$$\Uparrow$$

$$E_K^{(1)} = -\frac{1}{\epsilon(K)} \int dK' \chi^{(2)}(K'|K-K') \left(E_{K'}^{(0)} E_{K-K'}^{(0)} - < E_{K'}^{(0)} E_{K-K'}^{(0)} > \right)$$

drop the superscript (0).

$$
\begin{aligned}
&< E_{K'}E_{K-K'}E_{-K} >= -\frac{1}{\epsilon(K')}\int dK''\chi^{(2)}(K''|K'-K'') \\
&\times \left(\left\langle E_{K''}E_{K'-K''}E_{K-K'}E_{-K}\right\rangle - \left\langle E_{K''}E_{K'-K''}\right\rangle\left\langle E_{K-K'}E_{-K}\right\rangle\right) \\
&\qquad - \frac{1}{\epsilon(K-K')}\int dK''\chi^{(2)}(K''|K-K'-K'') \\
&\times \left(\left\langle E_{K''}E_{K-K'-K''}E_{K'}E_{-K}\right\rangle - \left\langle E_{K''}E_{K-K'-K''}\right\rangle\langle E_{K'}E_{-K}\rangle\right) \\
&\qquad - \frac{1}{\epsilon(-K)}\int dK''\chi^{(2)}(-K''|-K+K'') \\
&\times \left(\left\langle E_{K'}E_{K-K'}E_{-K''}E_{-K+K''}\right\rangle - \left\langle E_{K'}E_{K-K'}\right\rangle\left\langle E_{-K''}E_{-K+K''}\right\rangle\right).
\end{aligned}
$$

The closure: for homogeneous and stationary turbulence,

$$
\begin{aligned}
< E_K E_{K'} E_{K''} E_{K'''} >= \delta(K+K'+K''+K''') \\
\times \Big[\delta(K+K') < E^2 >_K < E^2 >_{K''} \\
+ \delta(K+K'') < E^2 >_K < E^2 >_{K'} \\
+ \delta(K'+K'') < E^2 >_K < E^2 >_{K'} \\
+ < E^4 >_{K;K+K';K+K'+K''}\Big].
\end{aligned}
$$

Ignoring irreducible four-body cumulant,

$$< E_{K'} E_{K-K'} E_{-K} >= -\frac{\chi^{(2)}(-K+K'|K)}{\epsilon(K')} < E^2 >_{K-K'} < E^2 >_K$$

$$- \frac{\chi^{(2)}(K|-K+K')}{\epsilon(K')} < E^2 >_{K-K'} < E^2 >_K$$

$$- \frac{\chi^{(2)}(-K'|K)}{\epsilon(K-K')} < E^2 >_{K'} < E^2 >_K$$

$$- \frac{\chi^{(2)}(K|-K')}{\epsilon(K-K')} < E^2 >_{K'} < E^2 >_K$$

$$- \frac{\chi^{(2)}(-K'|-K+K')}{\epsilon(-K)} < E^2 >_{K'} < E^2 >_{K-K'}$$

$$- \frac{\chi^{(2)}(-K+K'|-K')}{\epsilon(-K)} < E^2 >_{K'} < E^2 >_{K-K'} .$$

Symmetry Properties Associated With Nonlinear Susceptibility:

$$\chi^{(2)}(1|2) = -\frac{i}{2}\sum_a \frac{e_a}{m_a}\frac{\omega_{pa}^2}{k_1+k_2}\int dv\, \frac{1}{\omega_1+\omega_2-(k_1+k_2)\,v+i0}$$
$$\times \frac{\partial}{\partial v}\left[\left(\frac{1}{\omega_1-k_1 v+i0}+\frac{1}{\omega_2-k_2 v+i0}\right)\frac{\partial F_a}{\partial v}\right].$$

