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Rings of singularities

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Abstract. We show how to associate to a triple of positive integers (p_1, p_2, p_3) a two-dimensional isolated graded singularity by an elementary procedure that works over any field k (with no assumptions on characteristic, algebraic closedness or cardinality). This assignment starts from the triangle singularity $x_1^{p_1} + x_2^{p_2} + x_3^{p_3}$ and when applied to a Platonic (or Dynkin) triple produces the famous list of A-D-E-singularities. As another particular case the procedure produces Arnold's famous strange duality list consisting of the 14 exceptional unimodular singularities (and an infinite sequence of further singularities which are not treated in theses lectures).

To analyze the arising singularities we attach to each of them an abelian hereditary k-linear category $\mathcal{H}$ with Serre duality having a tilting object T, whose endomorphism ring is a canonical algebra (with three arms). These categories $\mathcal{H}$ has an interpretation as the category of coherent sheaves coh-$\mathbb{X}$ on a weighted projective line $\mathbb{X}$ whose weight type is just the triple of integers we started with.

In the focus of the lectures is the construction and analysis of three types of (usually not equivalent) triangulated categories which are naturally attached to coh-$\mathbb{X}$. These categories all have a tilting object and thus each one yields an explicit link to the representation theory of finite dimensional algebras. One of the three categories is the bounded derived category of coh-$\mathbb{X}$, the two others are obtained from two (usually different) Frobenius category structures on the category vect-$\mathbb{X}$ of vector bundles on $\mathbb{X}$. Due to a general result of Happel the associated stable categories are triangulated. Following work of Buchweitz (1987) they are equivalent to the stable categories of (suitably graded) Cohen-Macaulay modules, see [2].

The topics discussed in the final part of the lectures are related to recent work (2006) of Kajiura, Saito, Takahashi, Ueda and Lenzing–de la Peña. A key role in these developments is played by a theorem of Orlov (2005) dealing with the analysis of singularities by means of the triangulated category of (graded) singularities (=the stable derived category in Buchweitz's sense).

Throughout the lectures we pointed to the relationship of the topics under discussion to the other lectures of the Advanced School; we have kept these pointers in this written version, which is a slight expansion the actual lectures.

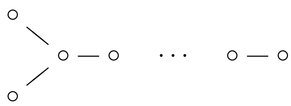
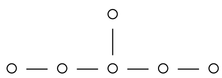
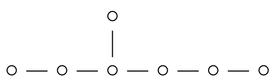
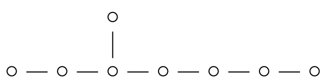
Contents

CHAPTER 1

From Dynkin diagrams to simple singularities

1. Introduction

An important aspect of singularity theory is incorporated in the following table of simple singularities:

Dynkin diagram Δ	simple singularity f_Δ
$\mathbb{A}_n$:	$zy + x^{n+1}$
$\mathbb{D}_n$: 	$z^2 + y^2x + x^{n-1}$
$\mathbb{E}_6$: 	$z^2 + y^4 + x^3$
$\mathbb{E}_7$: 	$z^2 + y^3x + x^3$
$\mathbb{E}_8$: 	$z^2 + y^3 + x^5$

The A-D-E-singularities for the base field $\mathbb{C}$

The above singularities should, for the moment, considered to be defined over the field of complex numbers, giving rise to the simple isolated singularities $R_\Delta = \mathbb{C}[x, y, z]/(f_\Delta)$[1]. As the Dynkin diagrams these singularities appear in many mathematical contexts where we only mention a few here. They appear

(1) in the classification of critical points of differential maps,
(2) rings of invariants under the natural action of finite subgroups of $\mathbf{SL}(2, \mathbb{Z})$ acting on $\mathbb{C}[[X, Y]]$. (A graded version with the action on $\mathbb{C}[X, Y]$ is also available). This links the topic with the ancient classification of regular or Platonic solids.
(3) in finite dimensional representation theory as orbit algebras of the Auslander-Reiten translation.

For further information on the omnipresence of Dynkin diagrams and singularities we refer to [8], [23] and [3].

[1] Simple singularities were discussed in I. Reiten's lecture on Auslander-Reiten theory and in O. Iyama's lectures on stable categories of Cohen-Macaulay modules. In G. Zwara's lectures on singularities of module varieties their appearance as singularities of module varieties was discussed.

A look (even a longer one) on the table does not reveal any building law. And, of course, in the setting discussed the equations f_Δ are far from being unique, since the real object of interest is the ring R_Δ which is not changed if we change the variables x, y, z by a linear base change with coefficients in $\mathbb{C}$. The first aim of these lectures is therefore to work in a graded setting in order

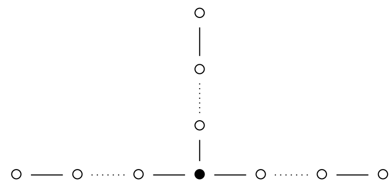
(1) to present an elementary method to generate the singularities f_Δ systematically, and basically produces a unique list,
(2) to work over an arbitrary field, and to design the construction as to be independent on any extra assumptions on this (characteristic, algebraically closedness).
(3) to recover from f_Δ or the associated graded ring $k[x, y, z]/(f_\Delta)$.

Later we are giving a more direct link to finite dimensional representation theory via associated abelian hereditary category and three related triangulated categories. The link is then established by means of appropriate tilting objects.

2. Dynkin diagrams

Assume we are given a triple (p_1, p_2, p_3) of integers $p_i \geq 0$. By the symbol $[p_1, p_2, p_3]$ we denote the star-shaped graph

$[p_1, p_2, p_3]$:



with base point, where the number p_i indicates the length of the ith branch (which for $p_i = 1$ degenerates to the base point). Here the length of the ith branch counts the number of vertices in the branch including the fat base point. In this notation a **Dynkin diagram** Δ is just a star $[p_1, p_2, p_3]$ satisfying the inequality

$$1/p_1 + 1/p_2 + 1/p_3 > 1. \tag{2.1}$$

We thus have $\mathbb{D}_n = [2, 2, n-2]$ with $n \geq 4$, $\mathbb{E}_6 = [2, 3, 3]$, $\mathbb{E}_7 = [2, 3, 4]$ and $\mathbb{E}_8 = [2, 3, 5]$. For $\mathbb{A}_n$ there is some ambiguity, since any triple $(p, q, 1)$ with $p + q - 1 = n$ produces the Dynkin diagram $\mathbb{A}_n$. Taking the base point into account, what we are going to do consistently, the ambiguity obviously disappears. Any triple (p_1, p_2, p_3) satisfying the inequality (2.1) we will call a **Dynkin triple** or, following F. Klein [11] a Platonic triple.

3. Triangle singularities

We work over an arbitrary field k and fix a triple $(p_1,\ p_2,\ p_3)$ of integers ≥ 1 integers, called **weight triple**. Let $\mathbb{L} = \mathbb{L}(p_1, p_2, p_3)$ be the abelian group given by generators $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$ and the defining relations $p_1\vec{x}_1 = p_2\vec{x}_2 = p_3\vec{x}_3 =: \vec{c}$. The element $\vec{c}$ is called the *canonical element* of $\mathbb{L}$. As is easily seen the group $\mathbb{L}$ has rank one, thus has shape $\mathbb{L} \cong \mathbb{Z} \oplus F$, where F is a finite (abelian) group. As a group, $\mathbb{L}$ is not particularly interesting. We are therefore putting additional structure on $\mathbb{L}$.

First of all $\mathbb{L}$ is an ordered group with the members from $\mathbb{N}\vec{x}_1 + \mathbb{N}\vec{x}_2 + \mathbb{N}\vec{x}_3$ forming its *positive cone*. Thus $\vec{x} \leq \vec{y}$ if and only if $\vec{y} - \vec{x}$ is a positive integral linear combination of the generators $\vec{x}_1, \vec{x}_2$ and $\vec{x}_3$. Putting $\bar{p} = \operatorname{lcm}(p_1, p_2, p_3)$ there is a uniquely defined homomorphism of groups, actually a homomorphism of ordered groups $\delta : \mathbb{L} \longrightarrow \mathbb{Z}$ sending each generator $\vec{x}_i$ to $\bar{p}/x_i$. We further note that $\delta : \mathbb{L} \to \mathbb{Z}$ is surjective and its kernel is the (finite) torsion group of $\mathbb{L}$. In order to deal with elements of $\mathbb{L}$ explicitly it is useful to have the following property: Each element $\vec{x}$ of $\mathbb{L}$ can uniquely written in **normal form**

$$\vec{x} = \sum_{i=1}^{3} \ell_1 \vec{x}_i + \ell \vec{c} \quad \text{with} \quad 0 \le \ell_i < p_i \quad \text{and} \quad \ell \in \mathbb{Z}. \tag{3.2}$$

Moreover if and element $\vec{x}$ is in normal form as above, then $\vec{x} \geq 0$ if and only if $\ell \geq 0$.

There is a further element of $\mathbb{L}$ which is important for reasons becoming clear only later. This is the **dualizing element** $\vec{\omega} = \vec{c} - (\vec{x}_1 + \vec{x}_2 + \vec{x}_3)$. For the moment we remark that the dualizing element is useful to determine how far the order $\leq$ on $\mathbb{L}$ is from a total order, since an element $\vec{x}$ of $\mathbb{L}$ either satisfies $\vec{x} \geq 0$ or else $\vec{x} \leq \vec{c} + \vec{\omega}$, a property whose proof we leave as a simple exercise. We are now in a position to introduce the **triangle singularity**[2]

$$h_{(p_1,p_2,p_3)} = x_1^{p_1} + x_2^{p_2} + x_3^{p_3} \tag{3.3}$$

over k and the associated algebra $S = k[x_1, x_2, x_3]/(x_1^{p_1} + x_2^{p_2} + x_3^{p_3})$. By forming the k-linear span of all monomials $x_1^{\ell_1} x_2^{\ell_2} x_3^{\ell_3}$ having the same degree $\vec{x} = \ell_1 \vec{x}_1 + \ell_2 \vec{x}_2 + \ell_3 \vec{x}_3$, we obtain a finite dimensional k-vectorspace $S_{\vec{x}}$ such that $S = \bigoplus_{\vec{x} \in S} S_{\vec{x}}$.

Proposition 1.1. *Assume (p_1, p_2, p_3) is a weight triple. Then the following properties hold:*

(a) *The k-algebra S is positively $\mathbb{L}$-graded by attaching degree $\vec{x}_i$ to each generator x_i. That is*

$$S = \bigoplus_{\vec{x} \geq 0} S_{\vec{x}}, \; S_0 = k, \; S_{\vec{x}} \cdot S_{\vec{y}} \subseteq S_{\vec{x}+vy} \text{ for all } \vec{x}, \vec{y} \in \mathbb{L}.$$

Moreover, the homogeneous components $S_{\vec{x}}$ of S are finite dimensional over k.

(b) *Restricting the grading of S to the subgroup $\mathbb{Z}\vec{c}$ we obtain the* ***heart***

$$H = S_{|\mathbb{Z}} = \bigoplus_{n \geq 0} S_{n\vec{x}} = k[x_1^{p_1}, x_2^{p_2}]$$

of S which 'is' the polynomial algebra in the 'variables' $x_1^{p_1}$ and $x_2^{p_2}$, viewed to be homogeneous of degree one. Accordingly $H = \bigoplus_{n \geq 0} H_n$ with H_n of k-dimension $n+1$ for $n \geq 0$.

(c) *If $\vec{x} = \ell_1 \vec{x}_1 + \ell_2 \vec{x}_2 + \ell_3 \vec{x}_3 + \ell \vec{c}$ has normal form, with $\ell \geq 0$, then*

$$S_{\vec{x}} = x_1^{\ell_1} \cdot x_2^{\ell_2} \cdot x_3^{\ell_3} \cdot H_\ell \tag{3.4}$$

$$\dim_k S_{\vec{x}} = \ell + 1. \tag{3.5}$$

Proof. Assertion (c) follows by collecting monomials having the same degree and using the relation $x_3^{p_3} = -(x_1^{p_1} + x_2^{p_2})$. Property (c) then implies assertions (a) and (b). $\square$

The next consequence explains the role of the order on $\mathbb{L}$.

