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Fractal/Multifractal Description of Seismicity

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FRACTAL / MULTIFRACTAL DESCRIPTION OF SEISMICITY

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Papers: Molchan & Kronrod , 2005, Geophys. J. Int., **162**, 899-960
2007, PAGEOPH, **164**, 75-96
2009, Geophys. J. Int. (accepted)

MOTIVATION

Contradictions in fractality:

- Fractality is considered as a ***physical*** property of seismicity ***but*** its characteristics (fractal dimensions, d_q) have little confidence;

- Correlation fractal dimension, d_2, dominates in applications, e.g. $d=d_2$ in the relations

  $\#\{\text{ event of } m>M \text{ in a box of size } L\} \propto 10^{-bM}L^{d}$,

  $\#\{\text{ inter-event time in a box of size } L\} \propto L^{-d}$

$\Rightarrow$"mono" - or "multi" - fractality?

the best *d*?

- fractality & self-similarity

  Key question (usually out of discussion):

  the non-trivial range of scale (L_{-}, L_{+}), $L_{+}/L_{-} \geq 10$, in the scaling laws related to fractality?

Outline

- Multifractality

  - the theoretical aspect

  - the empirical aspect

- Fractality of regional seismicity

- Scaling: inter-event time and seismicity rate

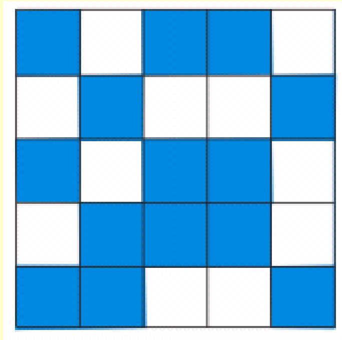
Multifractality: theoretical aspect

Main notion

- *box-dimension* of set S

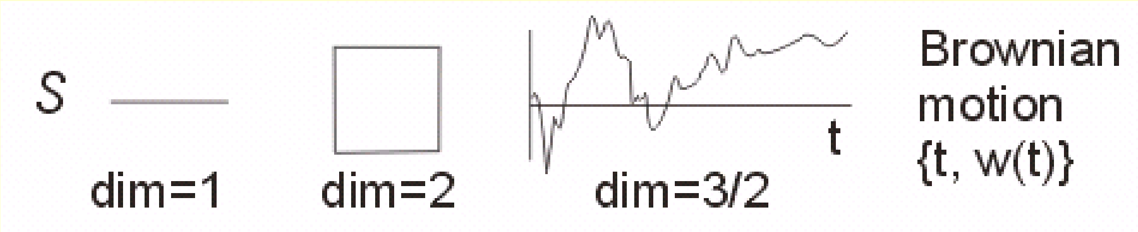
{■} box-covering of S

■ box of size L, Δ_L



$$\dim_b(S) = \lim_{L \downarrow 0} \frac{\log \#\{\Delta_L\}}{\log L}$$

Examples:



- *seismicity measure* (main object of study):

$\lambda(dg)$ = rate of $m>m_c$ events in

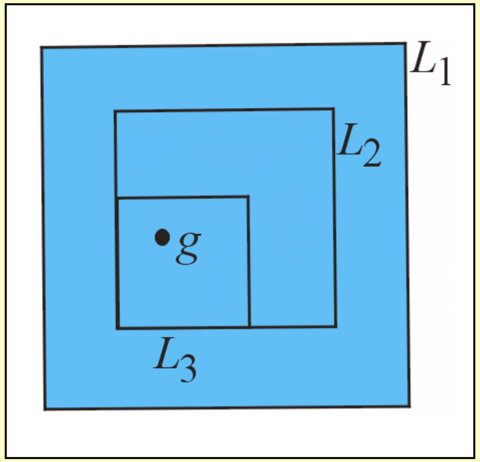
an elementary volume/area $dg \subset G$

λ reflects the clustering of events and complex fault system.

Therefore λ has singularities

- *Singularities* of $\lambda(dg)$: α-type at $g \in G$

$$\lambda(\Delta_L) = \int_{\Delta_L} \lambda(dg) \propto L^{\alpha},\ L \to 0,\ g \subset \Delta_L$$



- *multifractal spectrum* of $\lambda(dg)$: $\{\alpha, f(\alpha)\}$

  α is α-type singularity of $\lambda(dg)$

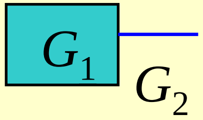
  $f(\alpha)=\dim_b(S_\alpha)$, $S_\alpha=\{g$: α-type singularity point$\}$

Trivial Example:

λ(dg), $g\in G=G_1\cup G_2$ has positive densities in

$G_1=[0,1]^2$ and $G_2=[0,1]$

$(\alpha, f(\alpha))=(2, 2); (1, 1); (\alpha, 0), \alpha\neq 1, 2$

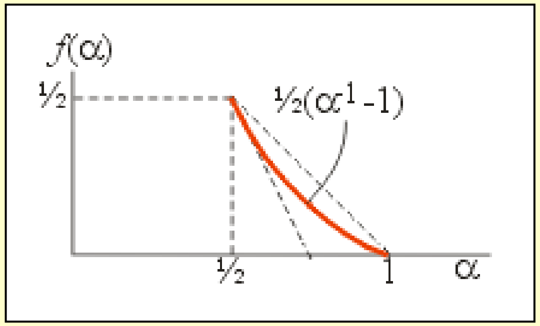


Nontrivial example. The model of sedimentation:

$$H(t)=\min_{s>t}\{as+w(s)\}$$ thickness of sedimentary layers at time t

w(t), Brounian motion

$\lambda(dt)=dH(t)$

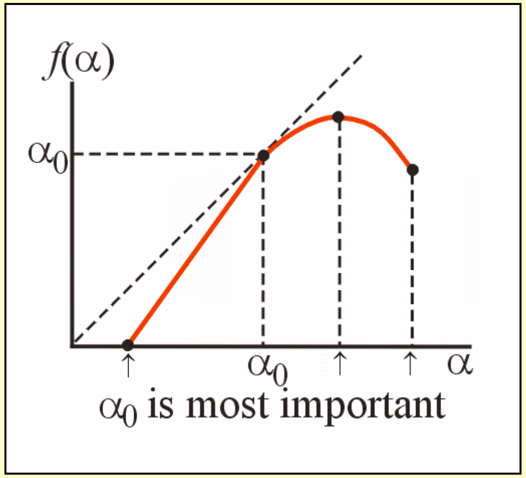


- *typical α-singularity* of $\lambda(dg)$: $\lambda(S_\alpha)>0$

  Young (1981): α_0 is typical $\Rightarrow$ $\alpha_0=f(\alpha_0)$,

  i.e. α_0 is fractal dimension

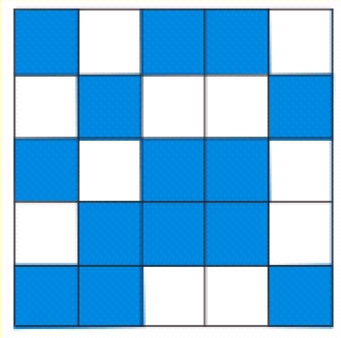
  α is not typical $\Rightarrow$ $\alpha>f(\alpha)$, $\lambda(S_\alpha)=0$



- How to find $(\alpha, f(\alpha))$?

