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Nuclear reactor dynamics - I

Piero Ravetto

Politecnico di Torino
Dipartimento di Energetica
Italy

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Preface

Piero Ravetto – Politecnico di Torino:

- Professor of nuclear reactor physics
- Chair of the nuclear and energy engineering program

Activities on reactor physics

Transport theory:

- Development of algorithms in P_N and S_N framework (FEM, BEM …)
- Treatment of high anisotropy problems and reduction of ray-effects
- Neutron propagation phenomena

Activities on reactor physics

Reactor dynamics:

- Development of algorithms and codes (quasi-statics, multipoint…)
- ADS dynamics
- Interpretation of experiments

Activities on reactor physics

Innovative reactor technology:

- Models and methods for fluid-fuel (molten-salt) systems
- Liquid-lead cooled reactors

PART I

- Introduction to problem
- The basics of the neutronic model
- The physical features of neutron kinetics
- A simple but interesting example to understand physics

Standard tasks of reactor physics 1.

Describe basic phenomena of neutron motion in material systems:

neutronic design of steady-state critical reactors

↓

Provide multiplication parameters and flux distributions

Standard tasks of reactor physics 2.

Short scale dynamic simulation

Provide information on transient behaviour in operational and accident conditions for stability and safety assessments

Standard tasks of reactor physics 3.

Long scale dynamic simulation

↓

Provide information on burn-up and nuclide evolution for fuel management

A new challenge of reactor physics

Neutronic design of innovative systems

↓

New features in static and dynamic simulations

Need to develop specific models and algorithms

Basic model for the neutronics of nuclear reactors: the transport equation

The equation is of deterministic nature, the balance is based on statistical principles

The equation for neutrons can be derived from the original non-linear equation for particles in a force field removing the force term and assuming neutron collisions only with a fixed bakground of nuclei (equation becomes linear)

Transport (kinetic) theory plays a fundamental role for all the standard and advanced tasks of reactor physicists

Ludwig E. Boltzmann (1844 – 1906)

Basics of transport theory for neutrons

To write the particle balance probabilities per unit neutron path are needed: cross sections $\Sigma(\mathbf{r}, E)$

Emission function is also needed $f(\mathbf{r}, E' \to E, \Omega' \cdot \Omega)$

Note: isotropic medium is supposed (does not imply isotropic emissions !)

[J.J Duderstadt, W. Martin, *Transport Theory*]

Basics of transport theory for neutrons

Neutron track length per unit volume, per unit energy, per unit solid angle, per unit time: neutron flux (velocity x density)

$$\varphi(\mathbf{r}, E, \Omega, t)$$

Basics of transport theory for neutrons

Elementary neutron current vector, neutrons crossing the unit oriented area at one space point per unit time, per unit energy, per unit solid angle

$$\Omega\varphi(\mathbf{r}, E, \Omega, t)$$

Integro-differential form (first order)

Local balance of particles

$$\frac{1}{v(E)}\frac{\partial\varphi(\mathbf{r},E,\Omega,t)}{\partial t}+\nabla\cdot\Omega\varphi(\mathbf{r},E,\Omega,t)+\Sigma(\mathbf{r},E)\varphi(\mathbf{r},E,\Omega,t)$$
$$=\int dE'\oint d\Omega'\Sigma(\mathbf{r},E')\varphi(\mathbf{r},E',\Omega',t)f(\mathbf{r},E'\rightarrow E,\Omega'\cdot\Omega)+S(\mathbf{r},E,\Omega,t)$$

$\varphi(\mathbf{r},E,\Omega,t=0)=\varphi_0(\mathbf{r},E,\Omega)$ Initial conditions

$\varphi(\mathbf{r}_S,E,\Omega_{in},t)=0$ Vacuum boundary conditions

Integro-differential form (second order, isotropic emissions)

even flux $\psi(\mathbf{r}, E, \Omega, t) = \frac{1}{2}\left[\varphi(\mathbf{r}, E, \Omega, t) + \varphi(\mathbf{r}, E, -\Omega, t)\right]$

odd flux $\chi(\mathbf{r}, E, \Omega, t) = \frac{1}{2}\left[\varphi(\mathbf{r}, E, \Omega, t) - \varphi(\mathbf{r}, E, -\Omega, t)\right]$

$$\left\{\begin{array}{l} \frac{1}{v(E)}\frac{\partial\psi(\mathbf{r}, E, \Omega, t)}{\partial t} - \Omega\cdot\nabla\frac{1}{\Sigma(\mathbf{r}, E)}\Omega\cdot\nabla\psi(\mathbf{r}, E, \Omega, t) \\ \qquad +\Sigma(\mathbf{r}, E)\psi(\mathbf{r}, E, \Omega, t) = \frac{1}{4\pi}Q(\mathbf{r}, E, t) \\ \chi(\mathbf{r}, E, \Omega, t) = -\frac{1}{\Sigma(\mathbf{r}, E)}\Omega\cdot\nabla\psi(\mathbf{r}, E, \Omega, t) \end{array}\right.$$

