COHEN–MACAULAY MODULES OVER SURFACE AND HYPERSURFACE SINGULARITIES

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In my talk a **surface singularity** R means a completion of the local ring of a singular point of an algebraic surface over an algebraically closed filed $\mathbb{k}$. It is called *normal* if R is an integrally closed domain. Equivalently, R is

(1) *isolated*, i.e. all rings $R_{\mathfrak{p}}$, where $\mathfrak{p}$ is a non-maximal prime ideal of R, are discrete valuation rings;
(2) *defined in codimension* 1, i.e. $R = \bigcap_{\mathfrak{p}} R_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through all prime ideals of height 1.

If R is a surface singularity, denote:

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\begin{aligned}
&S = \operatorname{Spec} R,\\
&o - \text{the closed point of } S \text{ (corresponding to the maximal ideal of } R),\\
&\check{S} = S \setminus \{o\}.
\end{aligned}
$$

We consider the problem of classification of Cohen–Macaulay modules over surface singularities. Recall some definitions.

A *Cohen–Macaulay module* over the ring R (more precisely, a *maximal Cohen–Macaulay module*, but we will omit the word "maximal") is such a module M that $\operatorname{Ext}^i_R(\mathbb{k}, M) = 0$ for $i < d$, where d is the Krull dimension of R. If R is a normal surface singularity, Cohen–Macaulay modules are just the *reflexive modules*, i.e. such that the natural map $M \to M^{**}$, where $M^* = \operatorname{Hom}_R(M, R)$, is an isomorphism.

A ring R is called

- *Cohen–Macaulay finite* if it only has finitely many indecomposable Cohen–Macaulay modules (up to isomorphism);
- *Cohen–Macaulay tame* if non-isomorphic indecomposable Cohen–Macaulay R-modules of any given rank form at most 1-parameter families;

- *Cohen–Macaulay wild* if for every finitely generated $\Bbbk$-algebra Λ there is an exact functor from the category of finite dimensional representations of Λ to the category of Cohen–Macaulay R-modules, which maps non-isomorphic modules to non-isomorphic ones and indecomposable to indecomposable. Evidently, in this case there are families of non-isomorphic indecomposable Cohen–Macaulay R-modules depending on any number of parameters. Moreover, when the rank tends to infinity, this number grows quadratically.

The well-known *Donovan–Freislich thesis* claims that there cannot be other cases. It has been confirmed in a lot of cases, and till now there are no contrary instances.

In what follows we suppose that $\operatorname{char} \Bbbk = 0$, though some results do not depend on this condition.

In 1978 J. Herzog [7] proved that if R is the so called *quotient singularity*, i.e. $R = \Bbbk[[x, y]]^G$, the ring of invariants of a finite subgroup $G \subseteq \mathrm{GL}(2, \Bbbk)$, then it is Cohen–Macaulay finite. Moreover, every indecomposable Cohen–Macaulay R-module is isomorphic to a direct summand of $\Bbbk[[x, y]]$ considered as R-module. Then H. Esnault (1985) [6] and M. Auslander (1986) [1], using quite different techniques, proved that any Cohen–Macaulay finite surface singularity is a quotient singularity. E. Brieskorn (1968) [3] classified all quotient singularities, then O. Riemenschneider (1981) [11] found the explicit equations defining them and J. Wunram (1987–88) [12, 13] described all Cohen–Macaulay modules over quotient singularities.

A special role play the *Du Val singularities*, i.e. such quotient singularities that $\mathrm{SL}(2, \Bbbk)$. They are just the *Gorenstein* quotient singularities, i.e. such that $\operatorname{inj.dim}_R R < \infty$ (then it equals 2). Moreover, they are also just *hypersurface* singularities, i.e. given by a unique equation: $R = \Bbbk[[x, y, z]]/(f)$. They also coincide with the so called *simple hypersurface singularities* in the sense of Arnold (in dimension 2). Here are the possibilities for the polynomial f (we use the Arnold notations; actually these singularities are closely related to the corresponding simple Lie algebras):

A_n	$x^2 - y^{n+1} + z^2$
D_n	$x^2 y - y^{n-1} + z^2$
E_6	$x^3 - y^4 + z^2$
E_7	$x^3 - xy^3 + z^2$
E_8	$x^3 - y^5 + z^2$

In 1988 C. Kahn [8] made an important step towards the study of Cohen–Macaulay modules over normal surface singularities. He related them to vector bundles over the projective curves arising in *resolution of singularities*.

Recall that a *resolution of the singularity* R (or of $S = \operatorname{Spec} R$) is a birational projective morphism $\pi : X \to S$ such that the scheme X is non-singular and π induces an isomorphism $\check{X} \to \check{S}$, where $\check{X} = X \setminus E$ and $E = \pi^{-1}(o)_{\mathrm{red}}$.

Kahn constructed a functor ρ_Z from the category of Cohen–Macaulay R-modules to the category of vector bundles over a certain closed subscheme (called a *reduction cycle*) $Z \subseteq X$ with $Z_{\mathrm{red}} = E$ and proved that rho_Z induces a 1-1 correspondence between the isomorphism classes of Cohen–Macaulay R-modules and some vector bundles over Z (called *full vector bundles*) singled out by some homological conditions.

