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Advanced Workshop on Evaluating, Monitoring and Communicating Volcanic and Seismic Hazards in East Africa

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GPS theory and principles

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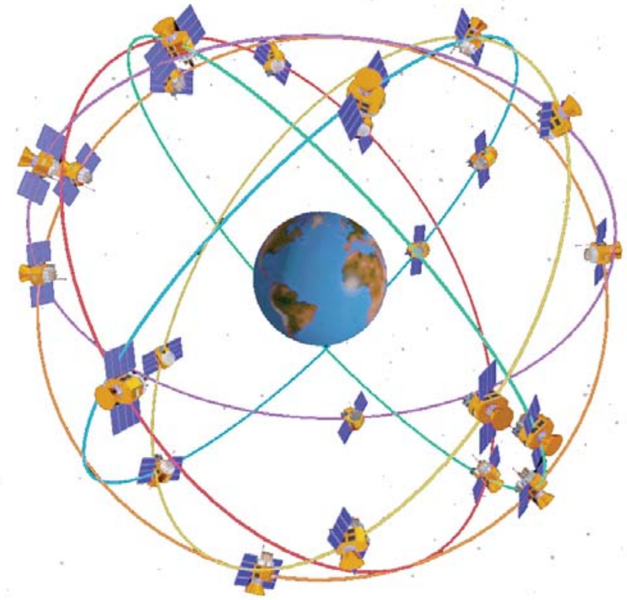
Basic GPS Theory and Principles



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Semester-long version of this lecture at:
http://web.ics.purdue.edu/~ecalais/teaching/ce511_eas591/

```
Position at epoch 2007.4869
X = 4908011.83893 +- 0.0023 m
Y = 4004007.32320 +- 0.0012 m
Z = -746405.43537 +- 0.0009 m
Lat = -6.76558574
Lon = 39.20792494
Height = 70.4757 m
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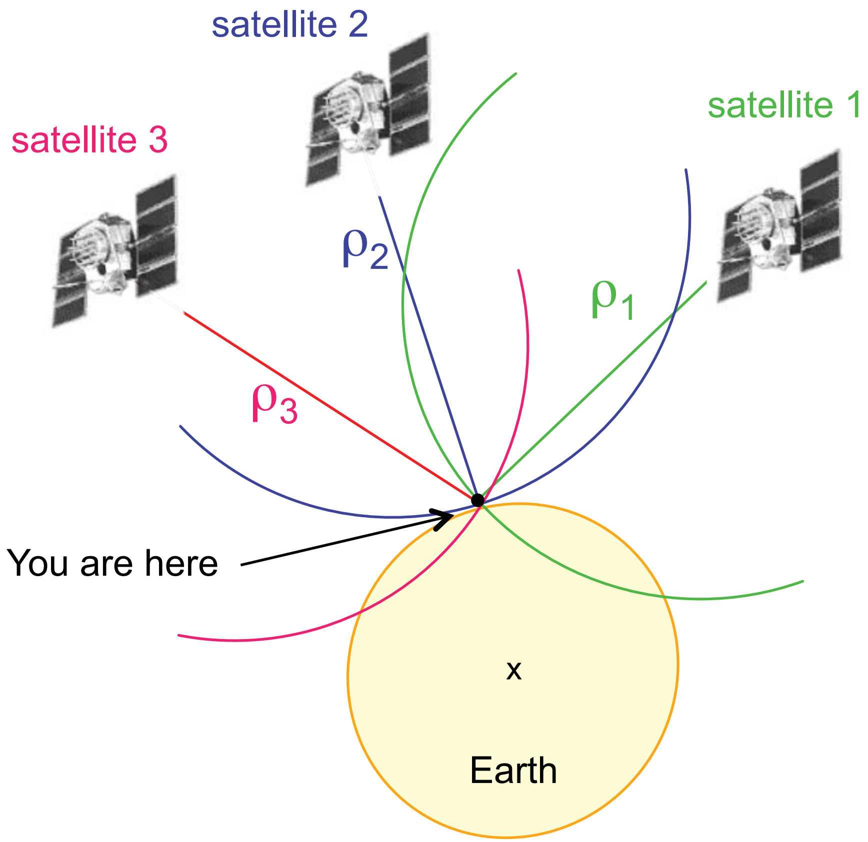
ICTP – Trieste, August 24, 2009

Basic Principle



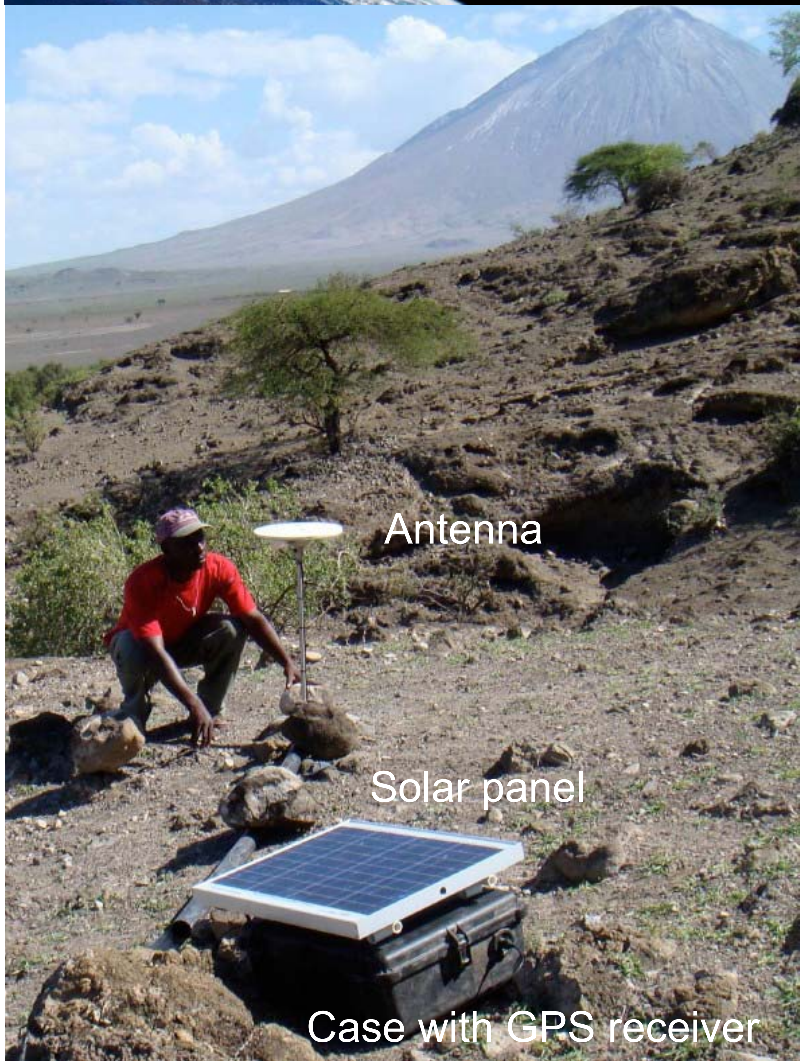
- Principle of GPS positioning:
  - Satellite 1 sends a signal at time t_{e1}
  - Ground receiver receives it signal at time t_r
  - The range measurement ρ_1 to satellite 1 is:
    - $\rho_1 = (t_r - t_{e1})$ x speed of light
    - We are therefore located on a sphere centered on satellite 1, with radius $\rho 1$
  - 3 satellites => intersection of 3 spheres
- The mathematical model is:

$$\rho_r^s = \sqrt{(X_s - X_r)^2 + (Y_s - Y_r)^2 + (Z_s - Z_r)^2}$$

  - If the position of the satellites in an Earth-fixed frame *(X_s, Y_s, Z_s)* is known…
  - …. one can solve for *(X_r, Y_r, X_r)* (if at least 3 simultaneous range measurements)

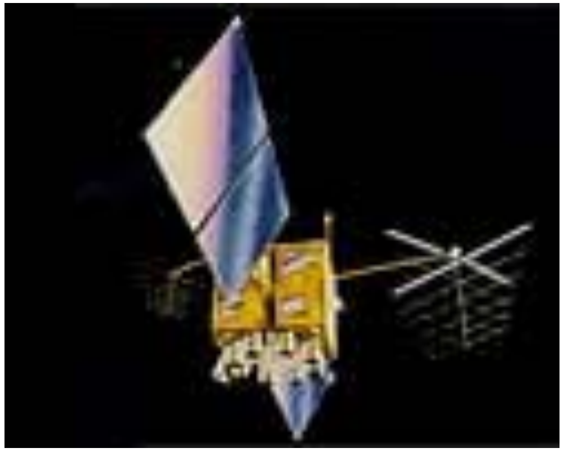
The GPS workflow

- What do GPS satellites do?