Space second order term

Integral form

Global balance of particles

$$\varphi(\mathbf{r}, E, \Omega, t) =$$

$$= \int_0^{Min[s_0(\mathbf{r},\Omega),vt]} ds \left[\int dE' \oint d\Omega' \Sigma(\mathbf{r} - s\Omega, E')\varphi(\mathbf{r} - s\Omega, E', \Omega', t - \frac{s}{v}) f(\mathbf{r} - s\Omega, E' \to E, \Omega' \cdot \Omega) \right.$$
$$\left. + S(\mathbf{r} - s\Omega, E, \Omega, t - \frac{s}{v}) \right] \exp\left(- \int_0^s ds' \Sigma(\mathbf{r} - s'\Omega, E) \right)$$

Integral form for isotropic emissions: Peierls equation

$$\Phi(\mathbf{r},E,t) = \frac{1}{4\pi}\int d\mathbf{r}'\left[\int dE' \oint d\Omega' \Sigma(\mathbf{r}',E')\Phi(\mathbf{r}',E',t-\frac{|\mathbf{r}-\mathbf{r}'|}{v})f(\mathbf{r}',E'\to E)\right.$$
$$\left. + S(\mathbf{r}',E,t-\frac{|\mathbf{r}-\mathbf{r}'|}{v})\right]\frac{\exp\left(-\int_0^{|\mathbf{r}-\mathbf{r}'|} ds'\,\Sigma(\mathbf{r}-s'\Omega,E)\right)}{|\mathbf{r}-\mathbf{r}'|^2}$$

The Monte Carlo approach

The full statistical simulation retrieves information on the solution of the integral equation

The simulation is performed on the basis of elementary interaction probability laws

The emission from fission

$$\frac{1}{v(E)}\frac{\partial \varphi(\mathbf{r},E,\Omega,t)}{\partial t} + \nabla \cdot \Omega \varphi(\mathbf{r},E,\Omega,t) + \Sigma(\mathbf{r},E)\varphi(\mathbf{r},E,\Omega,t)$$
$$= \int dE' \oint d\Omega' \Sigma_s(\mathbf{r},E')\varphi(\mathbf{r},E',\Omega',t) f_s(\mathbf{r},E' \to E,\Omega' \cdot \Omega)$$
$$+ \frac{\chi(\mathbf{r},E)}{4\pi}\int dE' \Sigma_f(\mathbf{r},E') \oint d\Omega' \varphi(\mathbf{r},E',\Omega',t) + S(\mathbf{r},E,\Omega,t)$$

The emission from fission: delayed neutrons

$$\frac{1}{v(E)}\frac{\partial\varphi(\mathbf{r},E,\Omega,t)}{\partial t}+\nabla\cdot\Omega\varphi(\mathbf{r},E,\Omega,t)+\Sigma(\mathbf{r},E)\varphi(\mathbf{r},E,\Omega,t)$$
$$=\int dE'\oint d\Omega'\Sigma_s(\mathbf{r},E')\varphi(\mathbf{r},E',\Omega',t)f_s(\mathbf{r},E'\to E,\Omega'\cdot\Omega)$$
$$+\frac{\chi(\mathbf{r},E)}{4\pi}(1-\beta)\int dE'\Sigma_f(\mathbf{r},E')\oint d\Omega'\varphi(\mathbf{r},E',\Omega',t)$$
$$+\sum_{i=1}^{R}\lambda_i C_i(\mathbf{r},t)\frac{\chi_i(\mathbf{r},E)}{4\pi}+S(\mathbf{r},E,\Omega,t)$$