Symmetry properties:

$$\chi^{(2)}(-1|-2) = \chi^{(2)*}(1|2),$$
$$\chi^{(2)}(1|2) = \chi^{(2)}(2|1).$$

Partial integrations,

$$\chi^{(2)}(1|2) = -\frac{i}{2}\sum_a \frac{e_a}{m_a}\frac{\omega_{pa}^2}{k_1+k_2}\int dv\, \frac{1}{\omega_1+\omega_2-(k_1+k_2)\,v+i0}$$
$$\times \frac{\partial}{\partial v}\left[\left(\frac{1}{\omega_1-k_1v+i0}+\frac{1}{\omega_2-k_2v+i0}\right)\frac{\partial F_a}{\partial v}\right]$$

$$= \frac{-i}{2}\sum_a \frac{e_a}{m_a}\omega_{pa}^2\int dv\, \frac{F_a}{(\omega_1-k_1v)(\omega_2-k_2v)[\omega_1+\omega_2-(k_1+k_2)\,v]}$$
$$\times \left(\frac{k_1}{\omega_1-k_1v}+\frac{k_2}{\omega_2-k_2v}+\frac{k_1+k_2}{\omega_1+\omega_2-(k_1+k_2)\,v}\right).$$

Useful symmetry relation:

$$\chi^{(2)}(1|2) = \chi^{(2)}(2|1) = -\chi^{(2)}(1+2|-2).$$

$$< E_{K'}E_{K-K'}E_{-K} >= \frac{2\chi^{(2)}(K'|K-K')}{\epsilon(K')} < E^2 >_{K-K'} < E^2 >_K + \frac{2\chi^{(2)}(K'|K-K')}{\epsilon(K-K')} < E^2 >_{K'} < E^2 >_K - \frac{2\chi^{(2)*}(K'|K-K')}{\epsilon^*(K)} < E^2 >_{K'} < E^2 >_{K-K'} .$$

$$0 = \epsilon(K) < E^2 >_K + 2\int dK' \left(\frac{\{\chi^{(2)}(K'|K-K')\}^2}{\epsilon(K')} < E^2 >_{K-K'} < E^2 >_K + \frac{\{\chi^{(2)}(K'|K-K')\}^2}{\epsilon(K-K')} < E^2 >_{K'} < E^2 >_K - \frac{|\chi^{(2)}(K'|K-K')|^2}{\epsilon^*(K)} < E^2 >_{K'} < E^2 >_{K-K'} \right).$$

Reintroduce the slow time dependence,

$$(\omega - \mathbf{k} \cdot \mathbf{v} + i\partial/\partial t)^{-1}.$$

This leads to

$$\epsilon(k,\omega) < E^2 >_{k\omega} \longrightarrow \epsilon\left(k, \omega + i\frac{\partial}{\partial t}\right) < E^2 >_{k\omega}$$
$$\rightarrow \left(\epsilon(k,\omega) + \frac{i}{2}\frac{\partial\epsilon(k,\omega)}{\partial\omega}\frac{\partial}{\partial t}\right) < E^2 >_{k\omega} .$$

$$\begin{aligned} 0 \; = \; & \frac{i}{2}\frac{\partial\epsilon(k,\omega)}{\partial\omega}\frac{\partial}{\partial t} < E^2 >_{k\omega} + \epsilon(k,\omega) < E^2 >_{k\omega} \\ & + 2\int dk' \int d\omega' \left(\frac{\{\chi^{(2)}(k',\omega'|k-k',\omega-\omega')\}^2}{\epsilon(k',\omega')} \left\langle E^2 \right\rangle_{k-k',\omega-\omega'} \left\langle E^2 \right\rangle_{k\omega} \right. \\ & + \frac{\{\chi^{(2)}(k',\omega'|k-k',\omega-\omega')\}^2}{\epsilon(k-k',\omega-\omega')} \left\langle E^2 \right\rangle_{k'\omega'} \left\langle E^2 \right\rangle_{k\omega} \\ & \left. - \frac{|\chi^{(2)}(k',\omega'|k-k',\omega-\omega')|^2}{\epsilon^*(k,\omega)} \left\langle E^2 \right\rangle_{k'\omega'} \left\langle E^2 \right\rangle_{k-k',\omega-\omega'} \right) . \end{aligned}$$