Renyi function:

$$R_L(q) = \sum_{i:\lambda(\Delta_L^i)>0} \left[\frac{\lambda(\Delta_L^i)}{\lambda(G)} \right]^q$$



$\{\Delta^i_L\}$, covering of G

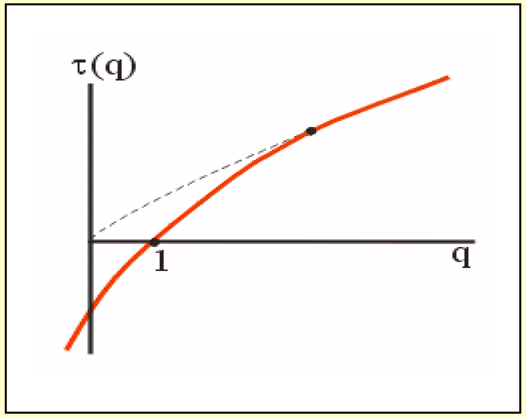
Multifractal case: $\log R_L(q)$ vs $\log L$ is linear for $L<<1$

Multifractal formalism:

$$\log R_L(q) \cong \tau(q)\log L\cdot(1+o(1)),\, L\to 0$$

$$\tau(q)=\min_{\alpha}(q\alpha - f(\alpha)),\, \tau(1)=0$$

If $\tau(q)$ is strictly concave then



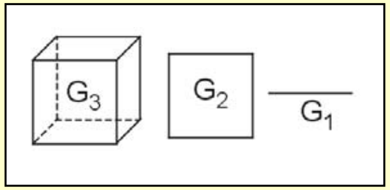
$$f(\alpha)=\min_q(q\alpha-\tau(q))$$

- $\{\tau'(q)\}\subset\{\alpha\}$, $\{\tau'(q)\}=\{\alpha\}$ (τ, strictly concave)

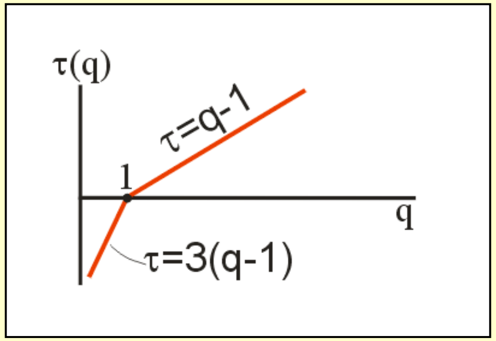
Hint: $f(\alpha)=\min_q(q\alpha-\tau(q))\Rightarrow q_{\min}:\alpha=\tau'(q_{\min})$

Trivial example.

$\lambda(dg)$ has densities in G_1, G_2, G_3



α=3; 2; 1



τ'=1; 3

- generalized dimensions by Grasberger & Procaccia

$$\boxed{d_q = \frac{\tau(q)}{q-1}} = \frac{\tau(q)-\tau(1)}{q-1} = \tau'(q^*) \subset \{\alpha\} \qquad d_1 = \lim_{q\downarrow 1}\frac{\tau(q)}{q-1} = \tau'(1)$$

q^* is between q and 1

$\Rightarrow$ $\{\tau'(q)\}$ and $\{d_q\}$ are different parametrization of $\{\alpha\}$

- $\tau'(q)$ and d_q are decreasing functions $\Rightarrow d_q \le d_0,\ q>0$

- typical singularity:

$$\alpha = \tau'(1) = d_1 = \dim_b(S_{\alpha=d_1})$$

Hint: α is typical, $\alpha = f(\alpha)$

$$\Downarrow$$

$$f(\alpha) = \min_q(q\alpha - \tau(q)) = \alpha \ \text{ if } \ q_{\min} = 1$$

$$\Downarrow$$

$$\alpha = \tau'(1) = d_1$$

$$\alpha = f(\alpha) = \dim_b(S_{\alpha=d_1})$$

Popular generalized dimensions: d_0, d_1, d_2

- $d_0 = -\tau(0) = \dim_b\{g\colon \lambda(dg)>0\}$, box/capacity dimension

  $\log \#\{\Delta^i_L\colon \lambda(\Delta_L)>0\} \cong d_o \log L,\ \ L\to 0$

  $\{\Delta^i_L\}$ is covering of G

- $d_1 = \tau'(1)$, information dimension (typical singularity)

  Entropy$_L$: $-\sum \lg P(\Delta^i_L)\cdot P(\Delta^i_L) \cong -d_1 \log L,\ \ L\to 0,$

  $P(dg) = \lambda(dg)/\lambda(G)$

- $d_2 = \tau(2)$, correlation dimension (singularity but not dimension)

  $\log \int \lambda(\Delta_L(g))\lambda(dg) \cong d_2 \log L,\ L\to 0$

Myth: d_2 is the best physics-oriented dimension

Monofractality: $d_q \equiv d_1$ for any q

Generalized Gutenberg-Richter law (*Kossobokov et al.)*

$$\log\lambda(\Delta_L) = A - Bm_c + C \log L$$

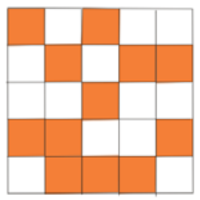
parameters		*in practice*
B	,	b-value in GR law
C, "fractal dimension",		$C=d_2$
range of L, $(L\_, L_+)$ **?**	,	$L_+<1000$ km
Δ_L , **?**	,	any box of size L in G

Applications: scaling, seismic risk analysis

Problem: ***C*** **- ?**

- If A, B, C are constant in G
  and Δ_L is any seismogenic box in G then $C=d_0$