The emission from fission: delayed neutrons

Additional equations are needed for delayed precursors (for solid fuel):

$$\frac{\partial C_i(\mathbf{r},t)}{\partial t} = \beta_i \int dE' \nu \Sigma_f(\mathbf{r},E') \oint d\Omega' \varphi(\mathbf{r},E',\Omega',t) - \lambda_i C_i(\mathbf{r},t)$$

Initial conditions $$C_i(\mathbf{r},t=0) = C_{i,0}(\mathbf{r})$$

The source-free steady-state problem and the eigenvalue

$$\frac{1}{v(E)}\frac{\partial\varphi(\mathbf{r},E,\Omega,t)}{\partial t}+\nabla\cdot\Omega\varphi(\mathbf{r},E,\Omega,t)+\Sigma(\mathbf{r},E)\varphi(\mathbf{r},E,\Omega,t)$$
$$=\int dE'\oint d\Omega'\Sigma(\mathbf{r},E')\varphi(\mathbf{r},E',\Omega',t)f(\mathbf{r},E'\to E,\Omega'\cdot\Omega)+S(\mathbf{r},E,\Omega,t)$$

The source-free steady-state problem and the eigenvalue

$$\nabla \cdot \Omega\varphi(\mathbf{r},E,\Omega) + \Sigma(\mathbf{r},E)\varphi(\mathbf{r},E,\Omega)$$
$$= \int dE' \oint d\Omega' \Sigma_s(\mathbf{r},E')\varphi(\mathbf{r},E',\Omega') f_s(\mathbf{r},E' \to E,\Omega' \cdot \Omega)$$
$$+ \frac{\chi(\mathbf{r},E)}{4\pi} \int dE' \nu\Sigma_f(\mathbf{r},E') \oint d\Omega' \varphi(\mathbf{r},E',\Omega') + S(\mathbf{r},E,\Omega)$$

$$\nabla \cdot \Omega\varphi(\mathbf{r},E,\Omega) + \Sigma(\mathbf{r},E)\varphi(\mathbf{r},E,\Omega)$$
$$= \int dE' \oint d\Omega' \Sigma_s(\mathbf{r},E')\varphi(\mathbf{r},E',\Omega') f_s(\mathbf{r},E' \to E,\Omega' \cdot \Omega)$$
$$+ \frac{\chi(\mathbf{r},E)}{4\pi} \int dE' \nu\Sigma_f(\mathbf{r},E') \oint d\Omega' \varphi(\mathbf{r},E',\Omega')$$

The source-free steady-state problem and the eigenvalue

$$\nabla \cdot \Omega\varphi(\mathbf{r},E,\Omega) + \Sigma(\mathbf{r},E)\varphi(\mathbf{r},E,\Omega)$$
$$= \int dE' \oint d\Omega' \Sigma_s(\mathbf{r},E')\varphi(\mathbf{r},E',\Omega') f_s(\mathbf{r},E' \to E,\Omega' \cdot \Omega)$$
$$+ \left[\frac{1}{k}\right] \frac{\chi(\mathbf{r},E)}{4\pi} \int dE' \nu\Sigma_f(\mathbf{r},E') \oint d\Omega' \varphi(\mathbf{r},E',\Omega')$$

$$\nabla \cdot \Omega\varphi(\mathbf{r},E,\Omega) + \Sigma(\mathbf{r},E)\varphi(\mathbf{r},E,\Omega)$$
$$= \left[\frac{1}{\gamma}\right] \left[\int dE' \oint d\Omega' \Sigma_s(\mathbf{r},E')\varphi(\mathbf{r},E',\Omega') f_s(\mathbf{r},E' \to E,\Omega' \cdot \Omega) \right.$$
$$\left. + \frac{\chi(\mathbf{r},E)}{4\pi} \int dE' \nu\Sigma_f(\mathbf{r},E') \oint d\Omega' \varphi(\mathbf{r},E',\Omega') \right]$$