  - Send a radio signal toward Earth at t_e
  - Radio signal contains:
    - Satellite number
    - Time of emission
    - Satellite position
- What do GPS receivers do?
  - Measure t_r
  - Decode the satellite signal
    - Read t_e
    - Read satellite position
  - Compute satellite receiver distance, or range ρ_r^s
- What do users do?
  - Set up the equipment…
  - Download the "GPS data" = range measurements
  - Compute position from at least 4 range measurements

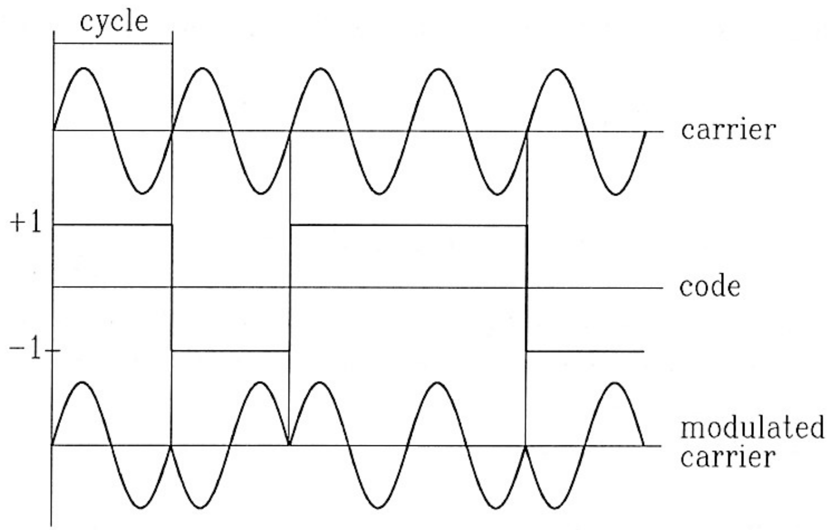


What do GPS satellites do?

- Maintain fundamental frequency *fo* = 10.23 Mhz using 2 or 4 atomic clocks (Ce/Rb):
  - Clock stability over 1 day = 10^{-13} (Rb) to 10^{-14} (Ce), ~1 ns/day
  - Clocks synchronized between all satellites
- Form two sinusoidal signals:
  - L1 *(fo x 154)* = 1.57542 GHz, wavelength 19.0 cm
  - L2 *(fo x 120)* = 1.22760 GHz, wavelength 24.4 cm
  - Called "GPS carrier frequencies"
- Broadcast a timing signal superimposed on carrier frequencies L1 and L2
- Broadcast ancillary information: e.g., satellite ephemerides



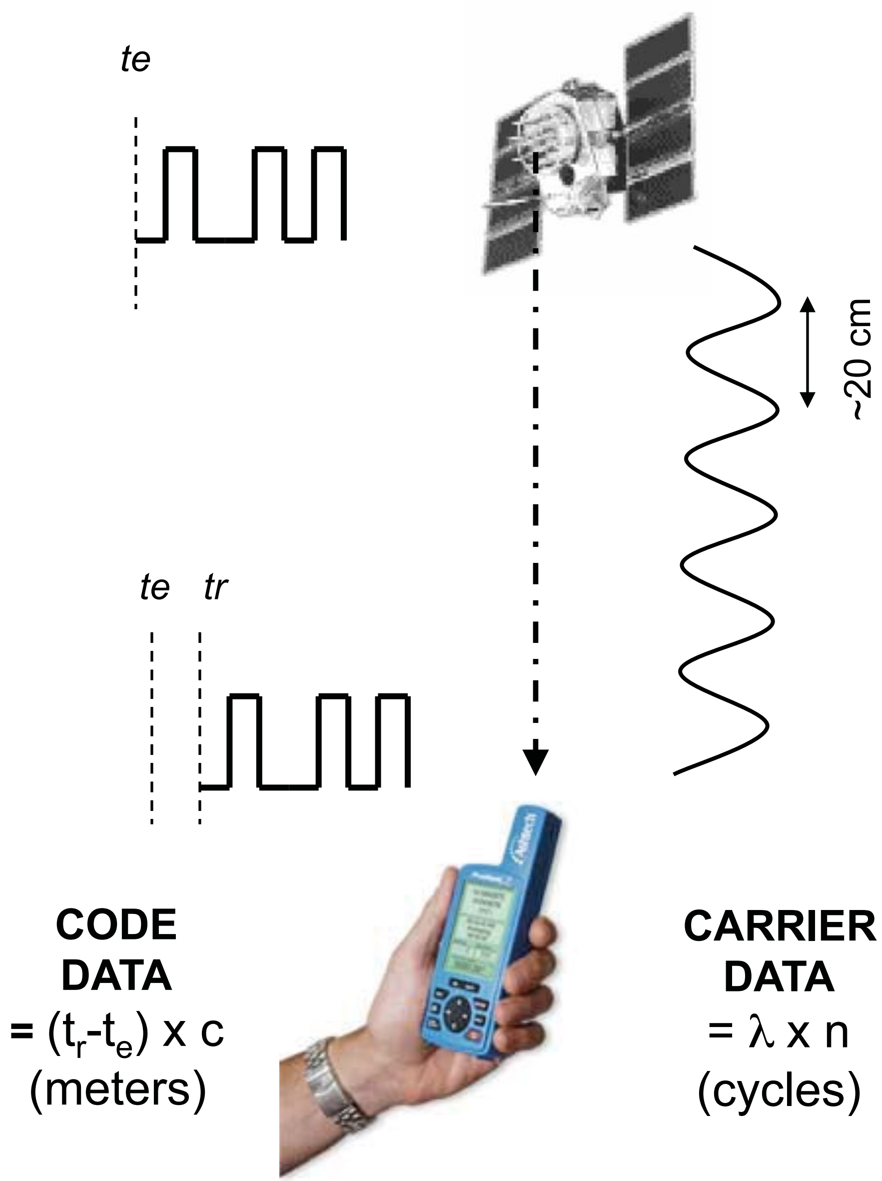
Block IIR satellite



Biphase modulation of the GPS carrier phase

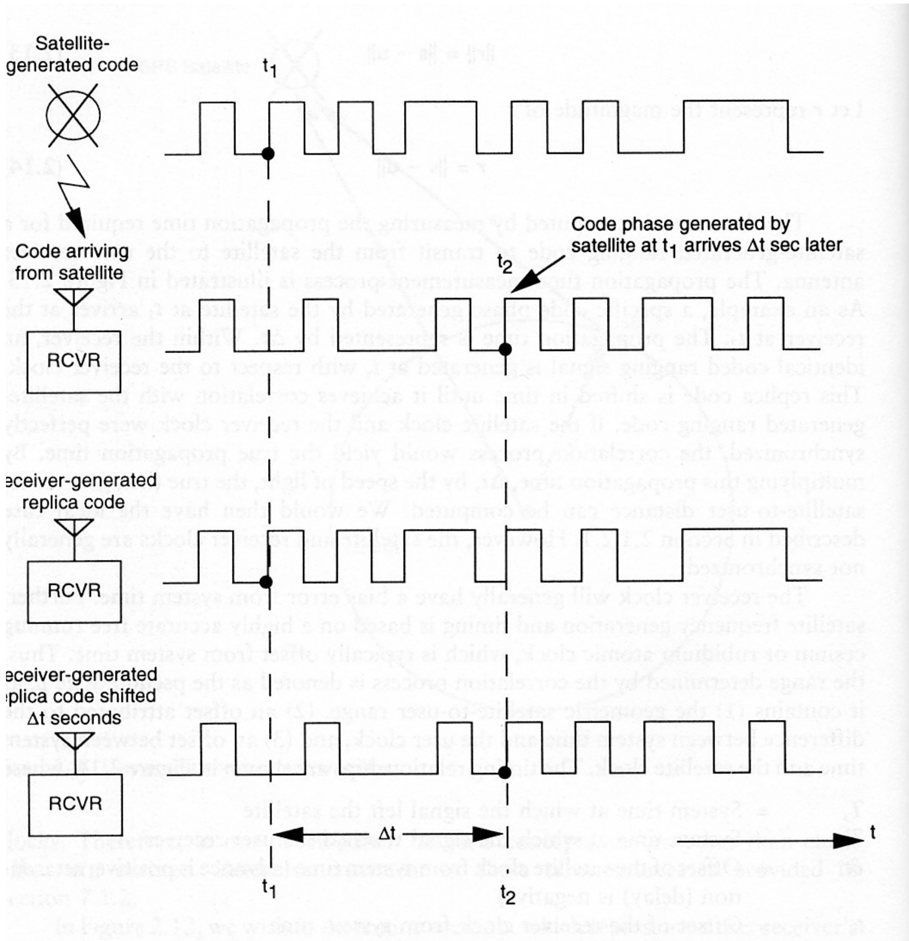
What do GPS receivers do?