Corollary 1.2. *For $\vec{x} \in \mathbb{L}$ we have $\vec{x} \geq 0$ if and only if $S_{\vec{x}} \neq 0$.* $\square$

Corollary 1.3. *As an $\mathbb{L}$-graded algebra S satisfies the following properties:*

(a) *S is* ***graded-integral****, that is, if x and y are non-zero homogeneous elements of S then also xy is non-zero.*

[2] Properly speaking, the name makes sense only if all $p_i \geq 2$. By abuse of language we extend the terminology to the present slightly more general setting.

(b) *The k-algebra S is **graded-factorial**, that is, each non-zero homogeneous element is a product of homogeneous prime elements. (Here, a homogeneous element p of S is called **prime** if $S/(p)$ is graded-integral.)*

(c) *The non-zero homogeneous prime elements of S form naturally a $\mathbf{P}^1(k)$-family.*

Proof. Concerning (a) it follows from formula (3.4) that each homogeneous element of S has the form $x_1^{l_1}x_2^{l_2}x_3^{l_3}h_l$, where h_l is a homogeneous element of the heart H of S which is an integral domain. Claim (b) follows in a similar way, observing that H is clearly graded-factorial. Using the known structure of homogeneous prime polynomials in H, claim (c) follows along the same lines. (If k is not algebraically closed, we have to interpret the projective line as a scheme, not as a variety.) □

Comments 1.4. (1) One should not mix the concepts "graded-integral" and "graded plus integral". For instance we have in characteristic two that the square of $x = x_1+x_2+x_3$ is zero in the $\mathbb{L}(2,2,2)$-graded algebra $S = k[x_1,x_2,x_3]/(x_1^2+x_2^2+x_3^2)$ which, as we have pointed out, is graded-integral. Note, in this context, that x is not a homogeneous element with regard to the $\mathbb{L}$-grading.

(2) A similar remark replies to all other "graded concepts". So a k-algebra R is a "graded field" (to be thought of as one word!) if each non-zero *homogeneous* element has a homogeneous inverse with respect to multiplication. In the graded sense therefore the algebra of Laurent polynomials $K = k[X, X^{-1}]$, (considered to be $\mathbb{Z}$-graded by attaching degree 1 to X) is a graded field. Similar care has to be taken, when dealing with graded modules. For instance the concepts "graded-injective" module and "graded plus injective" module will usually be different. For instance dealing with the $\mathbb{Z}$-graded polynomial algebra $k[X]$, where we give X degree one, the graded module K of Laurent polynomials, graded as above, is an injective object in the category of graded modules but K is far from being injective in the category of all $k[X]$-modules.

(3) If k is an algebraically closed field, then the algebras $S(p_1,p_2,p_3)$ exhaust the graded-factorial affine k-algebras of Krull dimension two which have three generators and are graded by a rank-one abelian group. This follows from a result of Kussin [12]. In that paper the more general situation of an arbitrary number of weights is treated, yielding the corresponding result. For simplicity we restrict in our lectures to three weights which is simplifying notation and nevertheless allows to cover the most interesting singularities.

Lemma 1.5. *The degree $\delta(\vec{\omega})$ is negative if and only if the triple (p_1,p_2,p_3) is — up to reordering — one of (1,1,1), (1,1,p), (1,p,2), (2,2,n), (2,3,3), (2,3,4) and (2,3,5). Moreover, we have $\delta(\vec{\omega}) = 0$ if and only if the triple is one of the triples (3,3,3), (2,4,4) and (2,3,6), called tubular. For all remaining triples, called wild, we have $\delta(\vec{\omega}) > 0$.* □

4. The simple singularity attached to a Dynkin diagram

We are now going to show how to associate to each Dynkin diagram, equivalently to each triple (p_1,p_2,p_3) of negative degree a simple singularity. This is simply done by restricting the $\mathbb{L}$-graded algebra to the (infinite cyclic) subgroup $\mathbb{Z}\vec{\omega}$. This restriction is defined to be the $\mathbb{Z}\vec{\omega}$-graded algebra $R = \bigoplus_{n\in\mathbb{Z}} S_{n\vec{\omega}}$. By our assumption on the degree of $\vec{\omega}$ we can have $n\vec{\omega} \geq 0$ only if $n \leq 0$. By means of the bijection $\mathbb{Z} \to \mathbb{Z}\vec{\omega}$, $n \mapsto n\vec{\omega}$, we may thus view R as the positively $\mathbb{Z}$-graded algebra with homogeneous components $R_n = S_{-n\vec{\omega}}$.

Theorem 1.6. *For any weight triple (p_1, p_2, p_3) with $\delta(\vec{\omega}) < 0$ let $\Delta = [p_1, p_2, p_3]$ denote the attached Dynkin diagram. Then the restriction of the grading of $S = S(p_1, p_2, p_3)$ to $\mathbb{Z}\vec{\omega}$ yields a $\mathbb{Z}$-graded algebra having a (minimal) system of three homogeneous generators x, y, z, all being monomials in x_1, x_2, x_3. With this choice of generators we have*

$$R := S_{|\mathbb{Z}\vec{\omega}} = k[x, y, z]/(f_\Delta)$$

where f_Δ is the simple graded singularity from the table below. Moreover, with the above assumptions, the singularity f_Δ can be chosen as a sum of monomials in x, y, z and then is unique.

Dynkin diagram Δ	*generators* (x, y, z)	$\deg(x, y, z)$	*relation* f_Δ	*deg*(f_Δ)
$\mathbb{A}_{p+q} = [p, q]$	$(x_1\, x_2, x_2^{p+q}, x_1^{p+q})$	$(1, p, q)$	$x^{p+q} - y\, z$	$p + q$
$\mathbb{D}_{2l+2} = [2, 2, 2l]$	$(x_3^2, x_1^2, x_1\, x_2\, x_3)$	$(2, 2l, 2l + 1)$	$z^2 + x(y^2 + y\, x^l)$	$4l + 2$
$\mathbb{D}_{2l+3} = [2, 2, 2l + 1]$	$(x_3^2, x_1\, x_2, x_1^2\, x_3)$	$(2, 2l + 1, 2l + 2)$	$z^2 + x(y^2 + z\, x^l)$	$4l + 4$
$\mathbb{E}_6 = [2, 3, 3]$	$(x_1, x_2\, x_3, x_2^3)$	$(3, 4, 6)$	$z^2 + y^3 + x^2\, z$	*12*
$\mathbb{E}_7 = [2, 3, 4]$	$(x_2, x_3^2, x_1\, x_3)$	$(4, 6, 9)$	$z^2 + y^3 + x^3\, y$	*18*
$\mathbb{E}_8 = [2, 3, 5]$	(x_3, x_2, x_1)	$(6, 10, 15)$	$z^2 + y^3 + x^5$	*30*
The simple graded surface singularities (arbitrary base field)				

We postpone a discussion of the proof for a moment in order to point out an interesting consequence.

Corollary 1.7. *Each algebra $R_\Delta = k[x, y, z]/(f_\Delta)$, with f_Δ from the above list where k is an arbitrary field, is an integral domain (both in the graded and ungraded sense). Accordingly the polynomial f_Δ is a prime element in $k[x, y, z]$ (both in the graded and ungraded sense).*

Just judging from the form of the relations f_Δ this not at all obvious, compare the example Comments 1.4, (2). With Theorem 1.6 at hand, this proof becomes very easy:

PROOF. Since S is graded integral, the same holds true for its restriction $R = S_{|\mathbb{Z}\vec{\omega}}$. Now, any positively $\mathbb{Z}$-graded algebra R which is graded integral is also integral. For this write two non-zero elements x, y as a sum of homogeneous elements $x = x_0 + x_1 + \cdots + x_m$ and $y = y_0 + y_1 + \cdots + y_n$ with leading terms x_m and y_n different from zero. Then the product of x and y has the non-zero leading term $x_m\, y_n$ and hence is non-zero. $\square$

We now sketch the **proof of Theorem 1.6**. The proof relies on two useful lemmas. Recall for this that the **Poincaré series** or Hilbert-Poincaré series of a positively $\mathbb{Z}$-graded algebra $R = \bigoplus_{n \geq 0} R_n$ with finite dimensional components R_n is the formal power series in x given as $P_R = \sum_{n \geq 0} \dim_k R_n x^n$.[3]

Lemma 1.8. *Let (p_1, p_2, p_3) be a weight triple where each p_i is ≥ 2. Let $\delta = \delta(\vec{\omega})$.*

(a) *If $\delta < 0$ then the Poincaré series of $R = S_{|\mathbb{Z}\vec{\omega}}$ with $R_n = S_{-n\vec{\omega}}$ is*

$$P_R = -\frac{1}{1-x} - \frac{1}{(1-x)^2} + \frac{1}{1-x} \sum_{i=1}^{3} \frac{1}{1 - x^{p_i}}. \tag{4.6}$$

(b) *If $\delta > 0$ then the Poincaré series of $R = S_{|\mathbb{Z}\vec{\omega}}$ with $R_n = S_{n\vec{\omega}}$ is*

$$x + \frac{1}{1-x} + \frac{x}{(1-x)^2} - \frac{x}{1-x} \sum_{i=1}^{3} \frac{1}{1 - x^{p_i}}. \tag{4.7}$$

[3] Recall that Hilbert-Poincaré series turned out to be a useful tool in D. Zacharia's lectures.

PROOF. We sketch the argument assuming $\delta < 0$ and all $p_i \geq 2$. In this case the element $-\vec{\omega} = x_1 + x_2 + x_3 - \vec{c}$ is already in normal form. For $n \geq 0$ the element $n\vec{x}_i =$ has normal form $(n - p_i[\frac{n}{p_i}]\vec{x}_i) + [\frac{n}{p_i}]\vec{c}$, where bracket notation $[q]$ denotes the integral part of a rational number q. For $n \geq 0$ it follows that the normal form of $-n\vec{\omega}$ is given by

$$-n\vec{\omega} = \sum_{i=1}^{3} \left(n - p_i \left[\frac{n}{p_i} \right] \right) \vec{x}_i + \left(-n + \sum_{i=1}^{3} \left[\frac{n}{p_i} \right] \right) \vec{c}$$

One then uses Proposition 1.1 and takes care of what happens for small values of n. The claim follows.

Concerning (b) the proof is similar, calculating this time the normal form of $n\vec{\omega}$ and using that the normal form of $\vec{\omega}$ is $(\sum_{i=1}^{3} (p_i - 1)\, \vec{x}_i) - 2\vec{c}$. □

Lemma 1.9. *Assume the algebra $A = k[u_1, u_2, u_3]/(f)$ is positively $\mathbb{Z}$-graded such that the generators u_i and the relation f are homogeneous of degree $c_i \geq 1$ and d, respectively. Then the Hilbert series P_A of A is given by the rational function*

$$P_A = \frac{1 - x^d}{(1 - x^{c_1})(1 - x^{c_2})(1 - x^{c_3})}. \tag{4.8}$$

PROOF. The polynomial ring $A_i = k[u_i]$ with $\deg(u_i) = c_i$ has Hilbert series $1/(1 - x^{c_i})$. As the tensor product of the A_i, the polynomial algebra $B = k[u_1, u_2, u_3]$ thus gets the Hilbert series $\sum_{n \geq 0} b_n x^n = \prod_{i=1}^{3} 1/(1 - x^{c_i})$. Finally, since f has degree d, we get exact sequences $0 \to B_{n-d} \xrightarrow{f\cdot} B_n \longrightarrow A_n \to 0$, yielding $\dim_k A_n = b_n - b_{n-d}$ and then $P_A = (1 - x^d) P_B$. This proves the claim. □

To prove the theorem, each row of the table is separately dealt with. For two cases we show the arguments involved. First we deal with the case $\mathbb{E}_8 = [2, 3, 5]$. Here, practically nothing is to show, since in this case $\mathbb{Z}\vec{\omega} = \mathbb{L}$, that is, up to renaming the grading group R coincides with S. In more detail we have $-6\vec{\omega} = \vec{x}_3$, $-10\vec{\omega} = \vec{x}_2$ and $14\vec{\omega} = \vec{x}_1$. The corresponding components are $R_6 = kx_3$, $R_{10} = kx_2$ and $R_{15} = kx_1$. Thus $x = x_3$, $y = x_2$ and $z = x_1$ are homogeneous generators for R satisfying the relation $f_\Delta = z^2 + y^3 + x^5$. Thus $R = k[x, y, z]/(f_\Delta)$ as claimed. Of course, in this case we do not need the two Lemmata stated above.