The source-free steady-state problem and the eigenvalue

$$\nabla \cdot \Omega\varphi(\mathbf{r},E,\Omega) + \left[\Sigma(\mathbf{r},E) + \frac{\alpha}{v}\right]\varphi(\mathbf{r},E,\Omega)$$
$$= \int dE' \oint d\Omega' \Sigma_s(\mathbf{r},E')\varphi(\mathbf{r},E',\Omega') f_s(\mathbf{r},E' \to E, \Omega' \cdot \Omega)$$
$$+ \frac{\chi(\mathbf{r},E)}{4\pi} \int dE' \nu\Sigma_f(\mathbf{r},E') \oint d\Omega' \varphi(\mathbf{r},E',\Omega')$$

$$\nabla \cdot \Omega\varphi(\mathrm{r},E,\Omega)$$
$$= \left[\frac{1}{\delta}\right]\left[\int dE' \oint d\Omega' \Sigma_s(\mathrm{r},E')\varphi(\mathrm{r},E',\Omega') f_s(\mathrm{r},E' \to E, \Omega' \cdot \Omega)\right.$$
$$\left. + \frac{\chi(\mathrm{r},E)}{4\pi} \int dE' \nu\Sigma_f(\mathrm{r},E') \oint d\Omega' \varphi(\mathrm{r},E',\Omega') - \Sigma(\mathrm{r},E)\varphi(\mathrm{r},E,\Omega)\right]$$

Eigenvalue: various formulations

Multiplication k

Collision γ

Time α

Density δ

Eigenvalue

The time eigenvalue can be defined to include delayed neutron information (ω-modes)

$$\nabla \cdot \Omega\varphi(\mathbf{r},E,\Omega) + \left[\Sigma(\mathbf{r},E) + \frac{\alpha}{v}\right]\varphi(\mathbf{r},E,\Omega)$$

$$= \int dE' \oint d\Omega' \Sigma_s(\mathbf{r},E')\varphi(\mathbf{r},E',\Omega') f_s(\mathbf{r},E' \to E, \Omega' \cdot \Omega)$$

$$+ \frac{1}{4\pi}\left[(1-\beta)\chi_p(\mathbf{r},E) + \sum_{i=1}^{R} \frac{\beta_i \lambda_i}{(\alpha+\lambda_i)}\chi_i(\mathbf{r},E)\right] \int dE' \nu\Sigma_f(\mathbf{r},E') \oint d\Omega' \varphi(\mathbf{r},E',\Omega')$$

What is a critical system?

A system for which a non-zero solution exists in the absence of any external source !

Hence: $k=1$

$\alpha=0$

What is a subcritical system?

A system for which fission production cannot compensate losses due to streaming (leakage through the boundary) and net removal through collisions; only by a source a steady-state can be established !

Hence: $k<1$

$\alpha<0$

What is a supercritical system?

A system for which fission production is larger than losses due to streaming (leakage through the boundary) and net removal through collisions; no steady-state can be established !

Hence: $k>1$

$\alpha>0$

The adjoint transport equation

Two ways to approach this problem:

1. Construct the adjoint equation and interpret it physically
2. Define the physical quantity "neutron importance" and construct its balance equation… observe it is the adjoint

The adjoint transport equation: approach 1.

Definition of adjoint operator requires definition of inner (scalar) product:

$$(g, f) = \langle g \,| f \rangle = \int d\mathbf{r} \int dE \oint d\Omega g(\mathbf{r}, E, \Omega) f(\mathbf{r}, E, \Omega)$$

The adjoint operator is such that:

$$(g, \Theta f) = \left(\Theta^{+} g, f\right)$$

The adjoint transport equation: approach 1.

For the operators of the transport equation:

$$\Sigma(\mathbf{r}, E)* \text{ is self-adjoint}$$

$$\left[\int dE' \oint d\Omega' \Sigma(\mathbf{r}, E') f(\mathbf{r}, E' \to E, \Omega' \cdot \Omega)*\right]^+$$

$$= \int dE' \oint d\Omega' \Sigma(\mathbf{r}, E) f(\mathbf{r}, E \to E', \Omega \cdot \Omega')*,$$

If functions of the adjoint space obey the following property

$$g(\mathbf{r}_s, E, \Omega_{out}) = 0,$$

$$[\nabla \cdot (\Omega *)]^+ = [-\nabla \cdot (\Omega *)]$$

The adjoint transport equation: approach 1.

Here is the adjoint equation:

$$
\begin{aligned}
& -\nabla \cdot \Omega\varphi^+(\mathbf{r}, E, \Omega) + \Sigma(\mathbf{r}, E)\varphi^+(\mathbf{r}, E, \Omega) \\
= \quad & \int dE' \oint d\Omega' \Sigma_s(\mathbf{r}, E)\varphi^+(\mathbf{r}, E', \Omega') f_s(\mathbf{r}, E \to E', \Omega \cdot \Omega') \\
& + \int dE' \frac{\chi(\mathbf{r}, E')}{4\pi} \nu\Sigma_f(\mathbf{r}, E') \oint d\Omega' \varphi^+(\mathbf{r}, E', \Omega') + S^+(\mathbf{r}, E, \Omega)
\end{aligned}
$$

$$\varphi^+(\mathbf{r}_s, E, \Omega_{out}) = 0$$

Has the adjoint function a physical meaning ?