- Maintain a (poor) clock
- Decode timing code = emission time:
  - Emission time - receive time = propagation time
  - Propagation time x speed of light = range
  - Easy, cheap, limited post-processing required
  - As precise as the time measurements, or ~1-10 m
- Track the incoming carrier signal:
  - If one could count the absolute number of phase n between sat and rcv…
  - … then n x wavelength = range
  - Difficult, requires significant post-processing
  - As precise as the phase detection, or ~1 mm



Code measurements

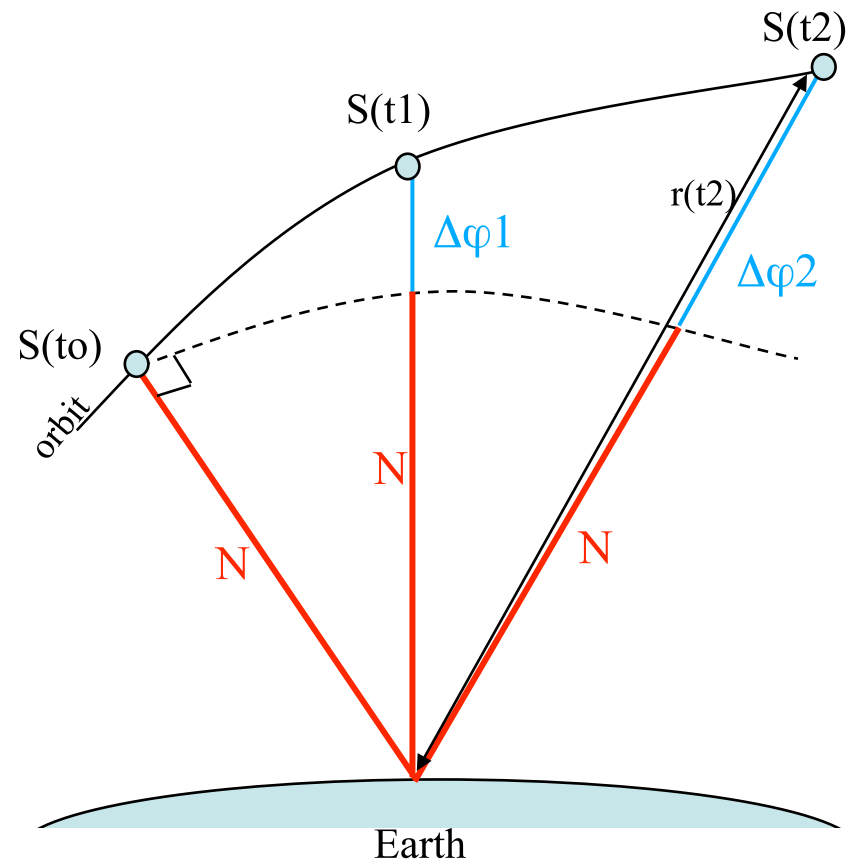
- Satellite clock time at transmission broadcast in form of 2 “codes”:
  - Coarse acquisition code = C/A code (open to civilians)
  - Precise code = P code (encrypted)
- Width of correlation function inversely proportional to signal bandwidth:
  - C/A code = 1 MHz bandwidth ⇒ peak 1 μsec wide = 300 m
  - P code = 10 MHz bandwidth ⇒ peak 0.1 μsec wide = 30 m
- Peak of correlation function can be determined to 1% of width
  - Range accuracy = 3 m for C/A code
  - Range accuracy = 0.3 m for P code
- Codes = low accuracy but absolute measurement



Principle of code measurement in a GPS receiver

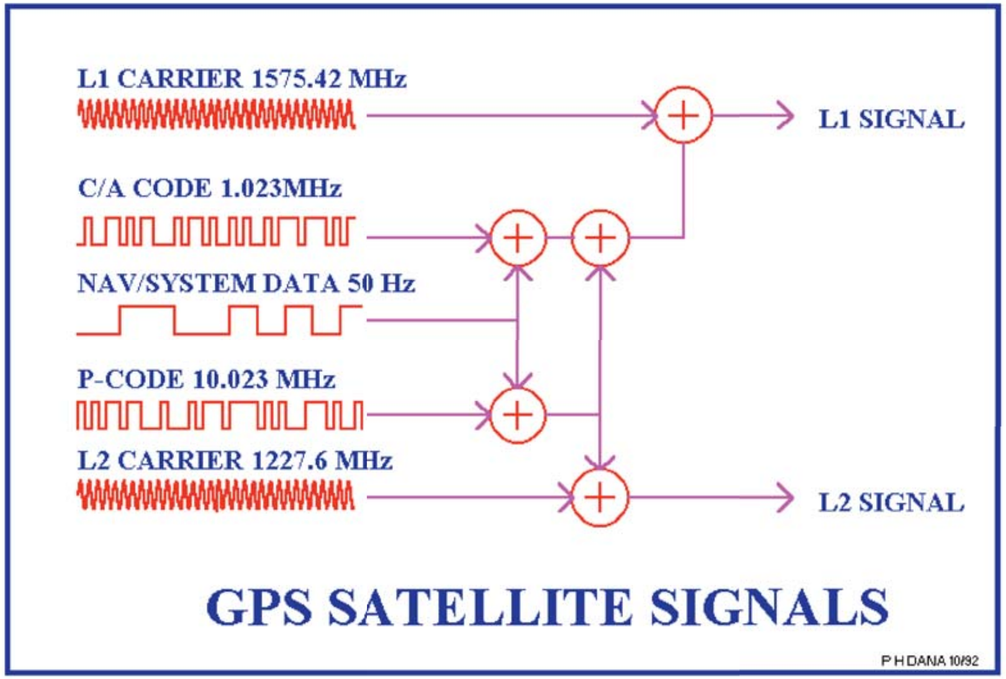
Carrier phase measurements

- Carrier phase can be converted to distance by multiplying by the wavelength ⇒ phase measurements are another way for measuring the satellite-receiver distance
- Phase can be measured to ~1% of the wavelength ⇒ range accuracy 2 mm for L1, 2.4 mm for L2
- Phase measurements are very precise, but ambiguous
- To fully exploit phase measurements, one **must** correct for propagation effects (several meters)



Receiver locked at t_o, counts the number of carrier phases $\Delta\varphi\ (t)$ since t_o, but the initial number of phases N at t_o is unknown = phase ambiguity. If no loss of lock, N constant over an orbit arc.