Next, we deal with case $\mathbb{E}_6 = [2, 3, 4]$. In calculating normal forms of $-n\vec{\omega}$, $n = 0, 1, 2, \cdots$ we first determine for small values of n those multiples $-n\vec{\omega}$ which are ≥ 0 and then by means of Proposition 1.1 (c) determine the members of R_n. Here we get

$$\begin{array}{rcll} -3\vec{\omega} & = & \vec{x}_1 & \text{hence } R_3 = kx_1 \\ -4\vec{\omega} & = & \vec{x}_2 + \vec{x}_3 & \text{hence } R_4 = kx_2x_3 \\ -6\vec{\omega} & = & \vec{c} & \text{hence } R_6 = kx_2^3 + kx_3^3. \end{array}$$

Restricting to monomials we have no choice in the first two cases, obtaining $x = x_1$ of degree 3 and $y = x_2x_3$ of degree 4. Concerning the third case we have three monomials in x_1, x_2, x_3 lying in R_6, namely x_1^2, x_2^3 and x_3^3. Since x_1^2 equals x^2, only the choices $z = x_2^3$ respectively $z = x_3^3$ make sense. Because of weight type $(2, 3, 3)$ these two choices are equivalent, and so let $z = x_2^3$. As is easily checked, the elements x, y, z are indeed generators for R (use some almost-periodicity of the expression $-n\vec{\omega}$ with "period" $6 = \mathrm{lcm}(2, 3, 3)$, resulting in some almost-periodic building law for R_n of the same "period"). The canonical homomorphism ϕ from the polynomial algebra $k[u_1, u_2, u_3]$ to R sending u_1, u_2, u_3 to x, y, z is therefore surjective; moreover ϕ is a homomorphism of graded algebras if we put $\deg(u_1, u_2, u_3) = 3, 4, 6$, respectively. Finally, x, y, z satisfy the relation $f_\Delta(x, y, z) = 0$ since using $x_1^2 + x_2^3 + x_3^3 = 0$

we get $z^2 = -x_2^3(x_1^2 + x_2^3) = -zx^2 - y^3$. Hence we obtain a surjective algebra homomorphism $\psi : k[u_1, u_2, u_3]/(f_\Delta(u_1, u_2, u_3)) \to R$ which preserves degrees. Lemmata 1.9 and 1.8 show that both algebras have the same Poincaré series, and it follows that ψ is an isomorphism, as claimed. $\square$

5. Conclusions

Comments 1.10. We comment on various aspects of the Theorem:

(1) In Theorem 1.6 we show uniqueness of the relation f_Δ if the minimal generators x, y and z for R_Δ are chosen from the set of monomials in x_1, x_2, x_3. However, any choice of a minimal triple of *homogeneous* generators x, y, z yields a valid relation g_Δ. For the types $\mathbb{E}_7 = [2, 3, 4]$ and $\mathbb{E}_8 = [2, 3, 5]$ each such system x, y, z is formed by monomials in x_1, x_2, x_3 up to multiplication with non-zero scalars. For the remaining cases we have a choice. This explains that in the literature one often finds different relations than those derived here.

(2) For instance, for type $\mathbb{E}_6 = [2, 3, 3]$ the usual form of the relation is given as $g = z^2 + y^4 + x^3$ and not as $f = z^2 + y^3 + x^2z$. We show how by simple base change equation f transforms into equation g provided the base field k is *algebraically closed of characteristic* $\neq 2$. First we note that for an arbitrary $\lambda \in k$ the elements x, y and $\bar{z} = z + \lambda x^2$ are again a minimal set of generators for R having degrees $3, 4$ and 6, respectively. Substitution into f yields the new relation $(\lambda^2 - \lambda)x^4 + (1 - 2\lambda)x^2 + y^3 + \bar{z}^2$. We now put $\lambda = 1/2$ such that the quadratic term in x disappears and introduce the new variable $\bar{x} = \mu x$ where μ is a 4th root of $-1/4$. This yields, as claimed, for new the generators $\bar{x}, y, \bar{z}$ the relation $\bar{z}^2 + y^3 + \bar{x}^4$. With similar arguments the relations f_Δ for $\mathbb{D}_n$ can be transformed into those from the previous list. Again this works for fields of characteristic $\neq 2$ which are algebraically closed.

(3) The degrees of the generators x, y, z and the degree of f_Δ are important numerical invariants of the graded singularity f_Δ. For instance, the degree of f_Δ equals the *Coxeter number* h_Δ of the Dynkin diagram Δ which equals the period of the Coxeter transformation of the Dynkin diagram Δ, and thus reflects important homological information of the Auslander-Reiten translation for mod-$k\Delta$ where $k\Delta$ is the path algebra: If $S_1, \ldots, S_n$ denote the simple $k\Delta$-modules (up to isomorphism) and $P_1, \ldots, P_n$ (resp. $I_1, \ldots, I_n$) are their projective covers (resp. injective hulls). Then the Coxeter transformation Φ of Δ is the automorphism of the Grothendieck group $\mathrm{K}_0(\text{mod-}k\Delta)$ sending each class $[P_i]$ to the class $-[I_i]$.

Another more conceptual description states that the Auslander-Reiten translation τ is a self-equivalence of the bounded derived category $\mathrm{D}^b(\text{mod-}k\Delta)$ induces the transformation Φ on the Grothendieck $\mathrm{K}_0(\mathrm{D}^b(\text{mod-}k\Delta)) = \mathrm{K}_0(\text{mod-}k\Delta)$. It is well-known that Φ is periodic in the Dynkin situation with the numbers $n+1, 2(n-1), 12, 18$ and 30 being the periods for $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7$ and $\mathbb{E}_8$, respectively. Here, the equality of the numbers $\deg f_\Delta$ and h_Δ occurs a surprising coincidence. Under a more advanced aspect we will return to this question in Lecture 4, where the relationship will be established on a conceptual level.

(4) The table contains further interesting information. The sum of the degrees of the generators x, y, z equals $1 + \deg(f_\Delta)$.

Summary 1.11. We summarize what we have achieved in the first lecture and also address some obvious questions:

(1) To each Dynkin diagram $\Delta = [p_1, p_2, p_3]$ the restriction of the $\mathbb{L}(p_1, p_2, p_3)$-graded triangle singularity $S = S(p_1, p_2, p_3)$ to the infinite cyclic group $\mathbb{Z}\vec{\omega}$, identified to $\mathbb{Z}$ via $n\vec{\omega} \leftrightarrow -n$, yields a positively $\mathbb{Z}$-graded k-algebra $R_\Delta = k[x, y, z]/(f_\Delta)$, where f_Δ is a homogeneous prime polynomial whose degree is the Coxeter number h_Δ of Δ. Moreover, R_Δ is a $\mathbb{Z}$-graded integral domain which is noetherian of

Krull dimension two. Unlike the $\mathbb{L}$-graded k-algebra S, the k-algebra R is no longer graded factorial (except for the Dynkin diagram $\mathbb{E}_9 = [2, 3, 5]$).

(2) Our treatment still leaves important questions open: What is the conceptual role of the grading group $\mathbb{L}$? What is the special role of the dualizing element $\vec{\omega}$ in $\mathbb{L}$ making the correspondence $\Delta \mapsto R_\Delta$ work? Sofar our correspondence $\Delta \mapsto R_\Delta$ looks a bit ad-hoc. On the other hand, we have seen that the restriction of S to the subgroup $\mathbb{Z}\vec{c}$ of $\mathbb{L}$ generated by the canonical element always yields the polynomial algebra $k[x, y]$, $x = x_1^{p_1}$, $y = x_2^{p_2}$, where x and y both get degree one. So what are the properties that are making ω so special that for $\vec{\omega}$ the correspondence works?

(3) In the next lecture we describe a setting, where the above questions get a natural answer. To give a brief indication already now, we first comment how to think of the $\mathbb{L}$-graded algebra S. For many questions it is natural to replace the $\mathbb{L}$-graded algebra by its **companion category** which is equipped with a natural shift-action of $\mathbb{L}$. This means to consider the k-linear category $[\mathbb{L}; S]$ given by the following data

- the objects are just the elements $\vec{x}$ of the grading group $\mathbb{L}$,
- the morphism space $\mathrm{Hom}(\vec{x}, \vec{y})$ equals $S_{\vec{y}-\vec{x}}$,
- composition of morphisms corresponds to the multiplication of S,
- An element $\vec{x} \in \mathbb{L}$ sends an object $\vec{y}$ to the object $\vec{y}(\vec{x}) := \vec{x} + \vec{y}$ and yields on morphisms the mapping
$$S_{\vec{z}-\vec{y}} = \mathrm{Hom}(\vec{y}, \vec{z}) \to \mathrm{Hom}(\vec{y}(\vec{x}), \vec{z}(\vec{x})) = S_{\vec{z}-\vec{y}},$$
corresponding to the identity map on $S_{\vec{z}-\vec{y}}$.

In Lecture 2 we are going to construct the category coh-$\mathbb{X}$ of coherent sheaves on the weighted projective line $\mathbb{X}$ of weight type (p_1, p_2, p_3), and we will see there that our companion category $[\mathbb{L}; S]$ is equivalent on to the category of all line bundles on $\mathbb{X}$. Under this equivalence, moreover, the action of $\vec{\omega}$ on $[\mathbb{L}; S]$ corresponds to the (restriction of the) Auslander-Reiten translation of coh-$\mathbb{X}$. In that sense the $\mathbb{L}$-graded algebra S embodies the properties of the Auslander-Reiten translation of the category coh-$\mathbb{X}$. It thus comes as a surprise that studying the simple singularity f_Δ thus means to study the Auslander-Reiten translation on the category coh-$\mathbb{X}$ and conversely.

(4) Recall in this context that weighted projective lines (also those with more than three weights) appeared in Happel's lectures in the following context: Assume that the base field k is algebraically closed, and $\mathcal{H}$ is a connected hereditary abelian k-linear category which is Ext-finite and has a tilting object. Then $\mathcal{H}$ is derived equivalent to the module category mod-$k\vec{\Delta}$ over the path algebra of a quiver $\vec{\Delta}$ or to a category coh-$\mathbb{X}$ of coherent sheaves on a weighted projective line, see [**7**]. If k is not algebraically closed then the class of weighted projective lines has to be enlarged to take many further cases into account, compare for this [**13**], [**17**] and [**22**].

CHAPTER 2

From singularities to diagrams

1. An analysis of the problem

We have seen in Lecture 1 how to attach to a Dynkin diagram $\Delta = [p_1, p_2, p_3]$, that is, to a weight triple (p_1, p_2, p_3) whose dualizing element satisfies $\delta(\vec{\omega}) < 0$ a simple $\mathbb{Z}$-graded surface singularity R by forming the restriction of the $\mathbb{L}$-graded triangle singularity $S = k[x_1, x_2, x_3]/(x_1^{p_1} + x_2^{p_2} + x_3^{p_3})$ to the subgroup $\mathbb{Z}\vec{\omega}$ of $\mathbb{L}$. Under the present assumptions the group $\mathbb{Z}\vec{\omega}$ is infinite cyclic; we identify $\mathbb{Z}\vec{\omega}$ with the integers by means of the correspondence $-n\vec{\omega} \leftrightarrow n$. In this way we have obtained a list (f_Δ), Δ Dynkin, of simple graded surface singularities, a list working for any field.

Remark 2.1. Certain aspects of the theory, nevertheless, still need clarification.