The adjoint transport equation

The neutron importance: the asymptotic power reached within the stationary system folowing the insertion of one neutron at a given point in phase space (Ussachev)
A balance (importance conservation) equation is written to verify that

$\varphi^{+}(\mathbf{r}, E, \Omega)$ is the neutron importance

The adjoint transport equation

The neutron importance is a basic tool for

- Reactor kinetics
- Variational methods
- Perturbation theory
- Sensitivity analysis

A digression on analytical methods

Why analytical methods?

- To grasp the mathematical nature of the problem
- To get full insight into physics
- To obtain reference solutions

The role of delayed neutrons

- Time-dependent analysis of nuclear systems can be done only taking account of delayed emissions from fission
- On the basis of elementary physics considerations, a multiplying system evolution is regulated by the exponential law $\exp((\delta k/\Lambda)t)$, where ....

The role of delayed neutrons

Λ is a "characteristic" time
- No delayed neutrons: 10^{-4} (thermal reactors) –
- 10^{-6} s (fast reactors… depends on the velocity of neutrons)
- With delayed neutrons: $\Lambda_{prompt} + \beta/\lambda \sim 10^{-1}$ s ($\lambda \sim 10^{-1}$ s^{-1})

Evolution is dominated by delayed neutrons
(for sub-prompt-critical systems)

Note: β is an important dynamic parameter (the *physical* fraction β is: for U235: 0.0065, for Pu239: 0.0022)… the effective fraction may be more or less … why?

Time-scales in the dynamics of nuclear reactors

- Prompt neutron (very fast) scale, connected to the lifetime of prompt neutrons (10^{-4} - 10^{-6} s)
- Delayed emission scale, connected to evolution of delayed neutron precursors (10^{-1} - 10^{1} s)
- Thermal-hydraulic scale (feedback), connected to the evolution of temperatures and hydraulic parameters (10^{-1} - 10^{2} s)
- Control scale, connected to the movement of masses in the system (control rods, poisons)
- Nuclide transmutation scale, connected to neutron transmutation phenomena ($>10^{2}$ s)

Time-scales in the dynamics of nuclear reactors

Very different time-scales

the physico-mathematical problem
is **stiff**

Time-scales in the dynamics of nuclear reactors

- We now focus our interest on the dynamics of nuclear systems during operational and accidental transients
  - Nuclide transmutation can be neglected, but still
    - Delayed emissions
    - Thermal feedback

    are to be considered.

Basic equations for neutron dynamics (1)

Boltzmann transport equation in presence of delayed emissions:

$$\begin{cases} \dfrac{\partial n(\mathbf{r},E,\Omega,t)}{\partial t} = \hat{\mathrm{B}}(t)n(\mathbf{r},E,\Omega,t) + \\ \qquad \displaystyle\sum_{i=1}^{6} \lambda_i \frac{\chi_i(E)}{4\pi} C_i(\mathbf{r},t) + S(\mathbf{r},E,\Omega,t) \\ \dfrac{\partial(\chi_i(E)C_i(\mathbf{r},t)/4\pi)}{\partial t} = \hat{\mathrm{M}}_i(t)n(\mathbf{r},E,\Omega,t) - \\ \qquad \lambda_i \dfrac{\chi_i(E)}{4\pi} C_i(\mathbf{r},t) \end{cases}$$

Basic equations for neutron dynamics (2)

$$\frac{\partial n(\mathbf{r},E,\Omega,t)}{\partial t} = \hat{\mathrm{B}}(t)n(\mathbf{r},E,\Omega,t) + \sum_{i=1}^{6}\lambda_i\frac{\chi_i(E)}{4\pi}C_i(\mathbf{r},t) + S(\mathbf{r},E,\Omega,t)$$

$$\hat{\mathrm{L}}(t) = -\Omega\cdot\nabla v(E) - \Sigma(r,E,t)v(E) + \int dE' \oint d\Omega' v(E')\Sigma_s(r,E'\rightarrow E,\Omega'\cdot\Omega,t)$$

$$\hat{\mathrm{M}}_p(t) = \sum_j \frac{\chi_p^j(E)}{4\pi}\int dE' \oint d\Omega' v(E')(1-\beta^j)\times \nu^j(E')\Sigma_f^j(\mathbf{r},E',t)$$