Dispersion relation,

$$\mathrm{Re}\,\epsilon(k,\omega) < E^2 >_{k\omega} = 0.$$

Wave kinetic equation,

$$\begin{aligned}
\frac{\partial}{\partial t} < E^2 >_{k\omega} = & -\frac{2\,\mathrm{Im}\,\epsilon(k,\omega)}{\partial\,\mathrm{Re}\,\epsilon(k,\omega)/\partial\omega} < E^2 >_{k\omega} \\
& - \frac{4}{\partial\,\mathrm{Re}\,\epsilon(k,\omega)/\partial\omega}\,\mathrm{Im}\int dk' \int d\omega' \\
& \times \left[\{\chi^{(2)}(k',\omega'|k-k',\omega-\omega')\}^2 \left(\frac{< E^2 >_{k-k',\omega-\omega'}}{\epsilon(k',\omega')} \right.\right. \\
& \left. + \frac{< E^2 >_{k'\omega'}}{\epsilon(k-k',\omega-\omega')} \right) < E^2 >_{k\omega} \\
& \left. - \frac{|\chi^{(2)}(k',\omega'|k-k',\omega-\omega')|^2}{\epsilon^*(k,\omega)} < E^2 >_{k'\omega'} < E^2 >_{k-k',\omega-\omega'} \right].
\end{aligned}$$

Particle kinetic equation,

$$\frac{\partial F_a}{\partial t} = -\frac{e_a}{m_a}\int dK\, \frac{\partial}{\partial v} < E_{-K}\, f_K^a >,$$

where

$$f_K = f_K^{(1)} + f_K^{(2)} + \cdots .$$

For evaluation of particle kinetic equation we only use $f_K^{(1)}$,

$$f_K = f_K^{(1)} = g_K\, F\, E_K = -i\,\frac{e_a}{m_a}\frac{E_{k\omega}}{\omega - kv}\frac{\partial F_a}{\partial v}.$$

Why? Because higher-order terms lead to divergence!

$$\boxed{\frac{\partial F_a}{\partial t} = \mathrm{Re}\, i\, \frac{e_a^2}{m_a^2}\frac{\partial}{\partial v}\int dk \int d\omega\, \frac{< E^2 >_{k\omega}}{\omega - kv + i0}\frac{\partial F_a}{\partial v}.}$$

$$\mathrm{Re}\,\epsilon(k,\omega) < E^2 >_{k\omega} = 0.$$

$$\frac{\partial F_a}{\partial t} = \mathrm{Re}\, i \frac{e_a^2}{m_a^2} \frac{\partial}{\partial v} \int dk \int d\omega \, \frac{< E^2 >_{k\omega}}{\omega - kv + i0} \frac{\partial F_a}{\partial v}.$$

$$\frac{\partial}{\partial t} < E^2 >_{k\omega} = - \frac{2\,\mathrm{Im}\,\epsilon(k,\omega)}{\partial\, \mathrm{Re}\,\epsilon(k,\omega)/\partial\omega} < E^2 >_{k\omega}$$
$$- \frac{4}{\partial\, \mathrm{Re}\,\epsilon(k,\omega)/\partial\omega} \mathrm{Im} \int dk' \int d\omega'$$
$$\times \left[\{\chi^{(2)}(k',\omega'|k-k',\omega-\omega')\}^2 \left(\frac{< E^2 >_{k-k',\omega-\omega'}}{\epsilon(k',\omega')} \right.\right.$$
$$\left. + \frac{< E^2 >_{k'\omega'}}{\epsilon(k-k',\omega-\omega')} \right) < E^2 >_{k\omega}$$
$$\left. - \frac{|\chi^{(2)}(k',\omega'|k-k',\omega-\omega')|^2}{\epsilon^*(k,\omega)} < E^2 >_{k'\omega'} < E^2 >_{k-k',\omega-\omega'} \right].$$

$$\mathrm{Re}\,\epsilon(k,\omega) < E^2 >_{k\omega} = 0, \qquad \Rightarrow \qquad \omega = \omega_k^\alpha \quad (\alpha = L, S).$$

$$< E^2 >_{k\omega} = \sum_\alpha \left[I_k^{+\alpha}\,\delta(\omega - \omega_k^\alpha) + I_k^{-\alpha}\,\delta(\omega + \omega_k^\alpha) \right]$$
$$= \sum_{\sigma=\pm 1} \sum_\alpha I_k^{\sigma\alpha}\,\delta(\omega - \sigma\omega_k^\alpha).$$

In the literature, the sign of wave phase speed σ is often not carefully taken care of!