Hint:



$$\lambda(G) = \lambda(\Delta_L) \cdot \#\{\Delta_L : \lambda(\Delta)>0\}$$

$$\text{const} = \lambda(\Delta_L) \cdot L^{-d_0}$$

- More realistic case:

  $\log R_L(q)$ vs $\log L$ *is linear for* $L=L_1, L_2,\ldots L_n$

Least Square Weighted estimate of C:

$$\sum_{i,j}\left[\log\lambda(\Delta^i_{L_j}) - a - C\log L_j\right]^2 w(\Delta^i_{L_j}) \Rightarrow \min_{a,C}$$

$a=A-Bm_c$, $\{\Delta^i_L\}$, covering of G

$w(\Delta) = [\lambda(\Delta)]^q,\ q>0 \Rightarrow C \cong \tau'(q)$

- $C \cong d_1$ (typical singularity) $\leftrightarrow$ $w(\Delta)=\lambda(\Delta)$

  $w(\Delta)$, $q\neq1$ filters the seismogenic points as $L\to0$

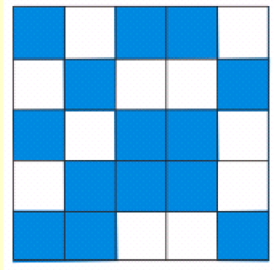
- coupling of $C\cong\tau'(q)$ and $w(\Delta)=\lambda^q(\Delta)$ is key point in understanding of results of multifractal analysis with finite/infinite number of scales.

Empirical multifractal analysis

Crucial postulate: seismicity is "self-similar",
i.e. "looks the same" in different scales

$$\Rightarrow \quad \log R_L(q) = const + \tau(q)\log L, \ L = L_1 \ldots L_n \in (L\_, L_+)$$

$$R_L(q) = \sum_{\lambda(\Delta)>0} \left[\frac{\lambda(\Delta_L^i)}{\lambda(G)} \right]^q$$



$\{\Delta^i{}_L\}$, covering of G

***Problem*:** estimation of $\tau(q)$

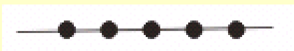
Math. approach

- $L_{\rm i} \to 0$, i=1,2...∞

Empirical approach

- $L_{\rm i}>0$, i=1,2…n, $L_{\rm i} \in (L\_, L_+)$?

$\lambda(dg)$ can look like multifractal differently in different scale ranges

Example: 

$S=\{1, 2, 3, \ldots\}$

$$\dim_b(S) = \begin{cases} 0, & L < 1 \\ 1, & L > 1 \end{cases}$$

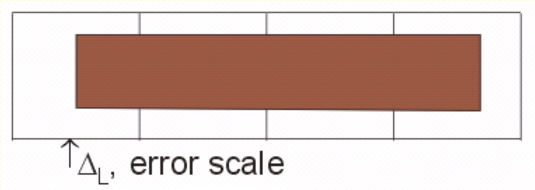
Math approach

- shape of Δ_L is arbitrary
- $\lambda(dg)$, multifractal
- boundary effect: no

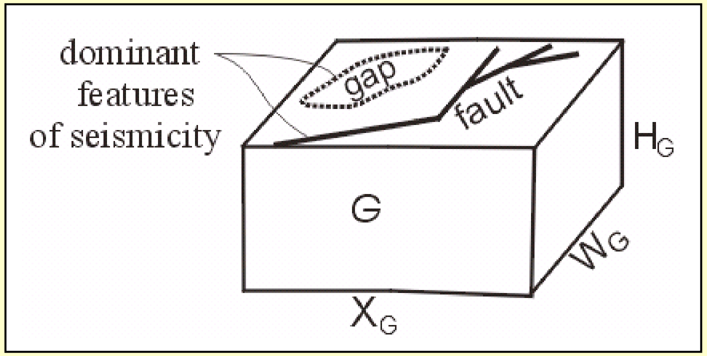
Empirical approach

- shape of Δ_L is a parameter:
  ball/cube → isotropic case
  parallelepiped → anisotropic case
- $\lambda(\Delta_L) = \#\{\text{event } (t,g,m) \in \Delta T \times \Delta_L \times [m_c,\infty]\}$,
  trivial multifractal
- yes

Example:



$$d_q \leq d_0 = 1$$



Upper bound

$$L < \min(X_G, W_G, H_G, L_{flt}, L_{gap})$$

Lower bound

- $L > \begin{cases} \delta g, \text{ location uncertainty} \\ L_{\overline{m}} \approx (\overline{m} - 4)/2, \text{ maximum rupture length of } (m < \overline{m}) - \text{events} \end{cases}$

$\overline{m} : \Pr\{m > \overline{m}\} = \varepsilon$

$\text{GR law} \Rightarrow \overline{m} = m_c + |\lg \varepsilon| / (b - \text{value})$

$L_{\overline{m}} < 1\text{km if } m_c = 3, \varepsilon = 10^{-b} \cong 0.1$

- $L > L_{\text{stbl}}$

Idea: $n(\Delta_L)$ = #{event in Δ_L} can't be small to identity the type of singularity of $\lambda(dg)$ at $g \in \Delta_L$

Goltz (1997): $< n(\Delta_L^i) > = \overline{n}_L > 5, \quad L > L_{\text{stbl}}$

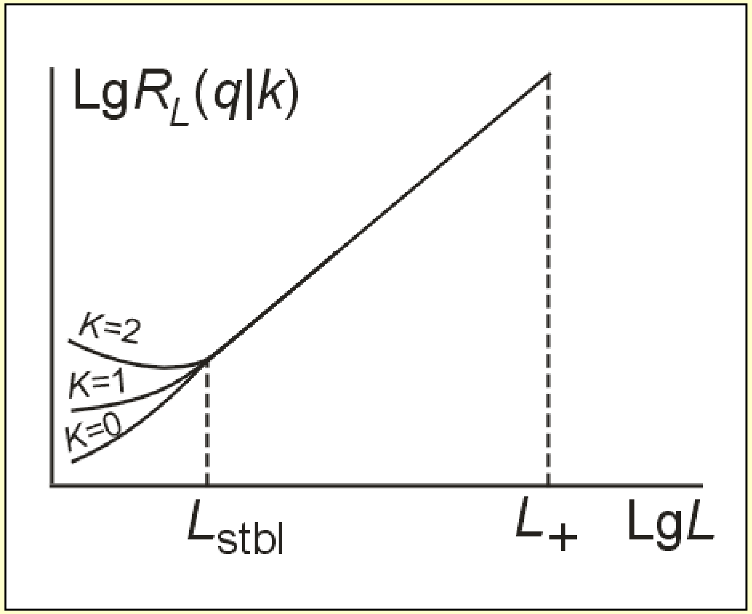
More flexible approach:

$\lg R_L$ *vs* $\lg L$ regression should be stable under

the operation of rejecting "half-empty" boxes, $n(\Delta_L) \le k,\ k = 1; 2; 3$

i.e.

$$\lg R_L(q) \cong \lg R_L(q|k),\ \ L_{\mathrm{stbl}} < L < L_+$$



$$R_L(q\,|\,k) = \sum_{i:\hat{n}(\Delta_L^i)\ge k} \left[\frac{\lambda(\Delta_L^i)}{\lambda_k(G)}\right]^q$$

$$\lambda_k(G) = \sum_{\hat{n}(\Delta_L^i)\ge k} \lambda(\Delta_L^i)$$

Fractality of Regional Seismicity

Regions with nontrivial fractal properties:

- 3D analysis, $L_+/L_- \cong 10$: Kamchatka, New Zealand

- 2D analysis,
  L_+/L_-=40-50: S.California, Garm (C.Asia), New Zealand, Kamchatka
  L_+/L_-=10-20: Greece, C.American Arc, Costa Rica

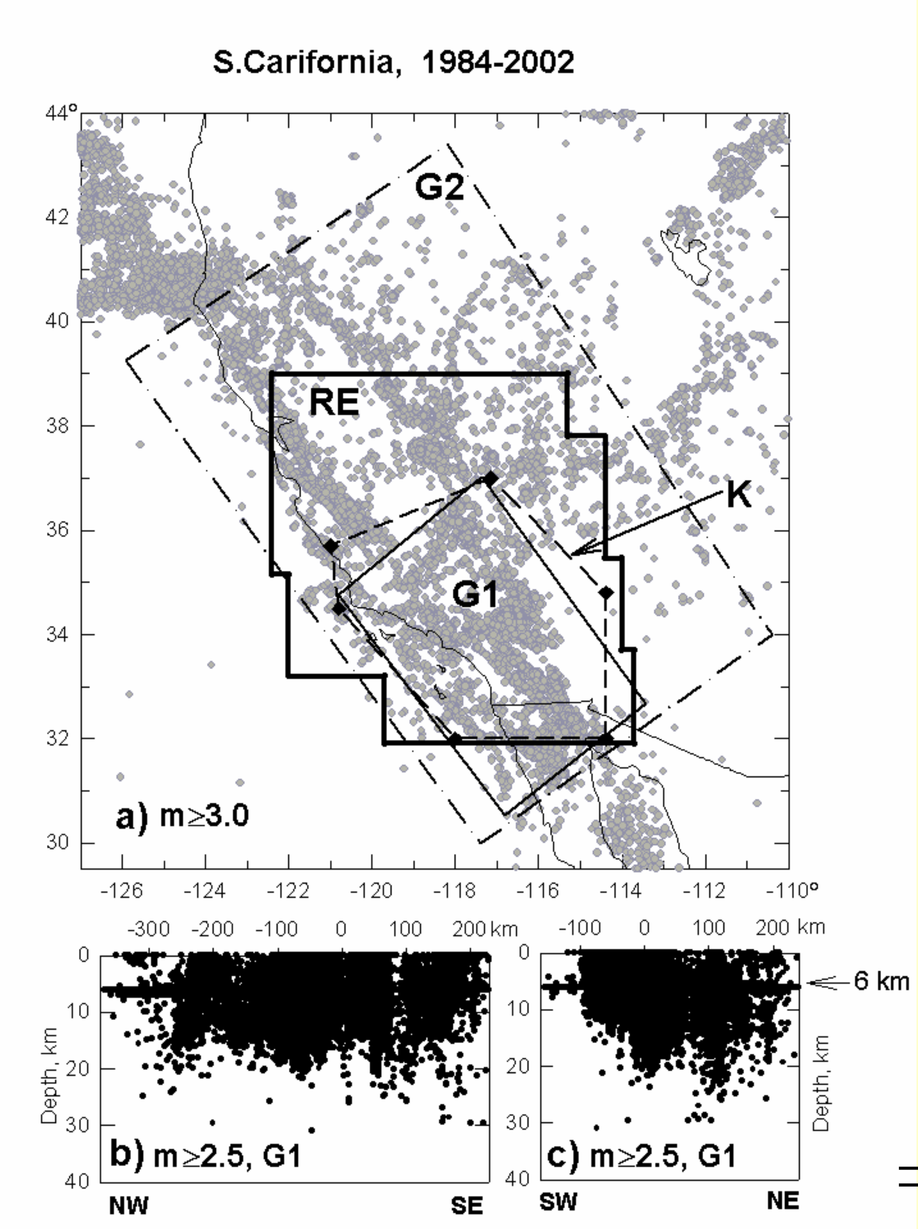
Excluded regions ($L_+/L_- < 10$)

Mid-ocean ridges, Kuril Islands (very narrow seismic belts)

Aleutian Islands, Tonga, Philippines Arc, Andes

Western Turkey, Alaska, South of W.Alps

Betica (Spain)