Moreover, he proved that if R is a *minimally elliptic singularity*, i.e. it is Gorenstein and $\dim_{\Bbbk} H^1(X, \mathcal{O}_X) = 1$, then $Z = E$ and the homological conditions are rather simple. Using this result and the Atiyah's description of vector bundles over elliptic curves, he obtained a classification of Cohen–Macaulay modules over *simple elliptic singularities*, i.e. such that E is an elliptic curve.

In 2001 G.-M. Greuel and I [4] found out which projective curves are tame and which ones are wild with respect to the classification of vector bundles. Using this result we proved in 2003, together with I. Kashuba [5], that among the minimally elliptic singularities only simple elliptic and *cusp singularities* are Cohen–Macaulay tame; all others are Cohen–Macaulay wild. Here a *cusp singularity* is such that E is a cycle of projective lines with transversal intersections. They were introduced in 1970 by Hirzebruch.

Certainly, this results can be easily transferred to the quotients of minimally elliptic singularities, i.e. to the case when $R = R_0^G$, where R_0 is minimally elliptic and G is a finite group of automorphisms. Such quotients actually coincide with the so called log-*canonical singularities* considered by Y. Kawamata [9].

Finally, I have extended the Kahn's considerations from minimally elliptic to all Gorenstein singularities. It has given the following result.

Theorem 1. *Let R be a normal Gorenstein surface singularity over a field of characteristic* 0*. It is*

- Cohen–Macaulay finite *if and only if it is a Du Val singularity;*
- Cohen–Macaulay tame *if and only if it is either simple elliptic or a cusp singularity;*
- Cohen–Macaulay wild *in all other cases.*

Using the technique of quotients, one easily extend this result to the $\mathbb{Q}$-*Gorenstein singularities*, i.e. such that the canonical module K_R has a finite order in the group of divisor classes (in Gorenstein case it is a principle ideal).

Moreover, it implies an analogous result for the *hypersurface singularities* of any dimension. Indeed, all of them are Gorenstein. Moreover, if such a singularity is given by an equation $f = 0$ and the corank of the quadratic part of f is bigger than 3, V. Bondarenko [2] proved that R is Cohen–Macaulay wild. Recall that, by a result of Y. Knörrer (1987 [10]), the quadratic part does not imply the Cohen–Macaulay type. Therefore, we get the following result.

Theorem 2. *Let R be an isolated hypersurface singularity over a field of characteristic* 0*. It is*

- Cohen–Macaulay finite *if and only if it is a simple singularity (i.e. of type A_n, D_n, E_6, E_7 or E_8);*
- Cohen–Macaulay tame *if and only if it is of type T_{pqr}, i.e. given by an equation*

$$x^p + y^q + z^r + cxyz + Q = 0,$$

  where $1/p+1/q+1/r \leq 1$ and Q is a non-degenerate quadratic form of all coordinates except x, y, z;
- Cohen–Macaulay wild *in all other cases.*

References

[1] M. Auslander. Rational singularities and almost split sequences. Trans. Amer. Math. Soc. 293 (1986), 511–531.

[2] V. V. Bondarenko. Matrix Problems in the Theory of Groups and Modules. Ph.D. Thesis. Kiev, 2007.

[3] E. Brieskorn. Rationale Singularitäten komplexer Flächen. Invent. math. 4 (1968), 336–358.

[4] Y. Drozd and G.-M. Greuel. Tame and wild projective curves and classification of vector bundles. J. Algebra, 246 (2001), 1–54

[5] Y. Drozd, G.-M. Greuel and I. Kashuba. On Cohen-Macaulay modules on surface singularities. Mosc. Math. J. 3 (2003), 397–418.

[6] H. Esnault. Reflexive modules on quotient surface singularities. J. Reine Angew. Math. 362 (1985), 63–71.

[7] J. Herzog. Ringe mit nur endlich vielen Isomorphieklassen von maximalen, unzerlegbaren Cohen-Macaulay Moduln. Math. Z. 163 (1978), 149-162.

[8] C. Kahn. Reflexive modules on minimally elliptic singularities. Math. Ann. 285 (1989), 141–160.

[9] Y. Kawamata. Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces. Ann. Math. 127 (1988), 93–163.

[10] H. Knörrer. Maximal Cohen–Macaulay modules on hypersurface singularities. I. Invent. Math. 88 (1987), 153–164.

[11] O. Riemenschneider. Zweidimensionale Quotientensingularitaeten: Gleichungen und Syzygien. Arch. Math. 37 (1981), 406-417.
[12] J. Wunram. Reflexive modules on cyclic quotient surface singularities. Singularities, Representation of Algebras, and Vector Bundles. Proc. Symp., Lambrecht 1985, Lect. Notes Math. 1273 (1987), 221-231.
[13] J. Wunram. Reflexive modules on quotient surface singularities. Math. Ann. 279 (1988), 583-598.

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