Leakage

Balance operator

Prompt multiplication

$$\hat{\mathrm{B}}(t) = \hat{\mathrm{L}}(t) + \hat{\mathrm{M}}_p(t)$$

Basic equations for neutron dynamics (3)

$$\frac{\partial(\chi_i(E)C_i(\mathbf{r},t)/4\pi)}{\partial t} = \hat{\mathrm{M}}_i(t)n(\mathbf{r},E,\Omega,t) - \lambda_i\frac{\chi_i(E)}{4\pi}C_i(\mathbf{r},t)$$

Delayed multiplication

$$\hat{\mathrm{M}}_i(t) = \sum_j \frac{\chi_i(E)}{4\pi}\int dE' \oint d\Omega' v(E')\beta_i^j \times \nu^j(E')\Sigma_f^j(r,E',t)$$

Operators can be time-dependent because of:

- effects independent of neutron flux (perturbations)
- non-linear effects (feedback)

Formal approach to neutron dynamics

$$\frac{\partial}{\partial t}\begin{bmatrix} n \\ \mathcal{E}_1 \\ \mathcal{E}_2 \\ \ldots \\ \mathcal{E}_6 \end{bmatrix} = \begin{bmatrix} \hat{B} & \lambda_1 & \lambda_2 & \ldots & \lambda_6 \\ \hat{M}_1 & -\lambda_1 & 0 & \ldots & 0 \\ \hat{M}_2 & 0 & -\lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \hat{M}_6 & 0 & 0 & \ldots & -\lambda_6 \end{bmatrix}\begin{bmatrix} n \\ \mathcal{E}_1 \\ \mathcal{E}_2 \\ \ldots \\ \mathcal{E}_6 \end{bmatrix} + \begin{bmatrix} S \\ 0 \\ 0 \\ \ldots \\ 0 \end{bmatrix}$$

In compact operator form:

$$\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle = \hat{\mathbb{K}}\left|\Psi(t)\right\rangle + \left|S(t)\right\rangle$$

Formal approach to neutron dynamics

Eigenstate (ω-modes) expansion method:

$$\hat{\mathbb{K}}\,|\phi_n\rangle = \omega_n\,|\phi_n\rangle$$

$$\langle\phi_n|\,\hat{\mathbb{K}}^{+} = \omega_n^{*}\,\langle\phi_n|$$

What are the characteristics of the spectrum (ω_n) ?
Can the eigenvalue problem be solved for some models ?

Formal approach to neutron dynamics

The general (formal) solution can be written as:

$$|\Psi(t)\rangle = \sum_{n}^{\infty} \left[\langle \phi_n | \, \Psi(0) \rangle \, e^{\omega_n t} + \int_0^t dt' \, \langle \phi_n | \, S(t') \rangle \, e^{\omega_n (t-t')} \right] |\phi_n\rangle$$

Can an analytical solution be obtained ?

A simple model

The point model and its solution

How to derive point equations consistently ?

Let us consider a simpler and easier problem: space one-group diffusion

Basic equations

one-group diffusion in homogeneous slab geometry with one delayed family and time-constant properties

A simple model

$$\begin{cases} \dfrac{1}{v}\dfrac{\partial \Phi(x,t)}{\partial t} = D\dfrac{\partial^2 \Phi(x,t)}{\partial x^2} - \Sigma_a \Phi(x,t) + \\ (1-\beta)\nu\Sigma_f \Phi(x,t) + \lambda C(x,t) + S(x,t), \\ \dfrac{\partial C(x,t)}{\partial t} = -\lambda C(x,t) + \beta\nu\Sigma_f \Phi(x,t). \end{cases}$$

boundary and initial conditions

$$\Phi(0, t > 0) = \Phi(H, t > 0) = 0,$$

$$\Phi(x, t = 0) = \Phi_0(x),$$

$$C(x, t = 0) = C_0(x).$$

A simple model

Exact solution by eigenfunction expansion

Helmholtz eigenfunctions (complete and orthogonal, most suitable base for the diffusion problem)

$$\frac{d^2\varphi_n(x)}{dx^2} = -B_n^2\varphi_n(x),$$

$$\varphi_n(0) = \varphi_n(H) = 0.$$

A simple model

Expansion of the solution:

$$\Phi(x,t) = \sum_{n=0}^{\infty} a_n(t)\varphi_n(x),$$

$$C(x,t) = \sum_{n=0}^{\infty} c_n(t)\varphi_n(x),$$

$$S(x,t) = \sum_{n=0}^{\infty} s_n(t)\varphi_n(x).$$

A simple model

where

$$s_n(t) = \int_0^H dx S(x,t)\varphi_n(x) \equiv (\varphi_n, S(t)) \,.$$

and

$$\Phi_0(x) = \sum_{n=0}^{\infty} a_{n0}\varphi_n(x) \equiv \sum_{n=0}^{\infty} (\varphi_n, \Phi_0)\,\varphi_n(x),$$

$$C_0(x) = \sum_{n=0}^{\infty} c_{n0}\varphi_n(x) \equiv \sum_{n=0}^{\infty} (\varphi_n, C_0)\,\varphi_n(x).$$

A simple model

Matrix form of the problem:

$$|x_n(t)\rangle = \left| \begin{array}{c} a_n(t) \\ c_n(t) \end{array} \right\rangle, \quad |s_n(t)\rangle = \left| \begin{array}{c} s_n(t) \\ 0 \end{array} \right\rangle,$$

$$\frac{d}{dt}|x_n(t)\rangle = M_n |x_n(t)\rangle + |s_n(t)\rangle\,,$$

where

$$M_n = \begin{pmatrix} v\left[(1-\beta)\,\nu\Sigma_f - DB_n^2\right] & v\lambda \\ \beta\nu\Sigma_f & -\lambda \end{pmatrix}.$$

A simple model

Solution is expressed in terms of eigen-vectors:

$$M_n |u_n\rangle = \omega_n |u_n\rangle$$
$$\langle u_n| M_n = \omega_n \langle u_n|$$

eigenvalues (**generalized inhour equation**):

$$\det (M_n - \omega_n \Im) = 0,$$

Note: spectrum is real !

A simple model

eigenvectors:

$$|u_n\rangle = \left| \begin{array}{c} 1 \\ \dfrac{\beta\nu\Sigma_f}{\omega_n + \lambda} \end{array} \right\rangle,$$

$$\langle u_n| = \left\langle 1 \;\; \frac{v\lambda}{\omega_n + \lambda} \right|$$

A simple model

analytical full closed-form solution:

$$|x_n(t)\rangle = \sum_{j=1}^{2} \frac{1}{\left\langle u_n^{(j)} \middle| u_n^{(j)} \right\rangle} \left[\left\langle u_n^{(j)} \right| x_n(0) \rangle \, e^{\omega_n^{(j)} t} + \int_0^t dt' \left\langle u_n^{(j)} \middle| s_n(t') \right\rangle e^{\omega_n^{(j)}(t-t')} \right] \left| u_n^{(j)} \right\rangle \equiv$$

$$\sum_{j=1}^{2} \left[b_{n0}^{(j)} e^{\omega_n^{(j)} t} + \int_0^t dt' \sigma_n^{(j)}(t') e^{\omega_n^{(j)}(t-t')} \right] \left| u_n^{(j)} \right\rangle .$$

A simple model

for the neutron flux:

$$\Phi(x,t) = \sum_{n=0}^{\infty} \left\{ \sum_{j=1}^{2} \left(1 + \frac{\beta v \Sigma_f v \lambda}{(\omega_n^{(j)} + \lambda)^2} \right)^{-1} \left[(\varphi_n, \Phi_0)\, e^{\omega_n^{(j)} t} + \frac{v\lambda}{\omega_n^{(j)} + \lambda} (\varphi_n, C_0)\, e^{\omega_n^{(j)} t} + \int_0^t dt' \, (\varphi_n, S(t'))\, e^{\omega_n^{(j)}(t-t')} \right] \right\} \varphi_n(x).$$

A simple model

A **point** reactor evolves according to the fundamental eigenfunction φ_0 only

$\Longrightarrow$ no space distortion during the transient

$\Longrightarrow$ the evolution is space-time separable

$\Longrightarrow$ any point is representative of the whole system

$\Longrightarrow$ the source must be distributed according to the fundamental eigenfunction