Transform fault zone

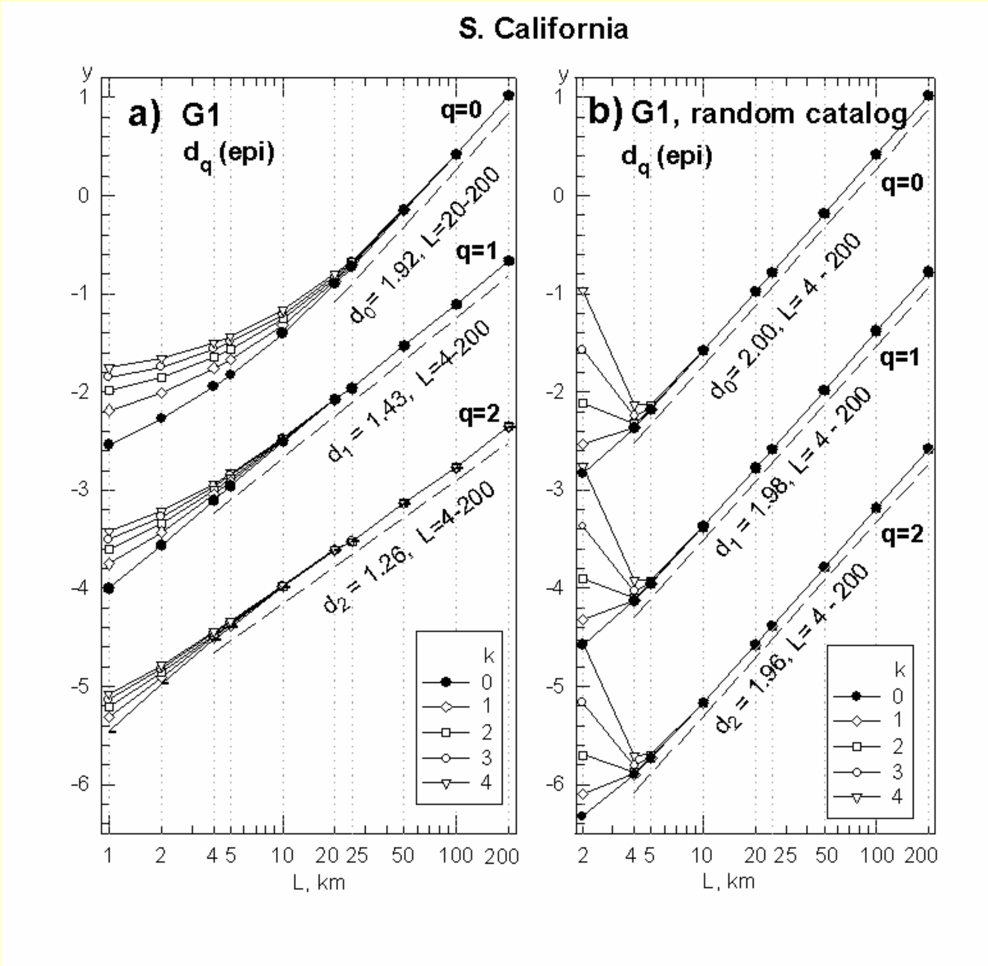
RE: relocated catalog
by Haukson et al.(2005)
location: $\delta g \leq 0.1$ km

- **G1** $\times[0, 15\text{ km}]$:
  $\bar{n}_L = 0.0116 L^3 < 1$
  $L = 0.1 - 4$ km
  $\Downarrow$
  3D: $(L_-, L_+) \subset (4, 15\text{ km})$
  $\Downarrow$
  $L_+/L_- \leq 3$ (trivial case)

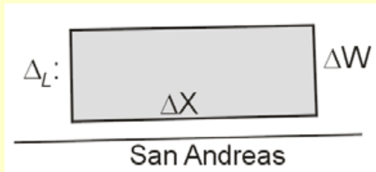
- #{event: h=6 km}=30%

$\Rightarrow$ 3D analysis is unreasonable

Conclusion: 2D analysis, $m \geq 2$

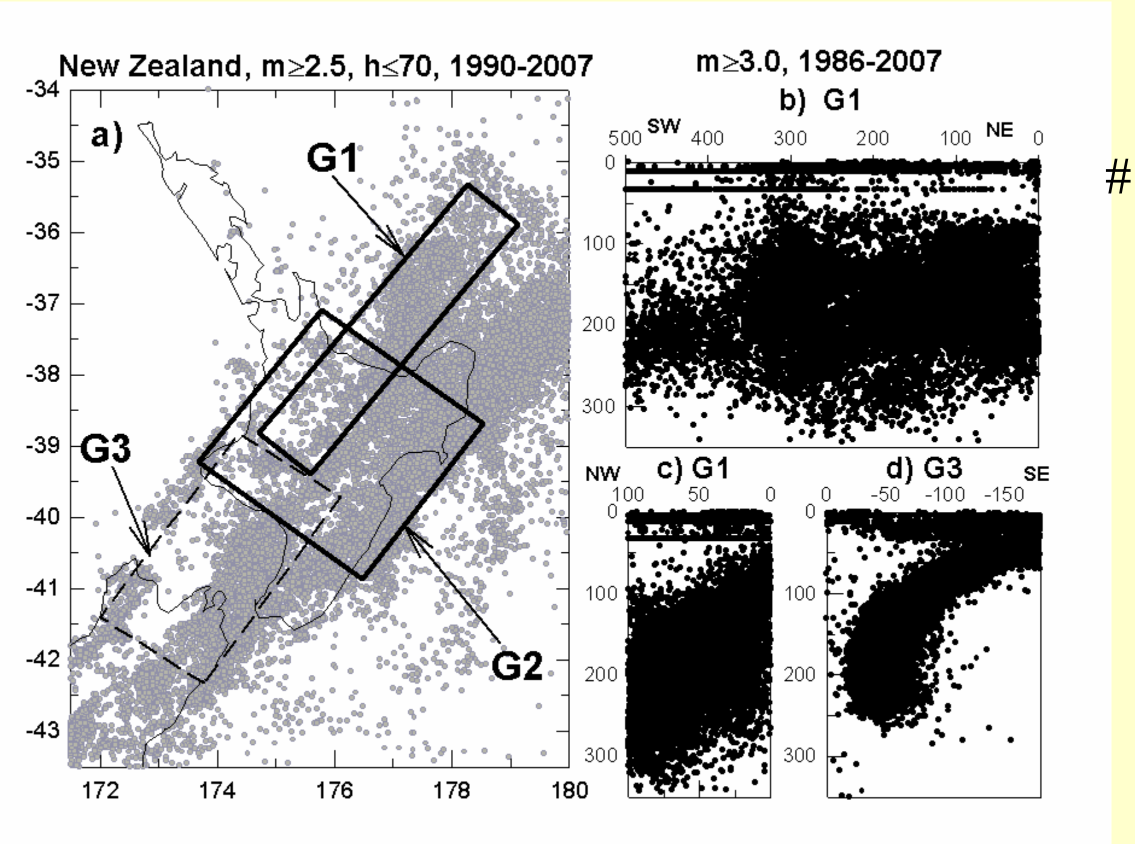


- $L_+/L\_ = 50$ (!)
- d_1, d_2 are relatively low
- $d_1 \neq d_2$ (multifractality) because d_1-d_2: 0.17 (real) >> 0.02 (random calalog)
- Shape of Δ_L is inessential:



$\Delta X/\Delta W = 0.5 – 5$

$\delta d_1 = \pm 0.03$, $\delta d_2 = (-0.03, 0.08)$



Subduction zone

Regional catalog:

#{event:h=5; 13; 28; 33 km}=23.4%

3D analysis: $m \geq 3$,

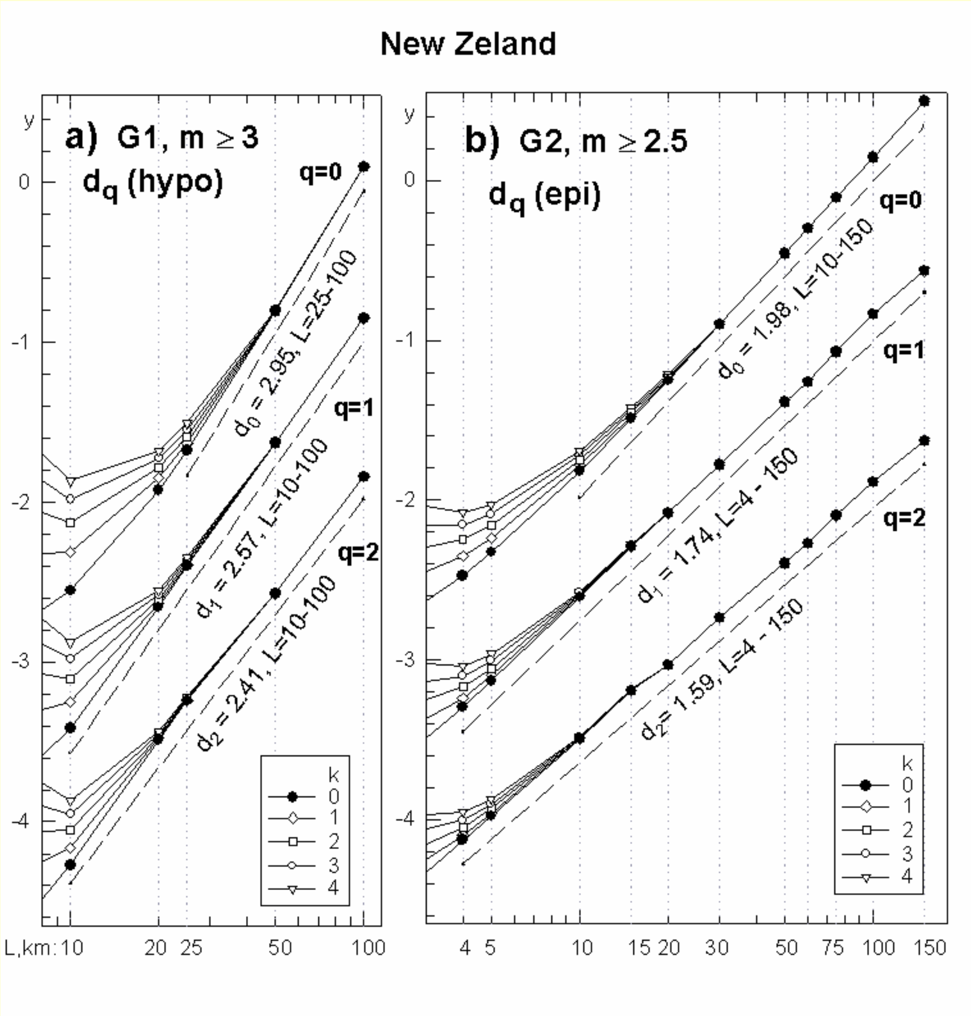
G1$\times(100 \leq H_G \leq 300$ km)

$= 500 \times 100 \times 200$ km,

2D analysis: $m \geq 2.5$,

$h \leq 70$ km

G2$=600 \times 300$ km



3D analysis:

$L_+/L\_=10$

2D analysis:

$L_+/L\_=38$ (**!**)

Harte(2001):

unsuccessful attempt of 2D analysis in Reg. **G3**

Conclusion

Is seismicity fractal?	It **looks** as a fractal for $m \geq (2\text{-}3.5)$; for $m \geq 4$?
mono- or multi-fractal?	**multi**, $d_0 \neq d_1 \neq d_2$
everywhere in space?	**NO**
nontrivial range of scale?	**yes**: $L_+/L\_=10$ – **50**, 2D case
shape of Δ_L?	insignificant
the best d_q?	d_1 is typical singularity & $\dim_b\{S_{\alpha=d_1}\}$
d_1(2D) d_2(2D)	~ 1.4 / ~ 1.3 [*S.California* transform fault] ~ 1.8 / $1.6\text{-}1.7$ [4 subduction zones[1)] 2 collision zones[2)]] [1)]Kamchatka, New Zealand,C.American Arc, Costa Rica [2)] Greece, Garm (Central Asia)

The use of multifractality

- nonproductive in form of generalised GR law for seismic risk
- productive in prediction of some scaling relations, e.g. $\lambda(\Delta_L)$, $\tau(\Delta_L)$

Optimal Spatial Scaling

- $\lambda(\Delta_L)$, seismicity rate of events in Δ_L
- $t(\Delta_L)$, inter-event time (time between successive events in Δ_L)

Δ_L, random box in G: $\Pr(\Delta_L)=w(\Delta_L)$, $\sum_i w(\Delta_L^i)=1$

Problem of scaling:

$$\lambda(\Delta_L)\propto L^{C_\lambda}, \quad C_\lambda\,? \qquad\qquad t(\Delta_L)\propto L^{-C_t}, \quad C_t-?$$

In practice: $C_\lambda=C_t=d_0$(Corral) $\qquad C_t=d_2$ (Bak et al.)

because $\lambda(\Delta_L)\cdot \mathrm{E}t(\Delta_L)=1$

Solution under conditions:

- $w_q(\Delta_L)\propto\lambda^q(\Delta_L)$
- multifractality of $\lambda(dg)$

Results

- scaling of means

$w(\Delta)$

$$<\lambda(\Delta_L^i)> = \sum_i \lambda^i(\Delta_L) w_q(\Delta_L^i) \propto L^{\tau(q+1)-\tau(q)} = \begin{cases} L^{d_0} & \text{uniform} \\ L^{d_2} & \lambda(\Delta) \end{cases}$$

$$<t(\Delta_L^i)> = \sum_i \mathrm{E}t(\Delta_L^i) w_q(\Delta_L^i) \propto L^{\tau(q-1)-\tau(q)} = \begin{cases} L^{-d_0} & \lambda(\Delta) \\ L^{-d_2} & \lambda^2(\Delta) \end{cases}$$