(1) We need a conceptual understanding why it is natural to consider the restriction of the $\mathbb{L}$-grading of S to $\mathbb{Z}\vec{\omega}$ and not to another infinite cyclic subgroup of $\mathbb{L}$. (Usually there are many such subgroups around.)
(2) Assume we are presented a (graded) singularity f_Δ from the list (but without the list itself and, of course, without the attached Dynkin label). How can we recover the Dynkin diagram, giving rise to it?
(3) More generally, and this time not restricting to the Dynkin triples, we may want to analyze the complexity (or shape) of an isolated graded surface singularity R by attaching somehow canonically suitable invariants which in the special case of Dynkin triples, should contain all the information on the Dynkin diagram in question.
(4) Actually in Idun Reiten's lectures[1] she discussed already a method how to do this. The context was not quite the same since she dealt with the ungraded complete simple singularities. In brief, starting with a simple isolated surface singularity f_Δ from our first table of singularities, Reiten was studying the ring $R = k[[x, y, z]]/(f_\Delta)$ and it's category $\mathrm{CM}(R)$ of Cohen-Macaulay modules and also the stable category $\underline{\mathrm{CM}}(R)$ obtained from $\mathrm{CM}(R)$ by factoring out all morphisms factoring through projective modules.. The main assertion in this context is that both categories have almost-split sequences and the AR-quiver in the first case is obtained from the extended Dynkin diagram $\tilde{\Delta}$, in the second case from the Dynkin diagram Δ itself by replacing each edge by a double arrow $\rightleftharpoons$, more specifically a 2-cycle.
(5) We will follow this pattern, establishing a graded version of the above and we will experience similar effects. The graded setting will also enable us to cover a larger range of phenomena.

2. Dynkin and extended Dynkin diagrams

Remark 2.2. For the discussion to follow, it is useful to have a clear conception of the natural bijection between Dynkin and extended Dynkin diagrams. (Note:

[1]The matter is going to be taken up again in Osamu Iyama's lectures.

they are not just lists of graphs! More structure is around.) The correspondence is given by looking at subadditive respectively additive functions[2].

(a) Recall that a positive integral function λ on a graph Δ is additive in a vertex v provided $2\,\lambda(v) = \sum_{v-p} \lambda(p)$, where the sum is over all vertices which are incident with v. Subadditivity in v means that we weaken the condition to $2\,\lambda(v) \geq \sum_{v-p} \lambda(p)$

(b) Dynkin diagrams are exactly the connected finite graphs such there is a unique (normalized) subadditive function which fails to be additive in a single vertex. Given a Dynkin graph Δ, let v be the vertex where a subadditive function fails to be additive. Attaching to v a new edge with a vertex yields an extended Dynkin diagram, denoted by $\tilde{\Delta}$.

(c) Extended Dynkin diagrams are exactly the finite connected graphs admitting an additive function. These function are all proportional, and it is possible to choose one, called normalized, attaining value 1. Deleting any vertex (and adjacent edges) then yields a Dynkin diagram.

(d) The two procedures in (b) and (c) are inverse to each other (on the level of isomorphism classes of graphs). We illustrate this by an example:

The Dynkin graph $\mathbb{D}_6$ admits the subadditive function depicted below:

$$\begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[dr, dash] & & & & \\ & 2 \arrow[r, dash] & 2 \arrow[r, dash] & \boxed{2} \arrow[r, dash] & 1 \\ 1 \arrow[ur, dash] & & & & \end{tikzcd}$$

It is additive, except in the framed vertex. Adding a new edge here, yields the extended Dynkin graph $\tilde{\mathbb{D}}_6$.

Conversely, the extended Dynkin graph $\tilde{\mathbb{D}}_6$ has a unique normalized additive function λ depicted below:

$$\begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[dr, dash] & & & & \boxed{1} \\ & 2 \arrow[r, dash] & 2 \arrow[r, dash] & 2 \arrow[ur, dash] \arrow[dr, dash] & \\ 1 \arrow[ur, dash] & & & & 1 \end{tikzcd}$$

There are four vertices v with $\lambda(v) = 1$. As we see it does not matter which one we delete. The four choices give rise to the "same" Dynkin diagram.

3. The Serre construction

We now describe how to attach to the $\mathbb{L}$-graded singularity S a hereditary K-linear category which is Hom-finite. (A similar construction will later be discussed for the $\mathbb{Z}$-graded algebra R if $\delta(\vec{\omega})$ is non-zero.)

First we form the abelian k-linear category of finitely generated $\mathbb{L}$-graded S-modules which we denote by $\mathrm{mod}^{\mathbb{L}}$-$S$. The objects of this category are the finitely generated $\mathbb{L}$-graded S-modules. We thus have $M = \bigoplus_{\vec{x}\in\mathbb{L}} M_{\vec{x}}$ such that $S_{\vec{x}}M_v y \subseteq M_{\vec{x}+\vec{y}}$ holds for all $\vec{y}$ and $\vec{y}$ from $\mathbb{L}$. It follows that all components $M_{\vec{x}}$ are finite-dimensional over k. Equipped with the degree-preserving morphisms (=morphism of degree zero) the category $\mathrm{mod}^{\mathbb{L}}$-$S$ is a k-linear abelian category with finite dimensional Hom-spaces.

An important feature of this category is the **action of the grading group $\mathbb{L}$ by shift**: If $M = \bigoplus_{\vec{x}\in\mathbb{L}} M_{\vec{x}}$ is a graded S-module and $y \in \mathbb{L}$ we define $M(\vec{y})$ to be the graded module with $M(\vec{y})_{\vec{x}} = M_{\vec{x}+\vec{y}}$. In particular, each indecomposable projective object in $\mathrm{mod}^{\mathbb{L}}$-$S$ has the form the modules $S(\vec{x})$ with $\vec{x} \in \mathbb{L}$. Viewed from a graded point of view the algebra S is **graded-local**[3], that is, it has a unique

[2]The matter was treated in I. Reiten's lectures.

[3]See previous remarks how to interpret concepts in the graded sense!

maximal graded ideal $\mathfrak{m} = (x_1, x_2, x_3)$. Accordingly $S/\mathfrak{m} = k$ is simple in $\mathrm{mod}^{\mathbb{L}}\text{-}S$; moreover each simple graded S-module has the form $k(\vec{x})$ for a unique $\vec{x}$ in $\mathbb{L}$. We conclude that a graded S-module has finite length if and only if it is finite dimensional. We denote by $\mathrm{mod}_0^{\mathbb{L}}\text{-}S$ the full subcategory of $\mathrm{mod}^{\mathbb{L}}\text{-}S$ consisting of all finite length objects. It is a **Serre subcategory**, that is, it is closed under subobjects, factor objects and extensions.

The setting allows us to deal with an $\mathbb{L}$-grading variant of the socalled **Serre construction** going back to Serre (1955). This is done by forming the **quotient category**[4]

$$\mathcal{H} = \frac{\mathrm{mod}^{\mathbb{L}}\text{-}S}{\mathrm{mod}_0^{\mathbb{L}}\text{-}S}$$

which is again an abelian category defined as follows (for details of the construction we refer to [4])

- the objects of $\mathcal{H}$ are just the objects of $\mathrm{mod}^{\mathbb{L}}\text{-}S$,
- the morphisms of $\mathcal{H}$ are obtained from the morphisms of $\mathrm{mod}^{\mathbb{L}}\text{-}S$ by **formally inverting** all morphisms having a kernel and a cokernel of finite length.
- the composition in $\mathcal{H}$ is induced by the composition in $\mathrm{mod}^{\mathbb{L}}\text{-}S$.

4. Coherent sheaves on a weighted projective line

The category $\mathcal{H}$ has an interpretation as the category of coherent sheaves on a weighted projective line $\mathbb{X}$ having three weighted points of weights (p_1, p_2, p_3). For this reason we are to some extent using sheaf-theoretic language for concepts related to $\mathcal{H}$: In the above setting we have a natural quotient functor $q : \mathrm{mod}^{\mathbb{L}}\text{-}S \to \mathcal{H}$ which is exact. Further, the $\mathbb{L}$-action on $\mathrm{mod}^{\mathbb{L}}\text{-}S$ induces an $\mathbb{L}$-action on $\mathcal{H}$, which we also denote in shift notation $(\vec{x}, X) \mapsto X(\vec{x})$. Moreover, we use the notation $\mathcal{O} = q(S)$ and, for reasons becoming transparent later, call this the *structure sheaf*. Moreover, we call the $q(S(\vec{x}) = \mathcal{O}(\vec{x}))$ the *twisted structure sheaves*. They will form a nice set of 'generators' for the category $\mathcal{H}$. We have the following result, see [5] for further details.

Theorem 2.3 (Geigle-L, '87)**.** *The category $\mathcal{H}$ has the following properties:*

(1) *$\mathcal{H}$ is a Hom-finite abelian category which is **noetherian**, that is, any ascending chain of subobjects becomes stationary.*

(2) *$\mathcal{H}$ satisfies **Serre duality** in the form*

$$\mathrm{D}\,\mathrm{Ext}^1(X, Y) = \mathrm{Hom}(Y, X(\vec{\omega})).$$

[This implies that the category $\mathcal{H}$ is hereditary and further that it has almost-split sequence with Auslander-Reiten translation τ given by twist with $\vec{\omega}$.]

(3) *The indecomposable objects from $\mathcal{H}$ come in two parts*

$$\begin{aligned} \mathcal{H}_0 &= \{X \in \mathcal{H} \,|\, \mathrm{length}(X) < \infty\} \\ \mathcal{H}_+ &= \{Y \in \mathcal{H} \,|\, Y \textit{ has no simple subobject}\}. \end{aligned}$$

Moreover, $\mathrm{Hom}(\mathcal{H}_0, \mathcal{H}_+) = 0$. *[Members of $\mathcal{H}_0$ will be called torsion (or finite length) sheaves, those of $\mathcal{H}_+$ will be called **bundles**.]*

(4) *We have natural isomorphisms* $\mathrm{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) = S_{\vec{y}-\vec{x}}$.

(5) *There is a $\mathbb{Z}$-linear form on $\mathrm{K}_0(\mathcal{H})$, called rank, which is 0 exactly on the objects of $\mathcal{H}_0$ and > 0 otherwise.*

(6) *For each line bundle L, that is, an indecomposable object of rank one, there exists a unique $\vec{x}$ from $\mathbb{L}$ such that L is isomorphic to $\mathcal{O}(\vec{x})$.*

[4] Under the name $Q\mathrm{mod}^{\mathbb{L}}\text{-}S$ this construction — in a $\mathbb{Z}$-graded setting did already appear in Dan Zacharia's lectures.

(7) *The indecomposables of $\mathcal{H}_0$ decompose into a $\mathbf{P}^1(k)$-family of (uniserial=standard stable) tubes with three distinguished ones having p_1, p_2, p_3 simple objects, respectively and the remaining ones containing exactly one simple.*

Proof. We give a few indications concerning the proof.

ad (1): Abelianness is a general feature of the quotient category with respect to a Serre subcategory. As is easy to see, noetherianness of $\mathrm{mod}^{\mathbb{L}}\text{-}S$ is preserved when passing to the quotient category.

ad (2): This is technically the most difficult part. On the other hand, it is a general technique in algebraic geometry. If one deals with a graded complete intersection S having a minimal set of homogeneous generators in degrees $a_1, \ldots, a_n$ and a minimal set of homogeneous relations in degrees $b_1, \ldots, b_m$, then one gets Serre duality in the form $\mathrm{D}\,\mathrm{Ext}^n(X, Y) = \mathrm{Ext}^{n-d}(Y, X(\omega))$ where $d = n - m - 1$ and $\omega = \sum_{i=1}^m d_i - \sum_{j=1}^n a_i$. This techniques either use a Koszul complex associated to the complete intersection or alternatively a minimal graded resolution of S. For details we refer to the literature.

ad (3): Let X be an object in $\mathcal{H}$. By noetherianness X has a largest noetherian subobject X_0 of finite length such we obtain a short exact sequence η: $0 \to X_0 \to X \to X/X_0 \to 0$ with X_0 from $\mathcal{H}_0$ and X/X_0 from $\mathcal{H}_+$. Invoking Serre duality one now shows that η splits which yields the result.

ad (4): This is another important feature following directly from graded factoriality of S.

ad (5): A quick way to define the rank is the following. Let $\mathcal{H}_0$ denote the full subcategory of $\mathcal{H}$ consisting of all objects of finite length. Then the quotient category $\mathcal{H}/\mathcal{H}_0$ is an abelian category where each object has finite length. For X in $\mathcal{H}$ now define the rank of X as the length of X in $\mathcal{H}/\mathcal{H}_0$. It is then easy to verify the properties (5).