GPS observables



- GPS receivers can record up to 5 observables :
  - $\varphi 1$ and $\varphi 2$: phase measurements on L1 and L2 frequencies, in cycles
  - C/A, P1, P2: pseudorange measurements, in meters

(+ doppler phase = $d\varphi/dt$)

```
02 11 30  0  0 30.0000000  0  8G14G 7G31G20G28G 1G25G11
  -7096034.24049  -5509904.97345  23971309.103    23971309.038    23971310.842
 -12570276.74149  -9768618.40046  23379169.469    23379168.448    23379172.496
  -4157689.84249  -3201324.38045  24195891.298    24195890.733    24195894.168
 -25480193.34249 -19826614.77248  20670858.774    20670857.983    20670861.191
  -5589280.20049  -4319738.39345  24553697.713    24553697.259    24553700.349
 -10252537.24449  -7918950.15946  23060092.127    23060091.841    23060095.687
  -4143445.15949  -2509987.53445  24581180.488    24581179.713    24581183.992
 -29659606.34049 -23089397.33548  20312382.965    20312382.530    20312384.719
```

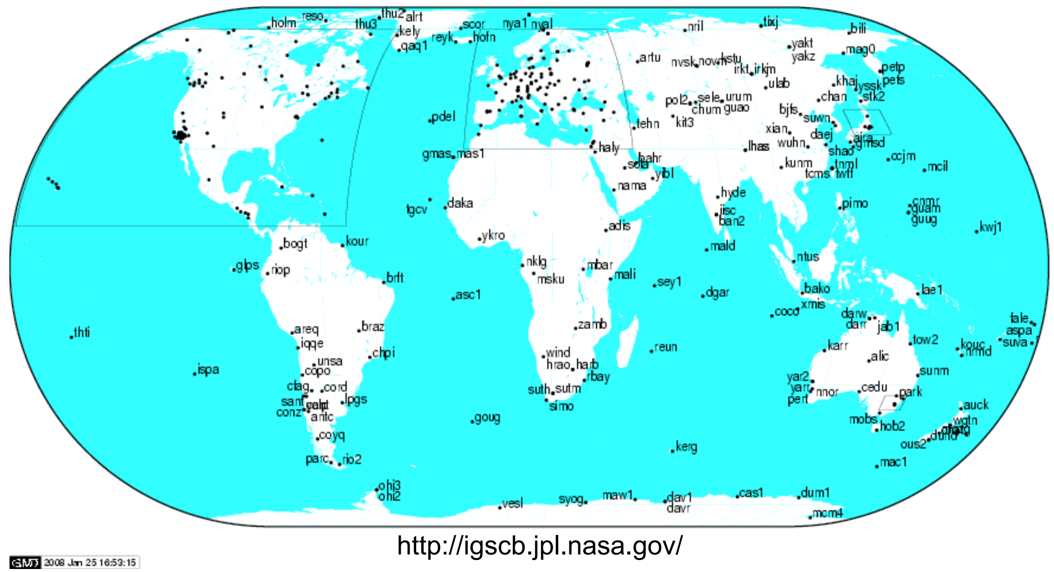
Block of RINEX data

- Data stored in proprietary format, can be converted to receiver independent exchange format = RINEX

GPS satellite orbits

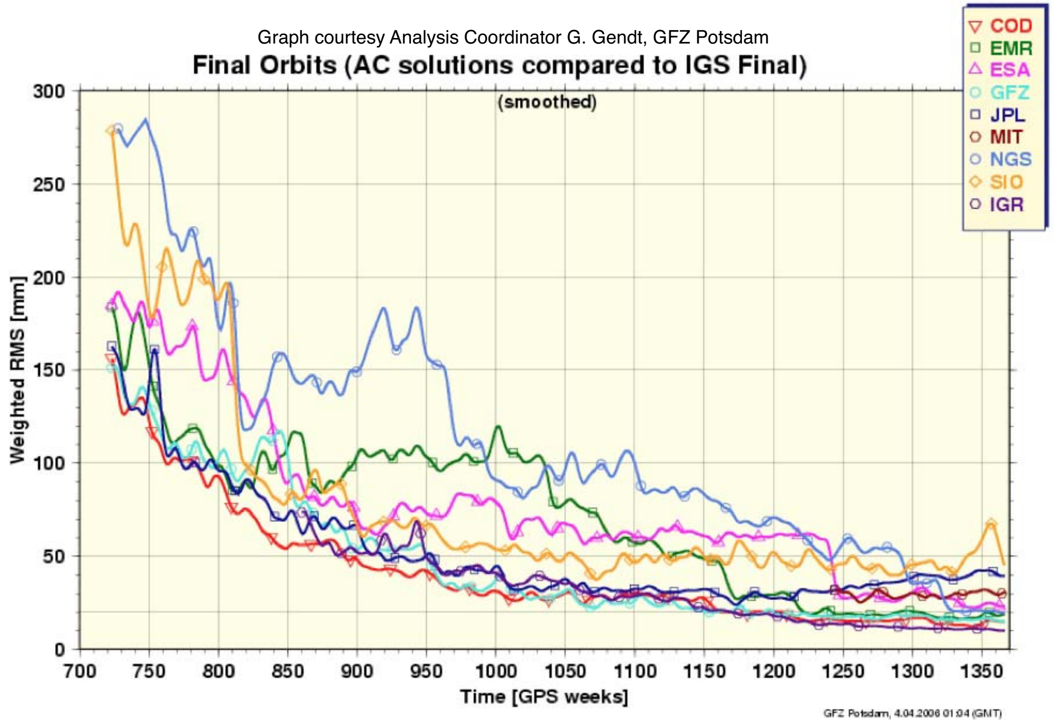
- Broadcast ephemeris:
  - Distributed to users via a "navigation message" included in the signal sent by the GPS satellites
  - Accuracy ~10 m
  - Ok for short baseline applications

- Precise orbits:
  - Provided by the IGS = international GNSS Service (under IAG)
  - Core network of >350 globally distributed, high-quality, continuous GPS stations:
    - Data processed by IGS analysis centers
    - Analysis center coordinator produces weighted average = IGS orbit
  - Accuracy < 5 cm



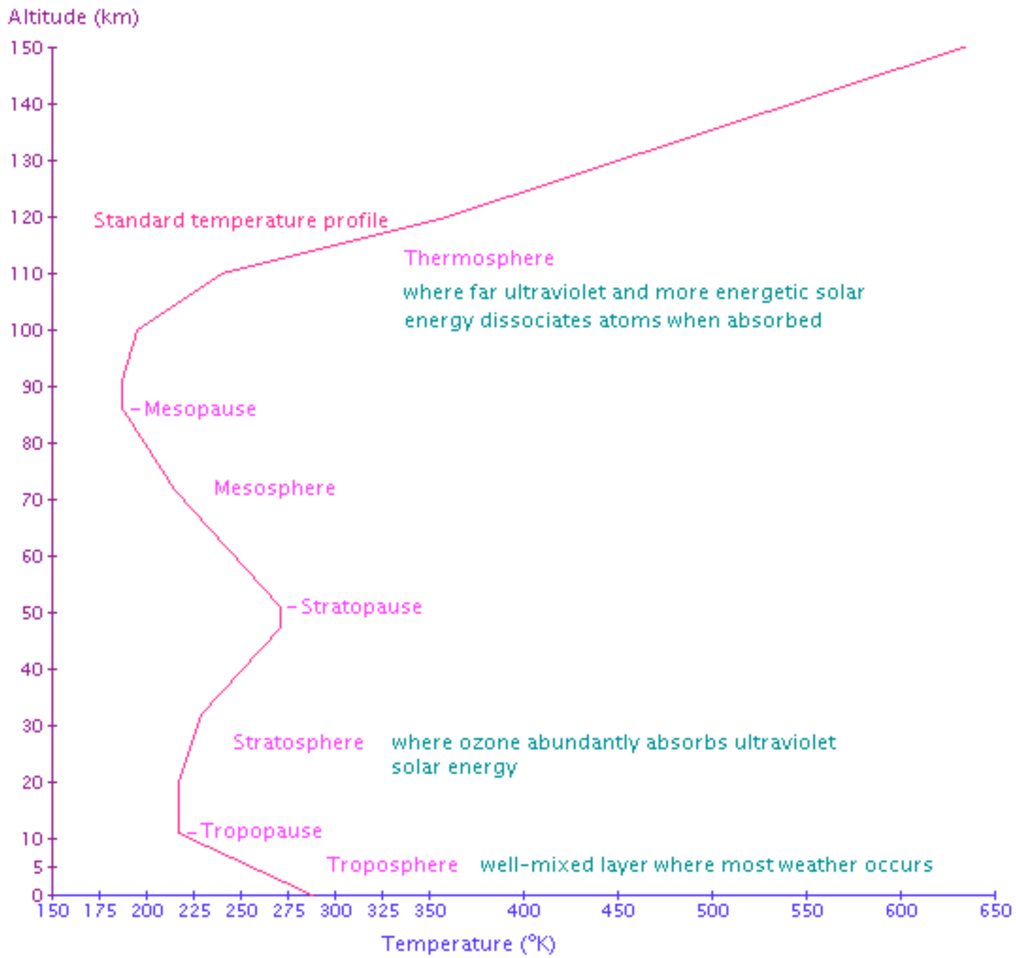
http://igscb.jpl.nasa.gov/

Graph courtesy Analysis Coordinator G. Gendt, GFZ Potsdam



GPS signal propagation

- L1 and L2 frequencies are affected by **atmospheric refraction**:
  - Ray bending (negligible)
  - Propagation velocity decrease (w.r.t. vacuum) ⇒ propagation delay
- In the **troposphere**:
  - Delay is a function of (P, T, H), 1 to 5 m
  - Largest effect due to pressure
- In the **ionosphere**: delay function of the electron density, 0 to 50 m
- The **refractive delay** biases the satellite-receiver range measurements, and, consequently the estimated positions (effect more pronounced in the vertical).



Observation models

Code (meters):

$$R_i^k(t) = \rho_i^k(t) + c(h^k(t) - h_i(t)) + I_i^k(t) + T_i^k(t) + MP_i^k(t) + \varepsilon$$

Carrier phase (cycles):

$$\Phi_i^k(t) = \rho_i^k(t) \times \frac{f}{c} + \left(h^k(t) - h_i(t)\right) \times f + I_i^k(t) + T_i^k(t) + MP_i^k(t) - N_i^k + \varepsilon$$

t = time of epoch
R = pseudorange measurement
Φ = carrier phase measurement
ρ = satellite-receiver geometric distance
c = speed of light
f = carrier frequency
h^k = satellite clock bias, h_i = receiver clock bias

I = ionospheric propagation error
T = tropospheric propagation error
MP = multipath
N = phase ambiguity
ε = other small errors, including receiver noise
(ranges in meters, time in seconds, phase in cycles)

With:

$$\rho_i^k = \sqrt{(X^k - X_i)^2 + (Y^k - Y_i)^2 + (Z^k - Z_i)^2}$$

X^k, Y^k, Z^k = satellite position
X_i, Z_i, Z_i = site position

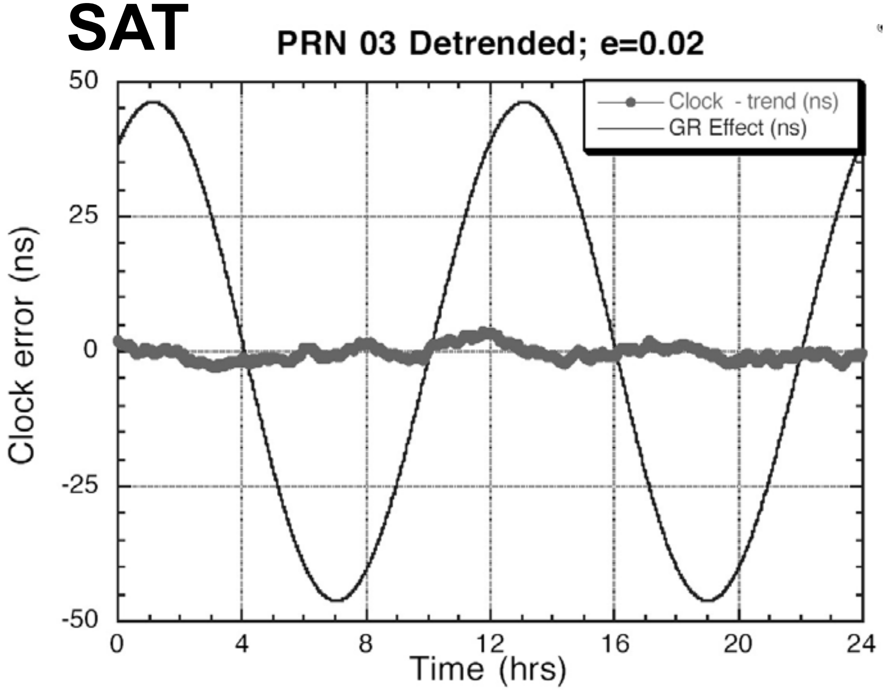
From GPS signal to position

$$R_i^k = \sqrt{(X^k - X_i)^2 + (Y^k - Y_i)^2 + (Z^k - Z_i)^2} + c(h^k - h_i) + I_i^k + T_i^k + MP + \varepsilon \quad (+N)$$

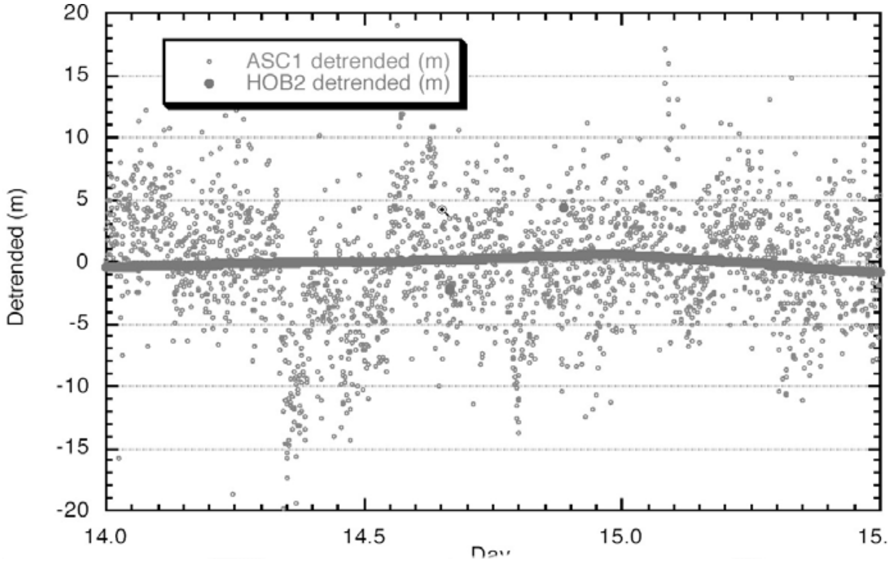
1. Measure satellite-receiver range
   - From code: absolute but not precise
   - From carrier phase: precise but not absolute
2. Obtain orbit information:
   - From broadcast navigation message: not precise
   - From IGS
3. Linearize square root term and write model equation for at least 4 ranges acquired at the same time from 4 different satellites.