Particle kinetic equation,

$$\frac{\partial F_a}{\partial t} = \frac{\partial}{\partial v}\left(D\,\frac{\partial F_a}{\partial v} \right),$$
$$D = \frac{\pi e_a^2}{m_a^2} \sum_{\sigma=\pm 1} \sum_\alpha \int dk\; I_k^{\sigma\alpha}\delta(\sigma\omega_k^\alpha - kv).$$

Wave kinetic equation

$$< E^2 >_{k\omega} = \sum_{\sigma=\pm 1} \sum_{\alpha} I_k^{\sigma\alpha}\, \delta(\omega - \sigma\omega_k^{\alpha}),$$

$$< E^2 >_{k'\omega'} = \sum_{\sigma'=\pm 1} \sum_{\beta} I_{k'}^{\sigma'\beta}\, \delta(\omega' - \sigma'\omega_{k'}^{\beta}),$$

$$< E^2 >_{k-k',\omega-\omega'} = \sum_{\sigma''=\pm 1} \sum_{\gamma} I_{k-k'}^{\sigma''\gamma}\, \delta(\omega - \omega' - \sigma''\omega_{k-k'}^{\gamma}),$$

$$\sum_{\sigma=\pm 1} \sum_{\alpha} \frac{\partial I_k^{\sigma\alpha}}{\partial t}\, \delta(\omega - \sigma\omega_k^{\alpha})$$

$$= - \sum_{\sigma=\pm 1} \sum_{\alpha} \frac{2\,\mathrm{Im}\,\epsilon(k, \sigma\omega_k^{\alpha})}{\partial\,\mathrm{Re}\,\epsilon(k, \sigma\omega_k^{\alpha})/\partial\sigma\omega_k^{\alpha}}\, I_k^{\sigma\alpha}\, \delta(\omega - \sigma\omega_k^{\alpha})$$

$$- 4\,\mathrm{Im} \int dk' \left[\sum_{\sigma=\pm 1} \sum_{\alpha} \frac{1}{\partial\,\mathrm{Re}\,\epsilon(k, \sigma\omega_k^{\alpha})/\partial\sigma\omega_k^{\alpha}} \right.$$

$$\times \left(\sum_{\sigma''=\pm 1} \sum_{\gamma} \frac{\{\chi^{(2)}(k', \sigma\omega_k^{\alpha} - \sigma''\omega_{k-k'}^{\gamma} | k-k', \sigma''\omega_{k-k'}^{\gamma})\}^2}{\epsilon(k', \sigma\omega_k^{\alpha} - \sigma''\omega_{k-k'}^{\gamma})}\; I_{k-k'}^{\sigma''\gamma} \right.$$

$$
+\sum_{\sigma'=\pm 1}\sum_{\beta}\frac{\{\chi^{(2)}(k',\sigma'\omega_{k'}^{\beta}|k-k',\sigma\omega_k^{\alpha}-\sigma'\omega_{k'}^{\beta})\}^2}{\epsilon(k-k',\sigma\omega_k^{\alpha}-\sigma'\omega_{k'}^{\beta})}I_{k'}^{\sigma'\beta}\Bigg)\; I_k^{\sigma\alpha}\,\delta(\omega-\sigma\omega_k^{\alpha})
$$

$$
-\frac{1}{\partial\,\mathrm{Re}\,\epsilon(k,\sigma'\omega_{k'}^{\beta}+\sigma''\omega_{k-k'}^{\gamma})/\partial(\sigma'\omega_{k'}^{\beta}+\sigma''\omega_{k-k'}^{\gamma})}
$$

$$
\times\sum_{\sigma',\sigma''=\pm 1}\sum_{\beta,\gamma}\frac{|\chi^{(2)}(k',\sigma'\omega_{k'}^{\beta}|k-k',\sigma''\omega_{k-k'}^{\gamma})|^2}{\epsilon^*(k,\sigma'\omega_{k'}^{\beta}+\sigma''\omega_{k-k'}^{\gamma})}
$$