ad (6): This uses the classification of homogeneous prime elements in S. If p is a homogeneous prime in S, then $q(S/(p))(\vec{x})$ is a simple object in $\mathcal{H}$, and each simple object S has this form. Here p is uniquely determined by S while $\vec{y}$ is not. □

Corollary 2.4. *We can recover the weight triple (p_1, p_2, p_3), hence the $\mathbb{L}$-graded algebra S from the category $\mathcal{H}$.*

Proof. This follows directly from part (7). □

Corollary 2.5. *The companion category $[\mathbb{L}; S]$ of the $\mathbb{L}$-graded algebra S is equivalent to the full subcategory $\mathcal{L}$ of* coh-$\mathbb{X}$ *which is formed by all the line bundles on $\mathbb{X}$.*

Proof. This is an immediate consequence of (4). □

CHAPTER 3

Link to algebras and Cohen-Macaulay modules

1. Singularities and finite dimensional algebras

Summarizing the present status, we have applied the Serre construction to the $\mathbb{L}$-graded algebra S and obtained the category $\mathcal{H} = \text{coh-}\mathbb{X}$ which is an abelian, Hom-fine k-linear category which is Krull-Schmidt and which has almost-split sequences. So $\mathcal{H}$ is already quite close to the features of a category of finite dimensional modules. This relationship is not only a formal one, it can be made very explicit since $\mathcal{H}$ also has a tilting object. (In fact it does have plenty of them!) Since $\mathcal{H}$ is hereditary it is convenient to say that an object T of $\mathcal{H}$ is a **tilting object** if the following two conditions are satisfied:

(a) T has no self-extensions, that is, $\operatorname{Ext}^1_{\mathcal{H}}(T,T) = 0$;
(b) T generates $\mathcal{H}$ homologically, that is, whenever $X \in \mathcal{H}$ satisfies $\operatorname{Hom}_{\mathcal{H}}(T,X) = 0 = \operatorname{Ext}^1_{\mathcal{H}}(T,X)$ then already $X = 0$.

Theorem 3.1. *Assume $\mathbb{X} = \mathbb{X}(p_1,p_2,p_3)$ is the weighted projective line given by the weight triple (p_1,p_2,p_3). Then the object*

$$T = \bigoplus_{0 \le \vec{x} \le \vec{c}} \mathcal{O}(\vec{x})$$

is a tilting object in coh-$\mathbb{X}$ *whose endomorphism ring $\Lambda = \operatorname{End}_{\mathcal{H}}(T)$ is the* ***canonical algebra*** *given by the same weight triple. (We write $\Lambda = \Lambda(p_1,p_2,p_3)$.) That is, Λ is given by the quiver*

$$\begin{tikzcd}
& \circ \arrow[r, "x_1"] & \circ \arrow[r, "x_1"] & \cdots \arrow[r, "x_1"] & \circ \arrow[dr, "x_1"] & \\
\circ \arrow[ur, "x_1"] \arrow[r, "x_2"'] \arrow[dr, "x_3"'] & \circ \arrow[r, "x_2"'] & \circ \arrow[r, "x_2"'] & \cdots \arrow[r, "x_2"'] & \circ \arrow[r, "x_2"'] & \circ \\
& \circ \arrow[r, "x_3"'] & \circ \arrow[r, "x_3"'] & \cdots \arrow[r, "x_2"'] & \circ \arrow[ur, "x_3"'] &
\end{tikzcd}$$

with three arms of lengths p_1,p_2,p_3 respectively with the single relation $x_1^{p_1} + x_2^{p_2} + x_3^{p_3} = 0$.

Proof. That T has no self-extensions uses Serre duality combined with the formula $\operatorname{Hom}(\mathcal{O}(\vec{x}),\mathcal{O}(\vec{y})) = S_{\vec{y}-\vec{x}}$. To show that T generates $\mathcal{H}$ homologically, the key point is to show that for each simple object S in $\mathcal{H}$ at least one of the $\mathcal{O}(\vec{x})$ with $\vec{x}$ in the range $0 \le \vec{x} \le \vec{c}$ admits a non-zero homomorphism to S. $\square$

Corollary 3.2. *The bounded derived categories of* $\mathrm{D}^b(\operatorname{mod}\Lambda)$ *and* $\mathrm{D}^b(\mathcal{H})$ *are triangle-equivalent.* $\square$

This result allows a number of strong consequences, since the abelian category $\mathcal{H}$ is hereditary, and hence the bounded derived category of $\mathcal{H}$ can be identified with the **repetitive category** of $\mathcal{H}$, as was shown in the lectures by Dieter Happel. Recall that the repetitive category is the disjoint union

$$\bigvee_{n \in \mathbb{Z}} \mathcal{H}[n] \quad \text{where each } \mathcal{H}[n] \text{ is a copy of } \mathcal{H},$$

with objects of $\mathcal{H}[n]$ written $X[n]$, and where morphism are given by

$$\mathrm{Hom}(X[m], Y[n]) = \mathrm{Ext}^{n-m}_{\mathcal{H}}(X, Y).$$

A particular consequence of this setting is

Corollary 3.3. *The category* $\mathrm{mod}\,\Lambda$ *of modules over the canonical algebra* Λ *is equivalent to (the additive closure of) the union of*

$$\{X \in \mathcal{H} \mid \mathrm{Ext}^1_{\mathcal{H}}(T, X) = 0\} \vee \{Y[1] \mid Y \in \mathcal{H} \text{ with } \mathrm{Hom}_{\mathcal{H}}(T, Y) = 0\}$$

viewed as a full subcategory of $\mathcal{H} \vee \mathcal{H}[1] \subset \mathrm{D}^b(\mathcal{H})$. □

Remark 3.4. We discuss briefly the relationship between the categories $\mathcal{H}$ and $\mathrm{mod}\,\Lambda$ as far it is relevant for the matter of singularities.

(1) As Corollary 3.3 states, the category $\mathcal{H}$ contains all the information on the category of Λ-modules (via the repetitive category of $\mathcal{H}$). In particular the representation type of $\mathcal{H}$ determines the representation type of Λ.

(2) The complexity of the classification problem for coh-$\mathbb{X} = \mathcal{H}$ is completely determined by the numerical invariant $\delta(\vec{\omega}) = \bar{p}\,(1 - (1/p_1 + 1/p_2 + 1/p_3))$. Since indecomposables of $\mathcal{H}_0$ are explicitly classified by means of 1-parameter families, indexed by the projective line, the complexity is determined by the category vect-$\mathbb{X} = \mathcal{H}_+$ of vector bundles on $\mathbb{X}$.

  (a) If $\delta(\vec{\omega}) > 0$, then the Auslander-Reiten quiver for the indecomposable vector bundles consists of a single component of shape $\mathbb{Z}\tilde{\Delta}$, where $\tilde{\Delta}$ is the extended Dynkin diagram corresponding to $\Delta = [p_1, p_2, p_3]$.

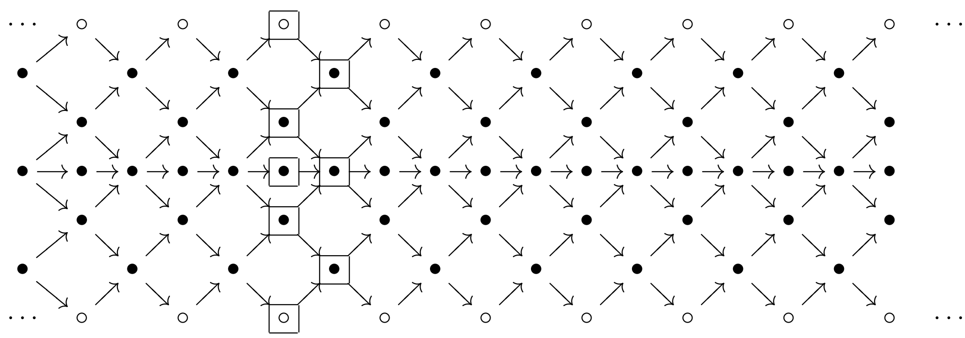
  (2) If $\delta(\vec{\omega}) = 0$, then the classification problem for coh-$\mathbb{X}$ is still tame (but complicated). The indecomposable vector bundles decompose into a rational family $(\mathcal{T}_q)_{q\in\mathbb{Q}}$ where, in turn, each $\mathcal{T}_1$ is a $\mathbf{P}^1(k)$-family of tubes, each one being of tubular type (p_1, p_2, p_3). We express this by saying that coh-$\mathbb{X}$, correspondingly Λ, has tubular type.[1]

  (3) If $\delta(\vec{\omega}) > 0$, then we deal with a wild situation. Here all AR-components for vect-$\mathbb{X}$ are of type $\mathbb{Z}\mathbb{A}_\infty$. Moreover, there is a natural bijection between the set of all such components to the set of all AR-components of regular modules for the path algebra of the star $[p_1, p_2, p_3]$, endowed with an arbitrary orientation, see [19].

2. Shape of the category of vector bundles

For the moment our main interest is in the case $\delta(\vec{\omega}) < 0$. We illustrate the situation by an example.

Example 3.5. For weight type $(2, 3, 4)$ the corresponding Dynkin diagram is $\mathbb{E}_7 = [2, 3, 4]$. Here the Auslander-Reiten quiver for vect-$\mathbb{X}$ is given by $\mathbb{Z}\tilde{\mathbb{E}}_7$, and therefore looks as follows:



[1] Tubular algebras play an important role in tame representation theory. They occurred in A. Skowroński's and in J.A. de la Peña's lectures.

Note that the line bundles form two τ-orbits sitting at the border of the component. We have marked the corresponding vertices by circles $\circ$ while the other vertices are marked by fat dots $\bullet$. We have 8 τ-orbits with corresponding orbit graph the extended Dynkin diagram $\tilde{\mathbb{E}}_7$. Since the rank, introduced in Theorem 2.3, is constant on τ-orbits it yields a function on $\tilde{\mathbb{E}}_7$ which turns out to be the unique (normalized) additive function for this extended Dynkin graph. The rank function on vect-$\mathbb{X}$ is thus determined by the following diagram with attached rank values

$$\begin{array}{ccccccccccccc} & & & & & & 2 & & & & & & \\ & & & & & & | & & & & & & \\ 1 & - & 2 & - & 3 & - & 4 & - & 3 & - & 2 & - & 1 \end{array}$$

We thus rediscover that there are two line bundle components sitting at the border of the AR-quiver.

More is true, also in the other cases of weight triples of Dynkin type $\Delta = [p_1, p_2, p_3]$. Namely, the category vect-$\mathbb{X}$, not just its Auslander-Reiten quiver is completely determined by the mesh category $\mathbb{Z}\tilde{\Delta}$, since the path category of this mesh category is equivalent to the category of indecomposable vector bundles on $\mathbb{X}$. This can be derived from the fact that coh-$\mathbb{X}$ has a tilting bundle T whose endomorphism ring is the path algebra $k\vec{\tilde{\Delta}}$ of the path algebra of an extended Dynkin quiver of type $\tilde{\Delta}$. For the case $[2, 3, 4]$ we have depicted such a tilting bundle above by framing the vertices of a section[2].

Concerning the position of a suitable tilting object we can be more specific. We have already defined the rank which is constant on τ-orbits and which is 0 on finite length sheaves and > 0 on non-zero bundles. There is another $\mathbb{Z}$-linear form with somewhat complementary properties, the **degree**. The degree is > 0 on objects of $\mathcal{H}_0$ and there it is also constant on τ-orbits. Moreover, it vanishes on $\mathcal{O}$. Each non-zero vector bundle X then has a well-defined **slope** given by $\mu X = \deg X / \operatorname{rk} F$. The following result is due to T. Hübner (unpublished), a proof can be found in [20]

Theorem 3.6. *Assume the weight triple (p_1, p_2, p_3) with $\delta(\vec{\omega})$. Let $\Delta = [p_1, p_2, p_3]$ be the corresponding Dynkin diagram. Then there is only a finite system $T_1, \ldots, T_n$ of pairwise nonisomorphic indecomposable vector bundles E with slope in the range $0 \leq \mu E < -\delta(\vec{\omega})$. Moreover, $T = T_1 \oplus \cdots T_n$ is a tilting object whose endomorphism algebra is isomorphic to the path algebra kQ of a quiver Q whose underlying graph $\underline{Q}$ is the extended Dynkin diagram $\tilde{\Delta}$.* □

If all the three weights are ≥ 2, then (as in the preceding example) Q has bipartite orientation.