- scaling of $\lambda(\Delta^i_L)$ and $t\,(\Delta^i_L)$ as the random variables

$$w(\Delta) \propto \lambda^q(\Delta) \Rightarrow \lambda(\Delta_L^i) \propto L^{\tau'(q)} \quad = \quad L^{d_1} \quad \lambda(\Delta)$$

$$t(\Delta_L) \propto L^{-\tau'(q)} \quad = \quad L^{-d_1} \quad \lambda(\Delta)$$

Testing

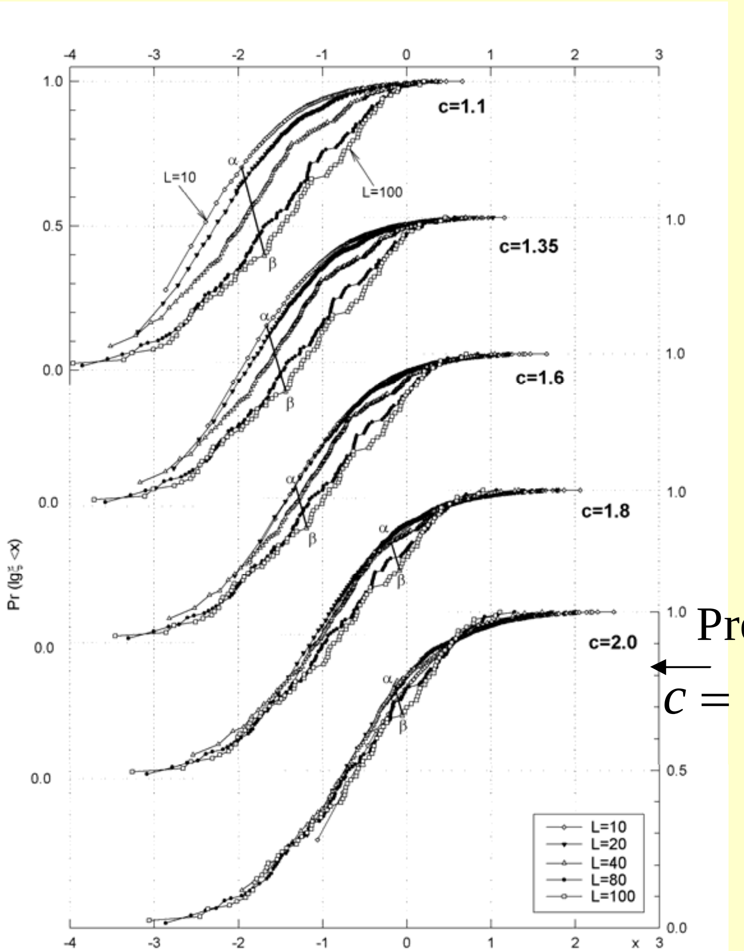
$\lambda(\Delta_L) \rightarrow \lambda(\Delta_L) L^{-c}$, renormalization

The set of distributions $\Pr\{\lambda(\Delta_L)L^{-c} < x\}$, $L \in (L\_, L_+)$
for fixed q must be the most tight if $c = \tau'(q)$

Scaling of $\lambda(\Delta_L)$: S.California (Reg. **G2**)