A simple model

Observations

$$\Phi(x,t) = \sum_{j=1}^{2} \left(1 + \frac{\beta v \Sigma_f v \lambda}{(\omega^{(j)}+\lambda)^2}\right)^{-1}$$

$$\left[(\varphi_0, \Phi_0)\, e^{\omega^{(j)}t} + \frac{v\lambda}{\omega^{(j)}+\lambda} (\varphi_0, C_0)\, e^{\omega^{(j)}t} + \right.$$

$$\left. \int_0^t dt' \left(\varphi_0, S(t')\right) e^{\omega^{(j)}(t-t')} \right] \varphi_0(x).$$

$$C(x,t) = \sum_{j=1}^{2} \left(1 + \frac{\beta v \Sigma_f v \lambda}{(\omega^{(j)}+\lambda)^2}\right)^{-1}$$

$$\left[(\varphi_0, \Phi_0)\, e^{\omega^{(j)}t} + \frac{v\lambda}{\omega^{(j)}+\lambda} (\varphi_0, C_0)\, e^{\omega^{(j)}t} + \right.$$

$$\left. \int_0^t dt' \left(\varphi_0, S(t')\right) e^{\omega^{(j)}(t-t')} \right] \frac{\beta \nu \Sigma_f}{\omega^{(j)}+\lambda} \varphi_0(x$$

A simple model

$\omega^{(j)}$ are solutions of the inhour equation (written in general form for 6 delayed families):

$$\det\left[M_0 - \omega I\right] = \frac{\omega\Lambda}{1+\omega\Lambda} + \frac{\omega}{1+\omega\Lambda}\sum_{i=1}^{6}\frac{\beta_i}{\omega+\lambda_i} - \rho = 0,$$

where

$$\rho = \frac{k_{eff}-1}{k_{eff}} \qquad \text{reactivity}$$

$$k_{eff} = \frac{\upsilon\Sigma_f/\Sigma_a}{1+L^2B_0^2} \qquad \text{multiplication constant}$$

$$\Lambda = \frac{1}{\upsilon\Sigma_a\left(1+L^2B_0^2\right)} \qquad \text{prompt lifetime}$$

A simple model

Features of the roots of the inhour equation $\omega^{(j)}$:

$\star$ all roots are real

$\star$ six of them are close to and approach each $-\lambda_i$ as subcriticality increases;

$\star$ the seventh one, $\omega^{(7)}$, is much larger in absolute value and negative and it determines the prompt response of the neutron population connected to the inverse of the prompt lifetime

A simple model

$\star$ with a constant source, asymptotically the solution is driven by the exponential, associated to the dominant root

$$\Phi_{as}(x,t) = \left(1 + \frac{\beta v \Sigma_f v \lambda}{(\omega^{(j)} + \lambda)^2}\right)^{-1}$$

$$\left[(\varphi_0, \Phi_0)\, e^{\omega^{(0)} t} + \frac{v\lambda}{\omega^{(0)} + \lambda} (\varphi_0, C_0)\, e^{\omega^{(0)} t} + \right.$$

$$\left. \frac{(\varphi_0, S)}{\omega^{(0)}} \left(1 - e^{\omega^{(0)} t}\right)\right] \varphi_0(x).$$

A simple model

$$C_{as}(x,t) = \left(1 + \frac{\beta v \Sigma_f v \lambda}{(\omega^{(j)} + \lambda)^2}\right)^{-1}$$

$$\left[(\varphi_0, \Phi_0)\, e^{\omega^{(0)}t} + \frac{v\lambda}{\omega^{(j)} + \lambda} (\varphi_0, C_0)\, e^{\omega^{(0)}t} + \right.$$

$$\left. \frac{(\varphi_0, S)}{\omega^{(0)}} \left(1 - e^{\omega^{(0)}t}\right)\right] \frac{\beta\nu\Sigma_f}{\omega^{(0)} + \lambda} \varphi_0(x).$$

Note:

⋆ the ratio of the precursor density to the neutron density is

$$\frac{C}{n} = \frac{vC}{\Phi} = \frac{v\beta\nu\Sigma_f}{\omega^{(0)} + \lambda} = \beta \frac{k_{eff}}{\Lambda} \frac{1}{\omega^{(0)} + \lambda},$$

which may assume values of the order of 10^3 - 10^4!

⋆ $-\lambda_1$ has a special role

⋆ the "averaging" of delayed families is a delicate task

Alternatively: the point model can be derived in a general fashion assuming a factorization of the neutron flux in the product of an amplitude and a (constant) shape function (Henry approach)