Corollary 3.7. *The path algebra kQ of a quiver Q without oriented cycles is derived equivalent to a canonical algebra if and only if the graph underlying Q is extended Dynkin.* □

3. From singularities to weights

After this digression on some representation-theoretic link of singularity theory, we come back to our main subject.

Theorem 3.8. *We assume a weight triple (p_1, p_2, p_3) with $\delta = \delta(\vec{\omega})$ different from zero. Let $R = S_{|\mathbb{Z}\vec{\omega}}$ considered as a $\mathbb{Z}$-graded k-algebra (with $\vec{\omega} \leftrightarrow -1$ for $\delta < 0$*

[2]The concept of a section in an Auslander-Reiten component was an important tool in the lectures of J.A. de la Peña and A. Skowroński

and $\vec{\omega} \leftrightarrow 1$ for $\delta > 0$). Then the restriction functor res : $\mathrm{mod}^{\mathbb{L}}$-$S \to \mathrm{mod}^{\mathbb{Z}}$-$R$, $M \mapsto M_{|\mathbb{Z}\vec{\omega}}$*, induces an equivalence*

$$\mathcal{H} = \frac{\mathrm{mod}^{\mathbb{L}}\text{-}S}{\mathrm{mod}_0^{\mathbb{L}}\text{-}S} \xrightarrow{\sim} \frac{\mathrm{mod}^{\mathbb{Z}}\text{-}R}{\mathrm{mod}_0^{\mathbb{Z}}\text{-}R}.$$

PROOF. We first observe that the restriction of $\mathbb{L}$-graded S-modules to $\mathbb{Z}$-graded R-modules preserves finite length, and thus induces a restriction functor for the two quotient categories. The main ingredients of the proof then are the following two facts:

(1) For each simple object E in $\mathcal{H}$ its image is non-zero (and then also simple).

(2) Each finitely generated $\mathbb{Z}\vec{\omega}$-graded R-module M extends to a finitely generated $\mathbb{L}$-graded S-module $\bar{M}$ (such that the restriction of $\bar{M}$ to $\mathbb{Z}\vec{\omega}$ equals M). This part of the proof uses left Kan-extension or, in different terminology, the graded tensor product $S \otimes_R -$. □

With Theorem 3.8 at hand, we have solved our problem to discover the Dynkin diagram from the $\mathbb{Z}$-graded simple surface singularity f_Δ.

Corollary 3.9. *Let f_Δ be a $\mathbb{Z}$-graded simple surface singularity f_Δ and $R = k[x,y,z]/(f_\Delta)$. Then the quotient category* $\mathrm{mod}^{\mathbb{Z}}$-$R/\mathrm{mod}_0^{\mathbb{Z}}$-$R$ *is equivalent to the category of coherent sheaves* coh-$\mathbb{X}$ *on the weighted projective line of weight type (p_1, p_2, p_3), where $\Delta = [p_1, p_2, p_3]$.* □

As we have seen before, the weight type of $\mathbb{X}$ can be recovered as the tubular type of $\mathcal{H}$, that is, by determining the τ-periods of the tubes in the AR-quiver of $\mathcal{H}_0$.

4. The link to graded Cohen-Macaulay modules

We start with a definition of graded maximal Cohen-Macaulay modules[3] for graded-local algebras of dimension two (like the $\mathbb{L}$-graded algebra S or the $\mathbb{Z}$-graded algebra R).

Definition 3.10. A finitely generated $\mathbb{L}$-graded module M is called (maximal) **Cohen-Macaulay** if

$$\mathrm{Hom}_S(E, M) = 0 = \mathrm{Ext}^1_S(E, M)$$

holds for each simple $\mathbb{L}$-graded S-module E. (Recall these are of the form $k(\vec{x})$.) By $\mathrm{CM}^{\mathbb{L}}$-S we denote the category of all $\mathbb{L}$-graded CM-modules, viewed as a full subcategory of $\mathrm{mod}^{\mathbb{L}}$-$S$.

A similar definition applies to $\mathbb{Z}$-graded R-modules. We remark here, that for algebras one always has the implications (hypersurface) ⇒ (complete intersection) ⇒ (Gorenstein) ⇒ (Cohen-Macaulay) in the graded and ungraded sense. Hence the algebra S is always graded Gorenstein. For $\delta(\vec{\omega}) \neq 0$ one can show the same for the $\mathbb{Z}$-graded algebra R. It is not so obvious for $\delta(\vec{\omega}) > 0$, while for $\delta(\vec{\omega}) < 0$ it follows from the list of simple graded singularities, which all are hypersurfaces.

Theorem 3.11. *Let (p_1, p_2, p_3) be a weight triple.*

(a) If q : $\mathrm{mod}^{\mathbb{L}}$-$S \to \mathrm{mod}^{\mathbb{L}}$-$S/\mathrm{mod}^{\mathbb{Z}}$-$R$ = coh-$\mathbb{X}$ *denotes the natural quotient functor, then q induces an equivalence q :* $\mathrm{CM}^{\mathbb{L}}$-$S \xrightarrow{\sim}$ vect-$\mathbb{X}$. *This equivalence sends the indecomposable projective $S(\vec{x})$ to $\mathcal{O}(\vec{x})$ and induces an equivalence between the category* $\mathrm{proj}^{\mathbb{L}}$-$S$ *of finitely generated $\mathbb{L}$-graded projective S-modules and the full subcategory $\mathcal{L}$ of* vect-$\mathbb{X}$*, consisting of all line bundles.*

[3]Cohen-Macaulay modules played a central role in O. Iyama's lectures. He did consider CM-modules for arbitrary Krull-dimension. Central tools were the depth of a module and the Auslander-Buchsbaum theorem.

(b) If we exclude the tubular weights, then the restriction functor from $\mathbb{L}$-graded S-modules to $\mathbb{Z}$-graded R-modules induces an equivalence

$$\mathrm{res} : \mathrm{CM}^{\mathbb{L}}\text{-}S \xrightarrow{\sim} \mathrm{CM}^{\mathbb{Z}}\text{-}R, \quad M \mapsto M_{|\mathbb{Z}\vec{\omega}}$$

This equivalence sends the indecomposable projective $R(n)$ to $\mathcal{O}(-n\vec{\omega})$ if $\delta(\vec{\omega}) < 0$ respectively to $\mathcal{O}(n\vec{\omega})$ if $\delta(\vec{\omega}) > 0$ and thus induces an equivalence between $\mathrm{proj}^{\mathbb{Z}}\text{-}R$ and the τ-orbit $\tau^{\mathbb{Z}}\mathcal{O}$.

Proof. The first assertion (a) follows from the existence of an inverse Γ : $\text{vect-}\mathbb{X} \to \mathrm{CM}^{\mathbb{L}}\text{-}S$ to q where $\Gamma(E) = \bigoplus_{\vec{x}\in\mathbb{L}} \mathrm{Hom}_{\mathbb{X}}(\emptyset(-\vec{x}), E)$. The point here is to prove that $\Gamma(M)$ is finitely generated over S which uses that E is a vector bundle. The first part of (b) then follows from Theorem 3.8. The remaining assertions in (a) and (b) then are obvious. □

The combination of Theorem 3.11 with Remark 3.4 immediately yields the following theorem.

Theorem 3.12. *Assume $[p_1, p_2, p_3]$ is a Dynkin diagram. With the notation introduced previously the k-linear categories*

$$\text{vect-}\mathbb{X}, \quad \mathrm{CM}^{\mathbb{L}}\text{-}S \quad \textit{and } \mathrm{CM}^{\mathbb{Z}}\text{-}R$$

can be naturally identified. Their Auslander-Reiten quiver forms a single component of shape $\mathbb{Z}\tilde{\Delta} = \{(n, v) \,|\, n \in \mathbb{Z},\ v \in \tilde{\Delta}_0\}$.

Moreover, in this component the indecomposable $\mathbb{Z}$-graded projective R-modules form a single τ-orbit lying at the boundary of the component if all weights are ≥ 2. □

As an immediate consequence we obtain the information on the shape of the Auslander-Reiten quiver of (maximal) CM-modules over the complete simple surface singularities[4].

Corollary 3.13. *Assume $\Delta = [p_1, p_2, p_3]$ is Dynkin and f_Δ is the corresponding 'singularity'. Then the Auslander-Reiten quiver of $\hat{R} = k[[x, y, z]]/(f_\Delta)$ is obtained from the extended Dynkin diagram $\tilde{\Delta}$ by replacing each edge $\circ - \circ$ by a 2-cycle $\circ \rightleftarrows \circ$.*

For instance the Dynkin diagram $\mathbb{E}_8$ with corresponding singularity $f_\Delta = z^2 + y^3 + x^5$ yields a category $\mathrm{CM}(k[[x, y, z]]/(f_\Delta))$ with Auslander-Reiten quiver

$$\begin{tikzcd}
 & & & & & \bullet \arrow[d, shift left] & & \\
\circ \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[r, shift left] & \bullet \arrow[l, shift left]
\end{tikzcd}$$

We have marked by $\circ$ the indecomposable projective module $\hat{R}$.

of the Corollary. The proof uses the completion functor $\hat{}$: $\mathrm{CM}^{\mathbb{Z}}\text{-}R \to \mathrm{CM}(\hat{R})$, $M \mapsto \prod_{n\in\mathbb{Z}} M_n$ as studied by Auslander-Reiten in [1]. In the present setting, completion preserves indecomposability and almost-split sequences; moreover two indecomposable graded modules have the same image if and only if they belong to the same τ-orbit of $\mathbb{Z}$-graded CM-modules. Hence the image of the completion functor is a finite component of $\mathrm{CM}(\hat{R})$. By a Brauer-Thrall type argument, it then follows that the functor is dense. This proves the claim. □

Corollary 3.14. *Let k be a field. Then the k-algebra $\hat{R} = k[[x, y, z]]/(x^2 + y^3 + z^5)$ is a factorial domain, that is, $\hat{R}$ is a domain and each non-zero element is a product of prime elements.*

[4]Some of the easier examples were treated in I. Reiten's lectures.

Proof. Since completion preserves the rank, it follows that $\hat{R}$ is the only CM-module over $\hat{R}$ having rank one. It is a well-known fact that, in the present setting, this property implies factoriality of $\hat{R}$. $\square$

Corollary 3.15. *Let Q be an extended Dynkin quiver associated with the Dynkin diagram Δ; we fix a vertex v of Q, where 'the' additive function for the graph underlying Q attains value 1. Denote by P the indecomposable projective kQ-module corresponding to the vertex v. Then the* ***orbit algebra***

$$\mathbb{A}(\tau^{-}, P) := \bigoplus_{n \geq 0} \mathrm{Hom}_{kQ}(P, \tau^{-n} P)$$

with multiplication $u_n \cdot v_m = \tau^{-m} u_n \circ v_m$ is a positively $\mathbb{Z}$-graded algebra which is isomorphic to the graded simple surface singularity $k[x, y, z]/(f_\Delta)$.

Proof. Using the identifications of the theorem, we may identify $\mathbb{A}(\tau^{-}, P)$ with the orbit algebra of the $\mathbb{Z}$-graded R-module R with regard to the grading shift $M \mapsto M(1)$, which obviously brings us back to the $\mathbb{Z}$-graded algebra R. $\square$

Remark 3.16. From a general perspective, the last result is quite remarkable. It tells us that a study of a (graded) simple singularities is equivalent to the analysis of the Auslander-Reiten translation for path algebras of an extended Dynkin quiver, or alternatively for the category of coherent sheaves on a weighted projective line whose weight triple determines the Dynkin diagram. Thus the simple singularities can be thought of as mathematical objects capturing the homological information on either category, there given by either Auslander-Reiten or Serre duality.

A K-theoretic shadow of this is contained in the list of simple graded singularities where we have seen in Comments 1.10 that the degree of the singularity f_Δ agrees with the Coxeter number h_Δ.