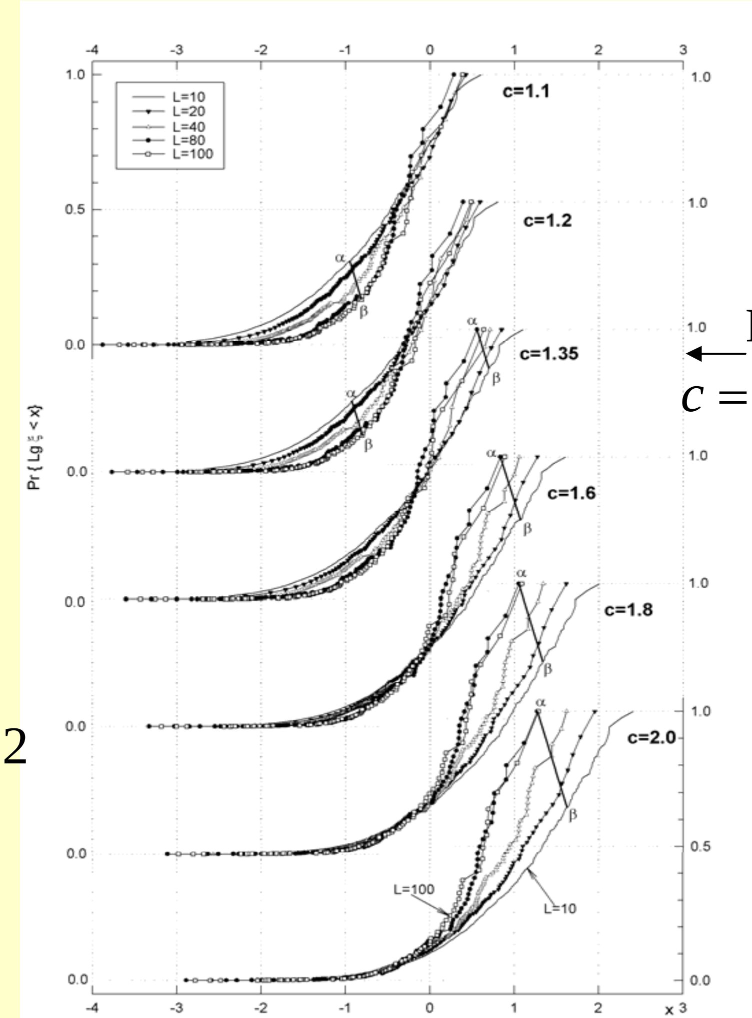
Pr(Δ), uniform

$\Pr(\Delta) \propto \lambda(\Delta)$



Predicted

$c = \tau'(0) \cong 2$



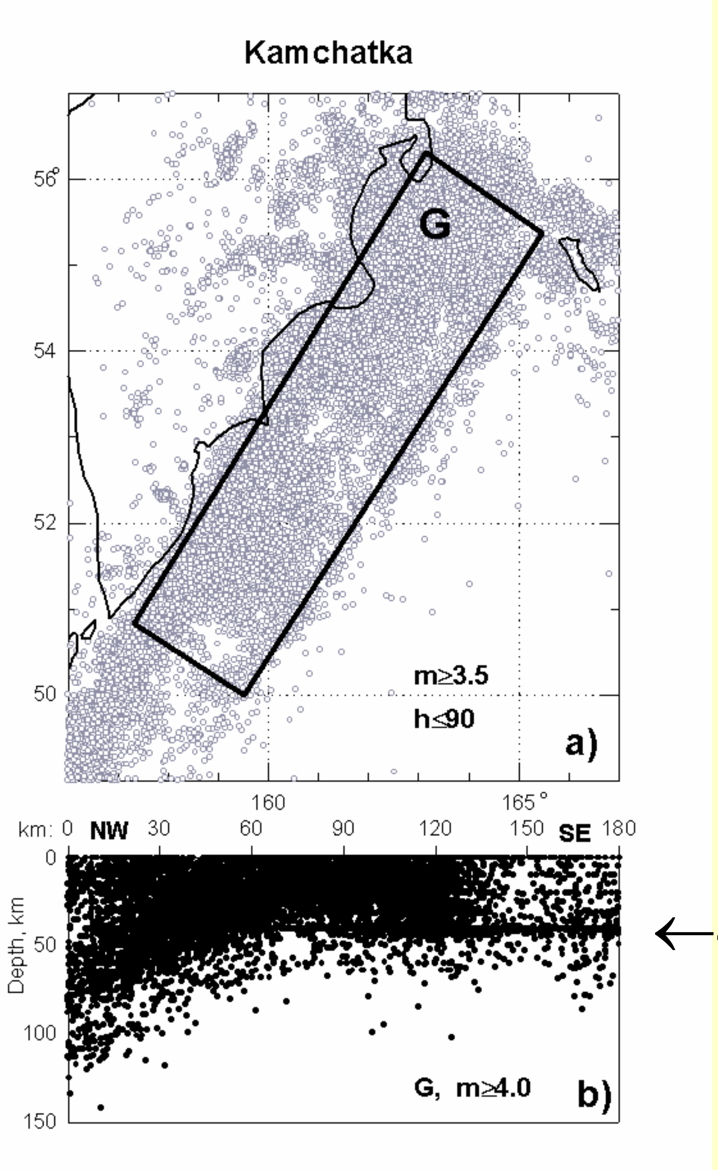
Predicted

$c = \tau'(1) \cong 1.35$

α — β Levy distance

vertical axis: $\Pr(\lambda(\Delta_L)L^{-c}<x)$, L=10-100 km

THANK YOU



Subduction zone

G: 720×180×[0, 90 km]

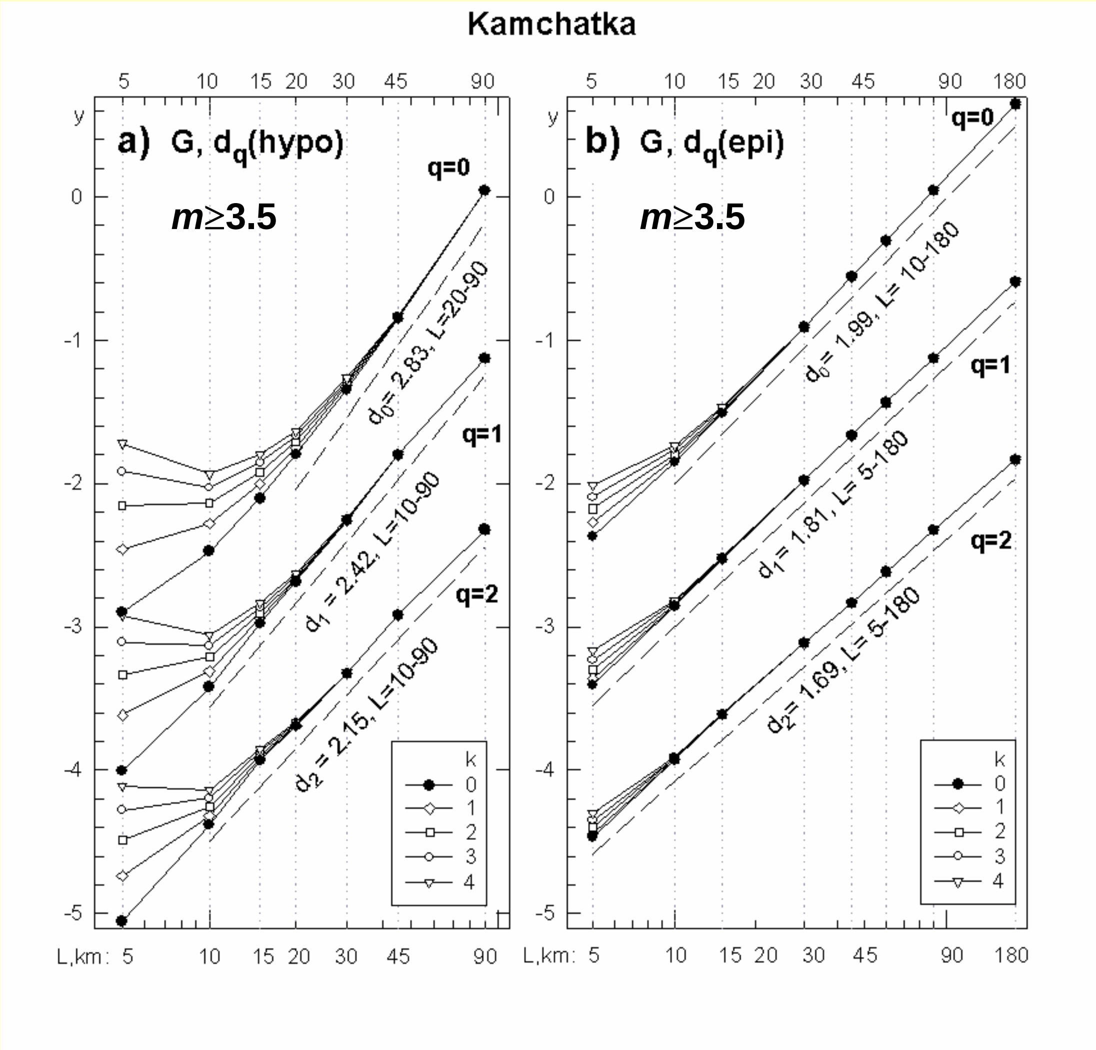
Composite catalog:
m≥3.5, 1962-2003

$L\_ \geq 10$ km:

$$\bar{n}(\Delta_{10\,\mathrm{km}}) = 2.3$$

δg=5 – 10 km

←40 km, 20% of events



- *m*=3.5 is relatively high level
- Similarity for *L*=10 – 180 km