4. Remaining terms:
   1. Clocks
   2. Propagation effects
   3. Ambiguities
   4. Other error sources (multipath, etc.)
5. Calculate solution in a given reference frame.

Clock errors

SAT



RCV



- Satellites:
  - S/A => ~200 ns
  - Currently ~5 ns = 1.5 m
  - IGS provides precise sat. clocks
- Receivers:
  - clocks errors can reach kms…
  - Sometimes well-behaved ⇒ can be modeled using linear polynomials.
  - Usually not the case…
  - Estimate receiver clocks at every measurement epoch (can be tricky with bad clocks)
  - Cancelled clock errors using a “trick”: double differencing

Double differences

- Combination of phase observables between 2 sats (k,l) and 2 rcvs (i,j):

$$\Phi_{ij}^{kl} = (\Phi_i^k - \Phi_i^l) - (\Phi_j^k - \Phi_j^l)$$

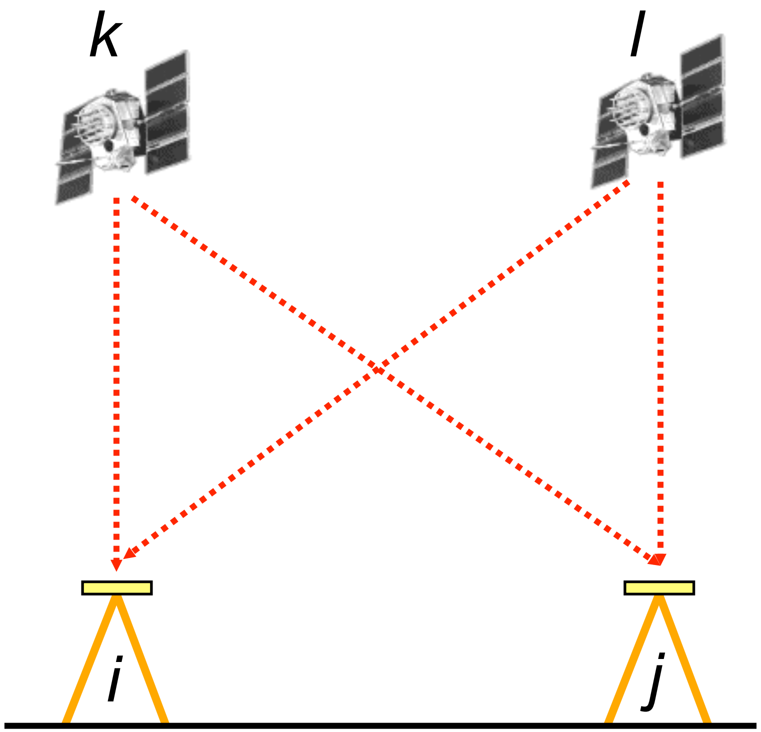
$$\Rightarrow \Phi_{ij}^{kl} = (\rho_i^k - \rho_i^l + \rho_j^k - \rho_j^l)*f/c - (h^k - h_i - h^l + h_i - h^k + h_j + h^l - h_j) - (N_i^k - N_i^l + N_j^k - N_j^l)$$

$$\Rightarrow \Phi_{ij}^{kl} = (\rho_i^k - \rho_i^l + \rho_j^k - \rho_j^l)*f/c - N_{ij}^{kl}$$

⇒ Clock errors $h_s(t)$ et $h_r(t)$ eliminated (but number of observations has decreased)

⇒ Any error common to receivers i and j will also cancel…!

- Atmospheric propagation errors cancel if receivers close enough to each other
- Short baselines provide greater precision than long ones.



Ionospheric refraction

- Ionospheric delay:
  - Proportional to ionospheric electron density along ray path
  - To first order: inversely proportional to the square of the frequency

$$I_1 = \frac{A}{cf_1^2} IEC$$

$$I_2 = \frac{A}{cf_2^2} IEC$$

- These properties can be used to form a new observable φ_{LC} :
  - Linear combination of φ_1 and φ_2
  - Independent from I
  - Requires dual-frequency GPS receivers.

$$\varphi_{LC} = \frac{f_1^2}{f_1^2 - f_2^2}\varphi_1 - \frac{f_1 \, f_2}{f_1^2 - f_2^2}\varphi_2$$

$$\varphi_{LC} = 2.546 \times \varphi_1 - 1.984 \times \varphi_2$$

Tropospheric refraction

- **Hydrostatic or “dry” delay:**
  - Molecular constituents of the atmosphere in hydrostatic equilibrium.
  - Can be modeled with a simple dependence on surface pressure (P_0=surface pressure (in mbar), λ= latitude, and H = height above the ellipsoid)

$$\Delta L_{hydro}^{zen} = \left(2.2768 \pm 0.0024 \times 10^{-7}\right)\frac{P_0}{f\left(\lambda,H\right)} \qquad f(\lambda,H) = 1 - 0.00266\cos\left(2\lambda\right) - 0.00028H$$

  - Standard deviation of current modeled estimates of this delay ~ 0.5 mm.

- **Non-hydrostatic or “wet” delay:**
  - Associated with water vapor that is not in hydrostatic equilibrium.
  - Very difficult to model because the quantity of atmospheric water vapor is highly variable in space and time (M_w and M_d the molar masses of dry air and water vapor)

$$\Delta L_{wet}^{zen} = 10^{-6}\left[\left(k_2 - \frac{M_w}{M_d}k_1\right)\int\frac{e}{T}dz + k_3\int\frac{e}{T^2}dz\right]$$

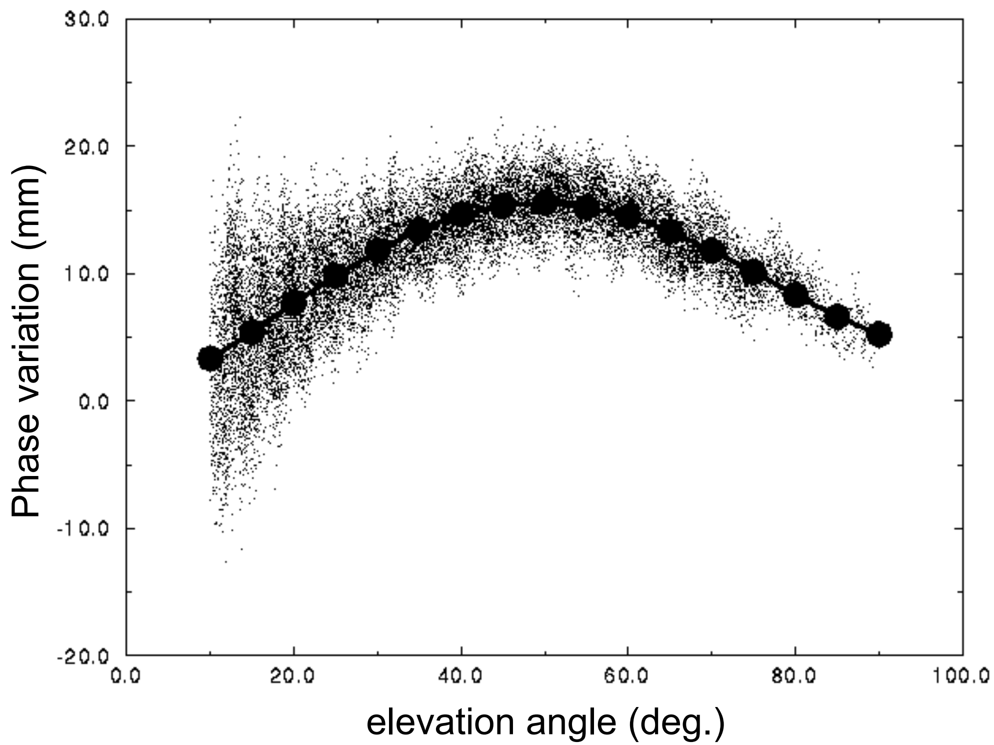
  - Standard deviation of current modeled estimates of this delay ~ 2 cm.