CHAPTER 4

Stable categories of vector bundles/ Cohen-Macaulay modules

1. Vector bundles as a Frobenius category

In this lecture we are dealing with recent joint work with J.A. de la Peña [18] and D. Kussin and H. Meltzer [15], see also [14] In the last lecture we have seen (Theorem 3.8) that for each non-tubular weight triple (p_1, p_2, p_3) we have a commutative diagram

$$\begin{tikzcd} \mathrm{CM}^{\mathbb{L}}\text{-}S \arrow[r, "\sim", "q"'] \arrow[d, "\mathrm{res}", "\cong"'] & \text{vect-}\mathbb{X} \arrow[d, equal] \\ \mathrm{CM}^{\mathbb{Z}}\text{-}R \arrow[r, "\sim", "q'"'] & \text{vect-}\mathbb{X} \end{tikzcd}$$

where q and q' are equivalences induced by the natural quotient functors. Since S and R are graded Gorenstein each of the categories $\mathrm{CM}^{\mathbb{L}}\text{-}S$ and $\mathrm{CM}^{\mathbb{Z}}\text{-}R$ inherits an exact structure from the ambient abelian categories of finitely generated graded modules $\mathrm{mod}^{\mathbb{L}}\text{-}S$ and $\mathrm{mod}^{\mathbb{Z}}\text{-}R$, respectively, which turns the two categories of graded CM-modules into Frobenius categories having the category of indecomposable graded projective modules as their indecomposable projective-injective objects. By transport of structure we thus obtain on vect-$\mathbb{X}$ two, usually different, structures of Frobenius categories.

In more detail, we arrive to the following setting

(a) From the $\mathbb{L}$-graded setting we obtain that vect-$\mathbb{X}$ is a Frobenius category with the system $\mathcal{L}$ of line bundles being the indecomposable-projective objects.
(b) In the non-tubular case we obtain from the $\mathbb{Z}$-graded setting that vect-$\mathbb{X}$ is a Frobenius category with the τ-orbit $\tau^{\mathbb{Z}}\mathcal{O}$ of the structure sheaf, that is a single τ-orbit of line bundles, being the indecomposable projective-injective objects.

Explanations 4.1. (1) A **Frobenius category** is defined to be an **exact category** which has sufficiently many (relative) projective and (relative) injective objects and where the projectives coincide with the injectives.

(2) The term exact category is used here in the sense of Quillen. An exact category $\mathcal{C}$, by definition, admits an embedding as a full, extension-closed subcategory into an abelian category $\mathcal{A}$. The exact structure on $\mathcal{C}$ then is induced from $\mathcal{A}$ consisting of all short exact sequences in $\mathcal{A}$ with all their terms in $\mathcal{C}$.

(3) More concretely, a sequence $\eta : 0 \to A \to B \to C \to 0$ in vect-$\mathbb{X}$ is exact with regard to the exact structure (a) if and only if for each line bundle L the sequence $\mathrm{Hom}(L, \eta)$ is exact. Serre duality then implies that it is equivalent to request exactness of $\mathrm{Hom}(\eta, L)$ for each line bundle L. By contrast, in case (b) η is exact if and only if $\mathrm{Hom}(\tau^n\mathcal{O}, \eta)$, equivalently $\mathrm{Hom}(\eta, \tau^n\mathcal{O})$, is exact for each integer n.

(4) By a result, due to Happel[1] [6] the associated stable categories

$$\text{vect-}\mathbb{X}/[\mathcal{L}] \quad \text{and} \quad \text{vect-}\mathbb{X}/[\tau^{\mathbb{Z}}\mathcal{O}]$$

are **triangulated**. Here a notation like vect-$\mathbb{X}/[\mathcal{L}]$ means the **factor category** of vect-$\mathbb{X}$ modulo the ideal generated by $\mathcal{L}$. In more detail this is the category having the same objects as vect-$\mathbb{X}$ with morphisms given by the quotient $\underline{\mathrm{Hom}}(X,Y) = \mathrm{Hom}(X,Y)/\{u : X \to Y \mid u \text{ factors through } \mathrm{add}(\mathcal{L})\}$.

(5) The stable categories (a) vect-$\mathbb{X}/[\mathcal{L}]$ and (b) vect-$\mathbb{X}/[\tau^{\mathbb{Z}}\mathcal{O}]$ are triangulated categories with **Serre duality** induced from the Serre duality of coh-$\mathbb{X}$. In particular, the categories (a) and (b) have almost-split triangles and the Auslander-Reiten translation is induced from the Auslander-Reiten translation of coh-$\mathbb{X}$.

It follows that the Auslander-Reiten quiver for (a) and (b) is obtained from the Auslander-Reiten quiver of vect-$\mathbb{X}$ by deleting in case (a) all orbits of line bundles and in case (b), assuming non-tubular type, by deleting just a single τ-orbit of line bundles.

(6) The stable categories of vector bundles (type (a) or (b)) will always have a tilting object. For $\delta(\vec{\omega}) > 0$ this will be a highly non-trivial matter. We are going to return to this aspect later.

In this lecture we will mainly concentrate on the stable category vect-$\mathbb{X}/[\tau^{\mathbb{Z}}\mathcal{O}]$ and hence use the abbreviation $\underline{\text{vect}}$-$\mathbb{X}$ for it.

Remark 4.2. We assume a non-tubular weight triple. What is then the role of $\underline{\text{vect}}$-$\mathbb{X}$, equivalently of $\underline{\mathrm{CM}}^{\mathbb{Z}}$-$R/[\mathrm{proj}^{\mathbb{Z}}$-$R]$? By old work of R. Buchweitz (1986), see [2], revived by D. Orlov in 2005, see [21], 'this' stable category is a measure for the complexity of the graded singularity R. It is equivalent moreover to the triangulated category of graded singularities of R defined as $\mathrm{D}^{\mathbb{Z}}_{sing}(R) = \mathrm{D}^b(\mathrm{mod}^{\mathbb{Z}}\text{-}R)/\mathrm{D}^b(\mathrm{proj}^{\mathbb{Z}}\text{-}R)$. For instance the polynomial algebra $R = k[x,y]$ with x and y homogeneous of positive degree, becomes graded-regular-local, yielding $\mathrm{CM}^{\mathbb{Z}}\text{-}R = 0$.

Because of the canonical equivalences between the categories $\underline{\text{vect}}$-$\mathbb{X}$, $\mathrm{CM}^{\mathbb{Z}}$-R and $\mathrm{D}^{\mathbb{Z}}_{sing}(R)$ it is advisable to think of all three as being incarnations of a single triangulated category. We will later encounter further triangulated categories which are triangle equivalent to the above, but in a non-canonical way.

2. Shape of the stable category, case $\delta < 0$

Concerning the existence of a tilting object we start with the case $\delta(\vec{\omega}) < 0$.

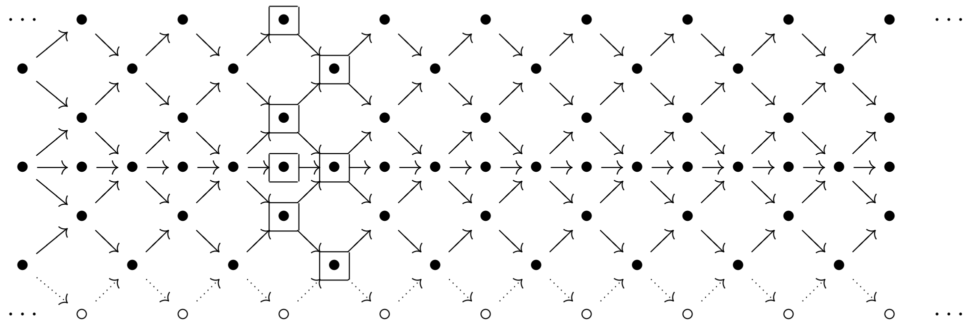
Theorem 4.3 (Kajiura-Saito-Takahashi '06)**.** *Assume a weight triple (p_1,p_2,p_3) such that $\Delta = [p_1,p_2,p_3]$ is a Dynkin diagram. Then the triangulated category $\mathrm{CM}^{\mathbb{Z}}$-R = $\underline{\text{vect}}$-$\mathbb{X}$ has a tilting object T such that $\mathrm{End}(T)$ is isomorphic to the path algebra $k\vec{\Delta}$ of a quiver $\vec{\Delta}$ with underlying graph Δ. In particular, we have equivalences*

$$\mathrm{CM}^{\mathbb{Z}}\text{-}R = \underline{\text{vect}}\text{-}\mathbb{X} \cong \mathrm{D}^b(\mathrm{mod}\, k\vec{\Delta}).$$

Proof. We will give two proofs which is different from the proof in [9] and actually is much shorter. The first proof is inspired by the proof of Theorem 3.6, where we have shown that the direct sum T of a representative system $T_1, \ldots, T_n$ of pairwise indecomposable bundles in the slope range $0 \le \mu(E) \le -\delta(\vec{\omega})$ is tilting in coh-$\mathbb{X}$. Now observe that the structure sheaf $\mathcal{O}$ belongs to this system. Let's assume that $\mathcal{O} = T_1$. It is then not difficult to check that $\bar{T} = T_2 \oplus \cdots T_n$ is a tilting object in $\underline{\text{vect}}$-$\mathbb{X}$ with endomorphism ring $k\vec{\Delta}$.

[1] This result of Happel was treated in D. Happel's lectures.

We next present another proof that perhaps is providing more insight in the mechanism. By way of illustration, we restrict to the weight type $(2, 3, 4)$, the other cases can be dealt with in a similar fashion. The relevant facts, we are going to use, all are present in Example 3.5. We recall that the category of indecomposable vector bundles for this weight type is given as the mesh category $k\mathbb{Z}\tilde{\mathbb{E}}_7$ depicted below:



We have depicted the vertices corresponding to the line bundles from the τ-orbit of $\mathcal{O}$ by small circles $\circ$ and all arrows starting or ending in such a vertex are marked by dotted arrows. Passing to the stable category $\underline{\text{vect}}\text{-}\mathbb{X}$ just kills the orbit $\tau^{\mathbb{Z}}\mathcal{O}$ and the morphisms factoring through a finite direct sum of those. On the level of mesh-categories this means to kill the marked vertices and the adjacent arrows, yielding the mesh category of $k\Delta$ of the Dynkin diagram Δ. It is a fundamental result by Happel [6] that the mesh category of $k\Delta$ is equivalent to the bounded derived category $\mathrm{D}^b(\mathrm{mod}\, k\vec{\Delta})$ for any orientation $\vec{\Delta}$ of Δ. It is further well-known that each slice in the AR-quiver $k\mathbb{Z}\Delta$ yields a tilting object in the triangulated category $\mathrm{D}^b(\mathrm{mod}\, k\vec{\Delta})$. In the above picture we have marked one such tilting object. $\square$

3. The case $\delta = 0$

Comment 4.4. What is going to happen for the tubular weight triples $(3, 3, 3)$, $(2, 4, 4)$ and $(2, 3, 6)$? In this case $\mathbb{Z}\vec{\omega}$ is a finite cyclic group and so restriction of S to $\mathbb{Z}\vec{\omega}$ does not yield a $\mathbb{Z}$-graded algebra. Moreover, it can be shown that the restriction $R = S_{|U}$ of S to any infinite cyclic subgroup U of $\mathbb{L}$ is never $\mathbb{Z}$-graded Gorenstein; in particular it will never be generated by three homogeneous elements.

The conclusion from this is that for tubular weight triples it only makes sense to study alternatively the stable category of vector bundles $\mathcal{T} = \text{vect-}\mathbb{X}/[\mathcal{L}]$ where one factors out all line bundles. Here, Ueda [24] that $\mathcal{T}$ is triangle-equivalent to the category $\mathrm{D}^b(\text{coh-}\mathbb{X})$. Ueda's proof uses an $\mathbb{L}$-graded version of a recent theorem of D. Orlov [21]. It also possible to directly construct a tilting object in $\mathcal{T}$ whose endomorphism algebra is the canonical algebra of the corresponding weight type, see [15].

Ueda's result (actually Orlov's result underlying Ueda's proof) looks paradoxical. Namely, we start with the category coh-$\mathbb{X}$ of coherent sheaves, then pass to the subcategory vect-$\mathbb{X}$ of vector bundles and in the next step make the category additionally smaller when passing to the stable category $\mathcal{T} = \text{vect-}\mathbb{X}/[\mathcal{L}]$. This, by Ueda's result, is triangle-equivalent to the bounded derived category $\mathrm{D}^b(\text{coh-}\mathbb{X})$ which, being equivalent to the repetitive category $\bigvee_{n\in\mathbb{Z}} \text{coh-}\mathbb{X}[n]$ looks much bigger than the category we started with. Note, in this context, that for tubular weight type the category coh-$\mathbb{X}$ has tame representation type.