$$
\times\, I_{k'}^{\sigma'\beta}I_{k-k'}^{\sigma''\gamma}\,\delta(\omega-\sigma'\omega_{k'}^{\beta}-\sigma''\omega_{k-k'}^{\gamma})\Big].
$$

$$
\frac{1}{\epsilon(k,\omega)}=\mathcal{P}\,\frac{1}{\epsilon(k,\omega)}-\sum_{\alpha}\sum_{\sigma=\pm 1}\frac{i\pi\,\delta(\omega-\sigma\omega_k^{\alpha})}{\partial\mathrm{Re}\,\epsilon(k,\sigma\omega_k^{\alpha})/\partial\sigma\omega_k^{\alpha}},
$$

$$
\frac{1}{\epsilon^*(k,\omega)}=\mathcal{P}\,\frac{1}{\epsilon^*(k,\omega)}+\sum_{\alpha}\sum_{\sigma=\pm 1}\frac{i\pi\,\delta(\omega-\sigma\omega_k^{\alpha})}{\partial\mathrm{Re}\,\epsilon(k,\sigma\omega_k^{\alpha})/\partial\sigma\omega_k^{\alpha}}.
$$

$$
\begin{aligned}
\frac{\partial I_k^{\sigma\alpha}}{\partial t} &= -\frac{2\,\mathrm{Im}\,\epsilon(k,\sigma\omega_k^\alpha)}{\partial\,\mathrm{Re}\,\epsilon(k,\sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha}\, I_k^{\sigma\alpha} \\
&- \frac{4}{\partial\,\mathrm{Re}\,\epsilon(k,\sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha}\,\mathrm{Im}\int dk' \\
&\times \Bigg[\sum_{\sigma''=\pm 1}\sum_\gamma \mathcal{P}\,\frac{\{\chi^{(2)}(k',\sigma\omega_k^\alpha-\sigma''\omega_{k-k'}^\gamma|k-k',\sigma''\omega_{k-k'}^\gamma)\}^2}{\epsilon(k',\sigma\omega_k^\alpha-\sigma''\omega_{k-k'}^\gamma)}\; I_{k-k'}^{\sigma''\gamma}\, I_k^{\sigma\alpha} \\
&+ \sum_{\sigma'=\pm 1}\sum_\beta \mathcal{P}\,\frac{\{\chi^{(2)}(k',\sigma'\omega_{k'}^\beta|k-k',\sigma\omega_k^\alpha-\sigma'\omega_{k'}^\beta)\}^2}{\epsilon(k-k',\sigma\omega_k^\alpha-\sigma'\omega_{k'}^\beta)}\, I_{k'}^{\sigma'\beta}\, I_k^{\sigma\alpha}\Bigg] \\
&+ \frac{4\pi}{\partial\,\mathrm{Re}\,\epsilon(k,\sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha}\sum_{\sigma',\sigma''=\pm 1}\sum_{\beta,\gamma} \\
&\times \mathrm{Im}\int dk'\,\Bigg[\{\chi^{(2)}(k',\sigma'\omega_{k'}^\beta|k-k',\sigma''\omega_{k-k'}^\gamma)\}^2 \\
&\times \left(\frac{I_{k-k'}^{\sigma''\gamma}\, I_k^{\sigma\alpha}}{\partial\,\mathrm{Re}\,\epsilon(k',\sigma'\omega_{k'}^\beta)/\partial\sigma'\omega_{k'}^\beta} + \frac{I_{k'}^{\sigma'\beta}\, I_k^{\sigma\alpha}}{\partial\,\mathrm{Re}\,\epsilon(k-k',\sigma''\omega_{k-k'}^\gamma)/\partial\sigma''\omega_{k-k'}^\gamma}\right) \\
&+ \frac{|\chi^{(2)}(k',\sigma'\omega_{k'}^\beta|k-k',\sigma''\omega_{k-k'}^\gamma)|^2}{\partial\,\mathrm{Re}\,\epsilon(k,\sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha}\, I_{k'}^{\sigma'\beta} I_{k-k'}^{\sigma''\gamma}\Bigg]\;\delta(\sigma\omega_k^\alpha-\sigma'\omega_{k'}^\beta-\sigma''\omega_{k-k'}^\gamma).
\end{aligned}
$$