Tropospheric refraction

- How to handle the range error introduced by tropospheric refraction?
  - Correct: using a priori knowledge of the zenith delay (total or wet) from met. model, WVR, radiosonde (not from surface met data…)
  - Model: ok for dry delay, not for wet…
  - Estimate:
    - → Introduce an additional unknown = zenith total delay
    - → Solve for it together with station position and time offset
    - → Even better: also estimate lateral gradients because of deviations from spherical symmetry
- If tropospheric delay is estimated, then GPS is also an atmospheric remote sensing tool!

Antenna phase center offsets

- Antenna phase center:
  - Point where the radio signal measurement is referred to
  - Not necessarily geometric antenna center
- No direct access to the antenna phase center:
  - We setup the antenna using its geometrical center
  - Need to correct for offset between APC and GC (1-2 cm)
- In addition, the position of the phase center varies with elevation and azimuth of the incoming signal:
  - Need for an azimuth/elevation dependent correction
  - The most common GPS antenna have been calibrated and correction tables are available for each model.

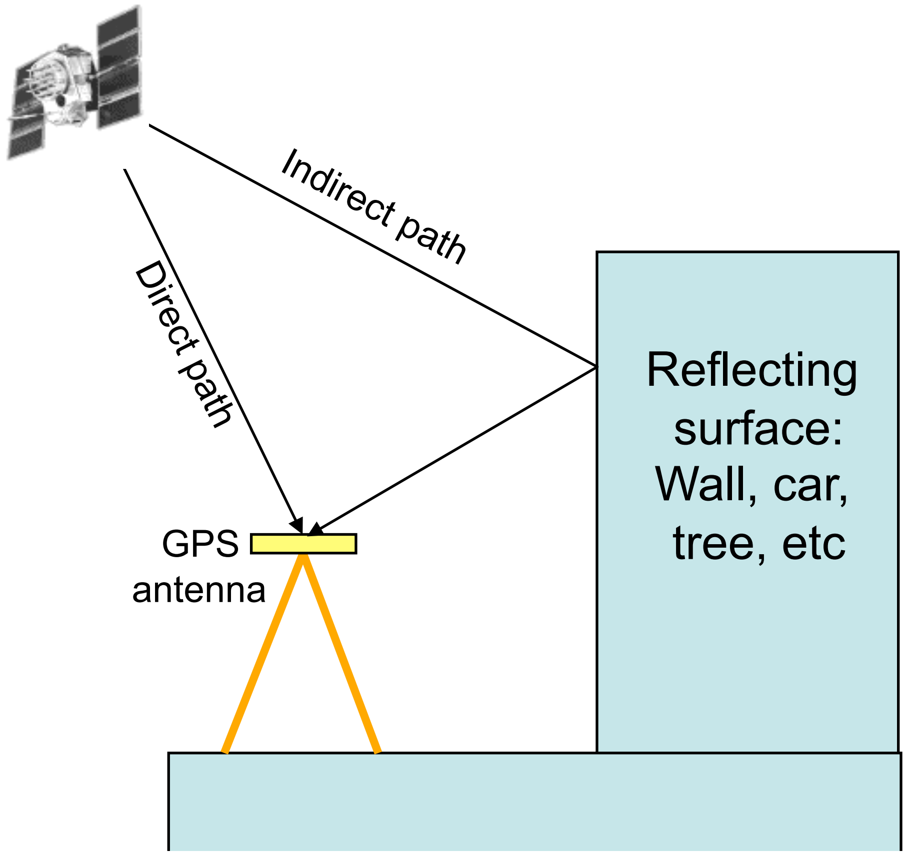


Phase residuals for an antenna calibration as a function of elevation. The phase variation is clearly evident. The solid curve is the polynomial fit to these data and the dots indicate the elevation increments used in the summary file.

(http://www.ngs.noaa.gov/ANTCAL/Files/summary.html)

Multipath

- GPS signal may be reflected by surfaces near the receiver
- Multipath errors:
  - Code measurements: up to several meters
  - Phase measurements: centimeter-level
- Multipath repeats daily because of repeat time of GPS constellation: can be used to filter it out
- Best solution: choose the location of the GPS sites carefully…!



Parameter estimation

- Linearization
  - Taylor's series
  - Compute a priori ("model") values for all parameters
  - Compute partial derivatives
- Prepare inverse problem
  - Write design matrix ("partial derivative matrix")
  - Write data vector
  - Write data covariance matrix
- Inversion
  - Weighted least squares
  - Unknown = adjustments
  - Covariance of unknowns

Model:

$$l_i^k = \sqrt{(X^k - X_i)^2 + (Y^k - Y_i)^2 + (Z^k - Z_i)^2} + c\delta_i + \ldots$$

Linearization:

$$f(X_i, Y_i, Z_i) = f(X_o, Y_o, Z_o)$$