4. Case $\delta > 0$, Arnold's strange duality list

We are now going to discuss what happens with the restriction procedure if we apply it to weight triples with $\delta(\vec{\omega}) > 0$. The following result is taken from [16] and [18] where additional information is available.

Proposition 4.5. *Let k be a field and assume (p_1, p_2, p_3) is a weight triple with $\delta(\vec{\omega}) > 0$. Let $R = S_{|\mathbb{Z}\vec{\omega}}$ be the $\mathbb{Z}$-graded restriction of the $\mathbb{L}$-graded triangle singularity S the subgroup $\mathbb{Z}\vec{\omega}$ which we identify with $\mathbb{Z}$ by the correspondence $\vec{\omega} \leftrightarrow 1$. Then the following holds:*

(a) *The algebra R is always graded Gorenstein.*
(b) *Exactly for the weights triples of* Arnold's strange duality list *the algebra R is generated by three homogeneous elements x, y, z and then has the form*
$$R = k[x, y, z]/(f)$$
where the generators x, y, z, the relation f and their degrees is given by the list below.

(p_1, p_2, p_3)	generators (x, y, z)	$\deg(x, y, z)$	relation f	$\deg f$	N	
$(2,3,7)$	(x_3, x_2, x_1)	$(6,14,21)$	$z^2 + y^3 + x^7$	42	12	
$(2,3,8)$	(x_3^2, x_2, x_1x_3)	$(6,8,15)$	$z^2 + x^5 + xy^3$	30	13	
$(2,3,9)$	(x_3^3, x_2x_3, x_1)	$(6,8,9)$	$y^3 + xz^2 + x^4$	24	14	
$(2,4,5)$	(x_3, x_2^2, x_1x_2)	$(4,10,15)$	$z^2 + y^3 + x^5y$	30	11	
$(2,4,6)$	$(x_3^2, x_2^2, x_1x_2x_3)$	$(4,6,11)$	$z^2 + x^4y + xy^3$	22	12	
$(2,4,7)$	$(x_3^3, x_2^2x_3, x_1x_2)$	$(4,6,7)$	$y^3 + x^3y + xz^2$	18	13	
$(2,5,5)$	(x_2x_3, x_1, x_2^5)	$(4,5,10)$	$z^2 + y^2z + x^5$	20	12	$\bullet$
$(2,5,6)$	$(x_2x_3^2, x_1x_3, x_2^4)$	$(4,5,6)$	$xz^2 + y^2z + x^4$	16	13	
$(3,3,4)$	(x_3, x_1x_2, x_1^3)	$(3,8,12)$	$z^2 + y^3 + x^4z$	24	10	$\bullet$
$(3,3,5)$	$(x_3^2, x_1x_2, x_3x_1^2)$	$(3,5,9)$	$z^2 + xy^3 + x^3z$	18	11	$\bullet$
$(3,3,6)$	$(x_2^3, x_1x_2x_3, x_2^3)$	$(3,5,6)$	$y^3 + x^3z + xz^2$	15	12	$\bullet$
$(3,4,4)$	$(x_2x_3, x_1^2, x_1x_2^4)$	$(3,4,8)$	$z^2 - y^2z + x^4y$	16	11	$\bullet$
$(3,4,5)$	$(x_2x_3^3, x_1^2x_3, x_1x_2^3)$	$(3,4,5)$	$x^3y + xz^2 + y^2z$	13	12	
$(4,4,4)$	$(x_1x_2x_3, x_1^4, x_2^4)$	$(3,4,4)$	$x^4 - yz^2 + y^2z$	12	12	$\bullet$

Arnold's strange duality list

Here, the bullet marks the cases where one has a choice for the monomial generators. Further, N denotes the sum of the three weights, whose mathematical significance will be revealed later.

Remark 4.6. (1) For each graded singularity f from Arnold's list, the original weight type can be recovered by the procedure discussed already for the weight triples of Dynkin type: The Serre construction, when applied to the $\mathbb{Z}$-graded algebra $R = k[x, y, z]/(f)$ yields back the category coh-$\mathbb{X}$ on the weighted projective line $\mathbb{X}(p_1, p_2, p_3)$, and the tubular type of coh-$\mathbb{X}$ just coincides with (p_1, p_2, p_3). In the classical context, where $k = \mathbb{C}$ this triple runs under the name of *Dolgachev numbers* of f.

(2) For the base field $k = \mathbb{C}$ this list is (equivalent to) Arnold's list of the 14 exceptional unimodular singularities.

This list, slightly extended by the so-called **Gabrielov numbers**, gives rise to what is called **Arnold's strange duality** which is also related to mirror symmetry. As pointed out before, the weight triples, we are using, will in this context be called **Dolgachev numbers**. On an ad-hoc bases we point out that the above list is equipped with an involution, keeping all weight triples (p_1, p_2, p_2) with $\sum_{i=1}^{3} p_i = 12$ fixed and otherwise sends a weight triple (p_1, p_2, p_3) (Dolgachev numbers) to the conjugate triple (p_1', p_2', p_3') (Gabrielov numbers) such that $\sum_{i=1}^{3} p_i + \sum_{i=1}^{3} p_i' = 24$ and moreover the degrees of the relations attached to the two weight triples are identical. We refer to the introductory account of Ebeling [3] for the definition and properties of Gabrielov numbers.

Our next theorem is taken from joint work with J.A. de la Peña [18]. We point out that the research by Kajiura-Saito-Takahashi [10] is related by subject and results; language and setting are however different. Before stating the result we need to introduce the concept of an **extended canonical algebra**. By definition, an extended canonical algebra $\bar{\Lambda} = \bar{\Lambda}(p_1, p_2, p_3)$ arises from a canonical $\Lambda = \Lambda(p_1, p_2, p_3)$ by attaching one new arrow (with a new vertex) to a vertex of the quiver of Λ, keeping the relation for Λ, and not introducing any new relation. The algebra given by the quiver

$$\begin{tikzcd}
 & \circ \arrow[r, "x_1"] & \circ \arrow[r, "x_1"] & \cdots \arrow[r, "x_1"] & \circ \arrow[dr, "x_1"] & & \\
\circ \arrow[ur, "x_1"] \arrow[r, "x_2"'] \arrow[dr, "x_3"'] & \circ \arrow[r, "x_2"'] & \circ \arrow[r, "x_2"'] & \cdots \arrow[r, "x_2"'] & \circ \arrow[r, "x_2"'] & \circ \arrow[r] & \star \\
 & \circ \arrow[r, "x_3"'] & \circ \arrow[r, "x_3"'] & \cdots \arrow[r, "x_2"'] & \circ \arrow[ur, "x_3"'] & &
\end{tikzcd}$$

having three arms of lengths p_1, p_2, p_3 respectively, and with the single relation $x_1^{p_1} + x_2^{p_2} + x_3^{p_3} = 0$ is thus an extended canonical algebra $\bar{\Lambda}(p_1, p_2, p_3)$. (We have marked the extension vertex by $\star$.) Any other attachment of a new arrow and a new vertex would have led to a derived-equivalent algebra, which indicates already some important feature of the extended canonical algebras.

Theorem 4.7. *For any weight triple (p_1, p_2, p_3) with $\delta(\vec{\omega}) > 0$ there exists a tilting object T in the stable category $\underline{\text{vect}}\text{-}\mathbb{X} = \text{vect-}\mathbb{X}/[\mathcal{L}]$ whose endomorphism ring is the extended canonical algebra $\bar{\Lambda}(p_1, p_2, p_3)$. Accordingly we have equivalences of triangulated categories*

$$\underline{\text{CM}}^{\mathbb{Z}}\text{-}R = \underline{\text{vect}}\text{-}\mathbb{X} \cong \mathrm{D}^b(\text{mod}\,\bar{\Lambda}),$$

and the Grothendieck group $\mathrm{K}_0(\underline{\text{vect}}\text{-}\mathbb{X})$ is finitely generated free of rank $\sum_{i=1}^{3} p_i$.

Proof. We fix a weight triple (p_1, p_2, p_3) with $\delta(\vec{\omega}) > 0$. Let us say first that we do not know any 'concrete' vector bundle T in vect-$\mathbb{X}$ producing a tilting object in the stable category $\underline{\text{vect}}$-$\mathbb{X}$. Our construction of such a tilting object T is thus done by a theoretical argument by using a recent theorem of D. Orlov: actually we need to apply the proof of Orlov's theorem to the present situation. Since the details are quite technical, we only describe the basic idea of the proof. It follows from our assumption on the weight type and from the construction of R that R is always graded Gorenstein and moreover, the so-called *Gorenstein parameter* equals -1. Orlov's theorem then states that there is an exceptional object E in $\underline{\text{vect}}$-$\mathbb{X}$ such that its right perpendicular category, that is, the triangulated subcategory consisting of all objects X such that $\underline{\text{Hom}}(E[n], X) = 0$ for each integer n, is equivalent to $\mathrm{D}^b(\text{coh-}\mathbb{X})$. Choosing in coh-$\mathbb{X}$ a tilting object T with endomorphism algebra $\Lambda = \Lambda(p_1, p_2, p_3)$ it then can be shown that the direct sum $\bar{T} = T \oplus E$ is a tilting object in $\underline{\text{vect}}$-$\mathbb{X}$.

It follows that the Grothendieck group of the triangulated category $\underline{\text{vect}}$-$\mathbb{X}$ is finitely generated free. Since, moreover, the quiver of the extended canonical algebra has $N = \sum_{i=1}^{3} p_i$ vertices, it follows that the rank of $\mathrm{K}_0(\underline{\text{vect}}\text{-}\mathbb{X})$ equals the sum N of the weights. □

The theorem has interesting applications; and up to now no other method is known to derive these assertions. We assume throughout that $\delta(\vec{\omega}) > 0$.

Corollary 4.8. *Each Auslander-Reiten component in $\underline{\text{vect}}$-$\mathbb{X} \cong \mathrm{D}^b(\text{mod}\,\bar{\Lambda})$ has shape $\mathbb{A}_\infty$.*

PROOF. The corresponding statement is known for the category vect-$\mathbb{X}$, see [19]. By a stability argument all line bundles form AR-orbits belonging to the boundary of $\mathbb{Z}\mathbb{A}_\infty$-components. It follows that after stabilization the components still have shape $\mathbb{Z}\mathbb{A}_\infty$. □

Corollary 4.9. *The set of Auslander-Reiten components of* $\underline{\mathrm{vect}}\text{-}\mathbb{X} \cong \mathrm{D}^b(\mathrm{mod}\,\bar{\Lambda})$ *is in natural bijection with the set of regular Auslander-Reiten components for any path algebra kQ of a quiver Q with underlying graph $[p_1, p_2, p_3]$.*

PROOF. For the set of AR-components for vect-$\mathbb{X}$ this is shown in [19]. By the previous argument, stabilization does not change the set of AR-components. □

Remark 4.10. (1) As for the simple graded singularities the degrees of the relations f from Arnold's list have an interpretation as the period of the Coxeter transformation for $\mathrm{D}^b(\mathrm{mod}\,\bar{\Lambda})$, equivalently as the period of the Coxeter transformation for the triangulated category $\underline{\mathrm{vect}}\text{-}\mathbb{X}$. In fact the two triangulated categories (which are equivalent) are Calabi-Yau of fractional Calabi-Yau dimension, yielding a conceptual reason for the observed periodicity.

(2) However, the fractional Calabi-Yau property for $\underline{\mathrm{vect}}\text{-}\mathbb{X}$ is not true for arbitrary weight triples. Weight type $(2, 3, 11)$ already yields an example.

(2) Finally, we remark that the stable categories of vector bundles of shape vect-$\mathbb{X}/[\mathcal{L}]$ are in a certain sense much better behaved than those discussed here. The main reason is that they have *more symmetry* since the Picard group $\mathbb{L}$ acts on them. For instance one then has tilting objects for each weight triple, given by an explicit construction. For details in this direction we refer to [15] and [14].

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