$$+ \frac{\partial f(X_o, Y_o, Z_o)}{\partial X_o}\Delta X_i + \frac{\partial f(X_o, Y_o, Z_o)}{\partial Y_o}\Delta Y_i + \frac{\partial f(X_o, Y_o, Z_o)}{\partial Z_o}\Delta Z_i$$
$$+ \frac{1}{2!}\frac{\partial^2 f}{\partial x^2} + \ldots$$

with:

$$f(X_o, Y_o, Z_o) = \sqrt{\left({}^jX(t) - X_o\right)^2 + \left({}^jY(t) - Y_o\right)^2 + \left({}^jZ(t) - Z_o\right)^2} = {}^j\rho_o(t)$$

For n satellites at a given epoch:

$$\begin{aligned}
{}^1l &= {}^1a_{Xi}\Delta X_i + {}^1a_{Yi}\Delta Y_i + {}^1a_{Zi}\Delta Z_i - c\delta_i \\
{}^2l &= {}^2a_{Xi}\Delta X_i + {}^2a_{Yi}\Delta Y_i + {}^2a_{Zi}\Delta Z_i - c\delta_i \\
{}^3l &= {}^3a_{Xi}\Delta X_i + {}^3a_{Yi}\Delta Y_i + {}^3a_{Zi}\Delta Z_i - c\delta_i \\
&\ldots \\
{}^nl &= {}^na_{Xi}\Delta X_i + {}^na_{Yi}\Delta Y_i + {}^na_{Zi}\Delta Z_i - c\delta_i
\end{aligned}$$

or: $\vec{L} = A\vec{X} + \vec{r} \qquad (\Sigma_L)$

Weighted least squares solution:

$$\vec{X} = \left(A^T \Sigma_L^{-1} A\right)^{-1} A^T \vec{L}$$

$$\Sigma_X = \left(A^T \Sigma_L^{-1} A\right)^{-1}$$

Precision

- Several positions at static site => time series can be plotted

- Scatter of daily positions to the weighted mean of the entire time series, called repeatability, is defined by:

$$wrms = \sqrt{\frac{\frac{N}{N-1}\sum_{i=1}^{N}\frac{(y_i-(a+bt_i))^2}{\sigma_i^2}}{\sum_{i=1}^{N}\sigma_i^2}}$$

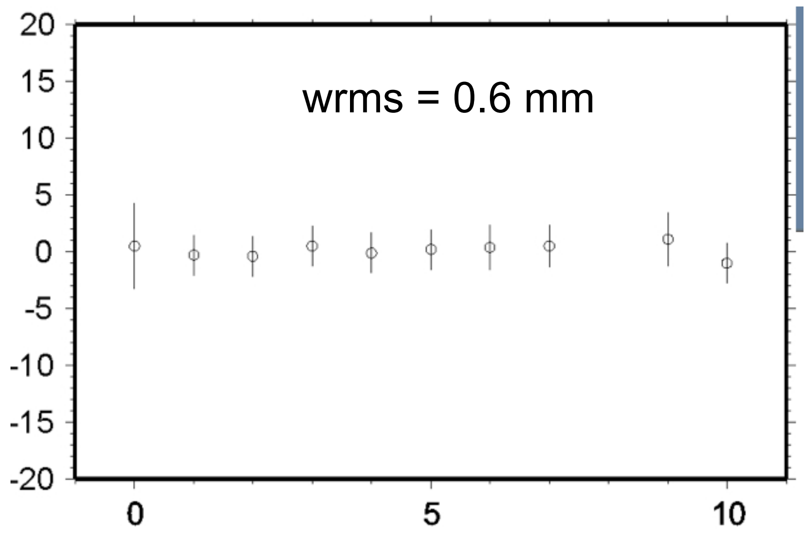
y_i *and* σ_i = position and associated formal error from the inversion

N = number of data points

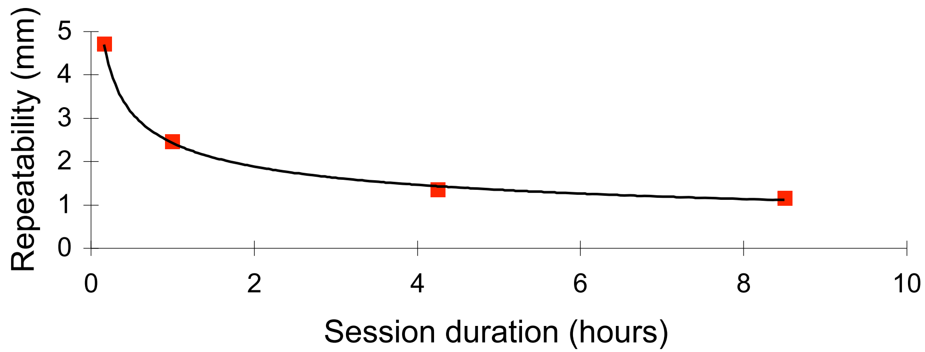
a, b = parameter of long-term slope, that term could also be the mean of y_i*'s.*

- Repeatability is a reliable measure of precision.

1.4 km-long baseline observed 10 days in a row, one solution every 24 hour (L1)

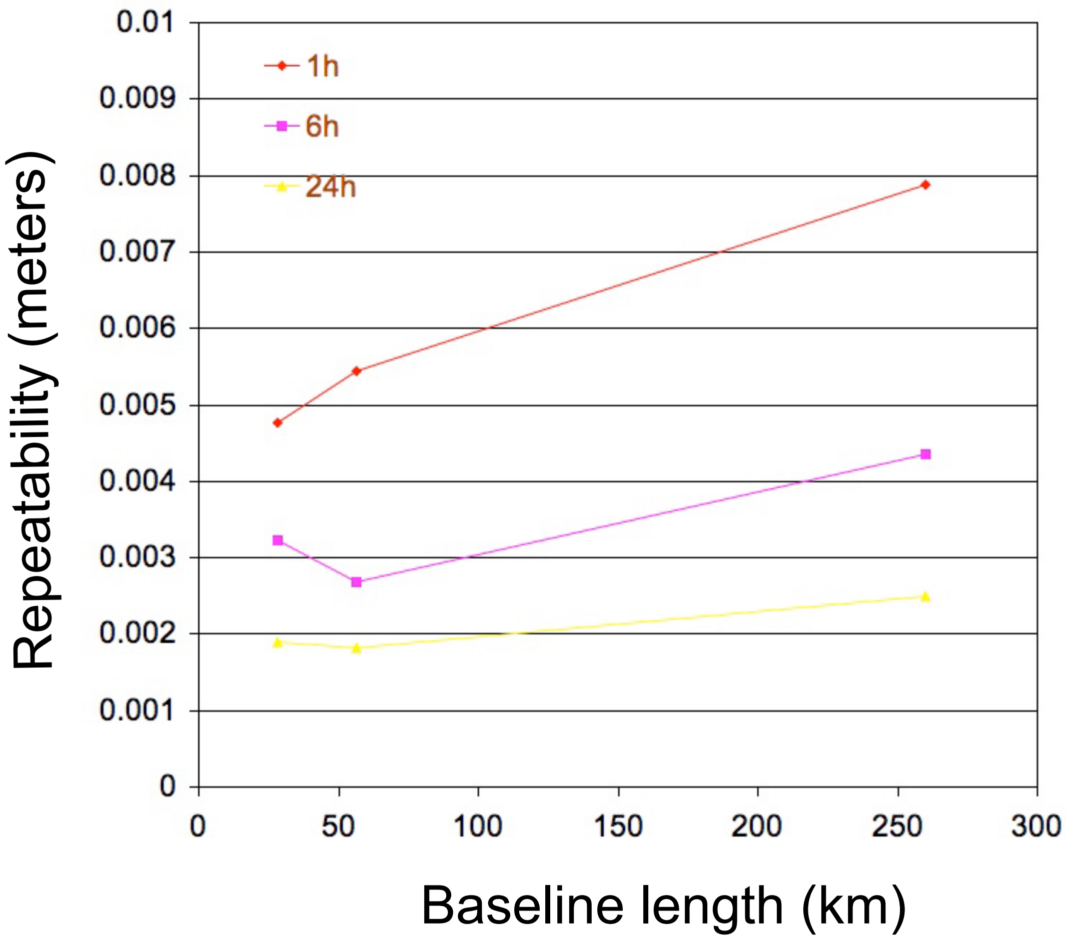


Same baseline, but decrease solution time span to 8 h, 4 h, 1 h, 15 mn



Precision

- Three baselines observed continuously during 30 days
- Length = 30, 60 and 260 km
- Sophisticated processing of phase data (LC)
  - 1, 6, and 24 hr sessions
  - Research software (GAMIT)
  - Precise IGS IGS, estimation of tropospheric parameters, etc.
- Precision depends on:
  - Network size
  - Observation time (for geophysics, minimum of 12 hours)




GUNTHER, I'M LOST! WHAT BUILDING IS THIS?
F-12, SECTOR M, LEVEL ONE. WHERE ARE YOU TRYING TO GO?
ROOM D-27
OK, THAT WOULD BE COORDINATE T-72. FROM HERE, GO OUT THE WEST DOOR, TURN LEFT AND GO 115 YARDS
HOW DO YOU KNOW THAT?
GLOBAL POSITIONING WATCH
GUNTHER, YOU ARE A MAJOR GEEK... BUT YOU'RE MY HERO