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Degenerations of algebras

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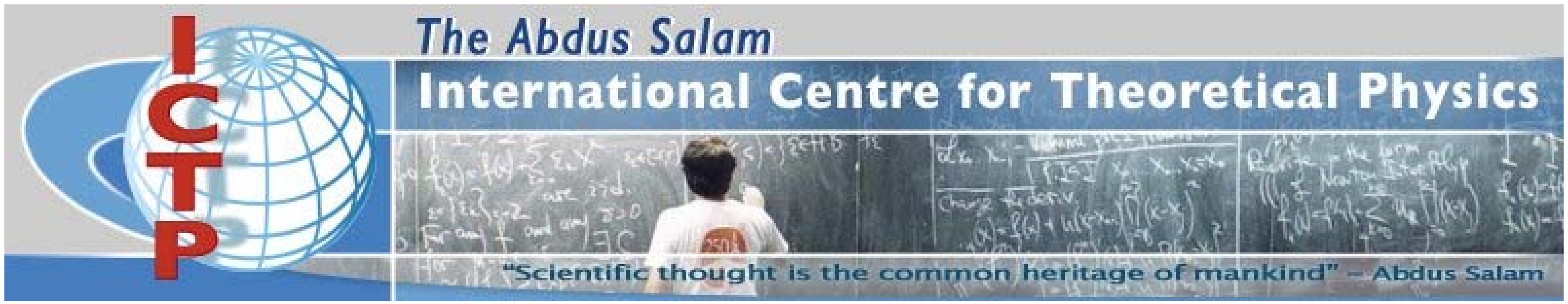

Degenerations of algebras.
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Degenerations of algebras.
Lecture 1. Notation.

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- Algebras are associative finite dimensional k-algebras with an identity. Here k is an algebraically closed field.
- Let A be an algebra. By mod_A we denote the category of finite dimensional (= finitely generated) left A-modules.
- We assume algebras A are basic and $A = kQ/I$, where Q is a *quiver* (=an oriented graph) with set of vertices Q_0 and set of arrows Q_1.
- Recall: the set of vertices Q_0 is the set of isoclasses of simple A-modules $\{1, \ldots, n\}$. Let S_i be a simple A-module representing the i-th class. Then there are as many arrows from i to j in Q as $\dim_k \text{Ext}^1_A(S_i, S_j)$. If A is basic, there is a surjective morphism $\nu : kQ \to A$ such that the ideal $\ker \nu$ is admissible, that is, $(\text{rad}\, A)^m \subset \ker \nu \subset (\text{rad}\, A)^2$ for some $m \geq 2$.

Degenerations of algebras.
Lecture 1. Notation: representations.

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kQ the *path algebra* has as k-basis the oriented paths in Q, including a trivial path e_s for each vertex $s \in Q_0$, with the product given by concatenation of the paths. We shall identify $A = kQ/I$ with a k-category whose objects are the vertices of Q and whose morphism space $A(s,t)$ is $e_t A e_s$.

A module $X \in \mathrm{mod}_{kQ}$ is a *representation* of Q with a vector space $X(s) = e_s X$ for each vertex $s \in Q_0$ and a linear map $X(\alpha)\colon X(s) \to X(t)$ for each arrow $s \to t$ in Q_1.

An A-module X is a k-linear functor $X\colon A \to \mathrm{mod}_k$.

The *dimension vector* of X is $\mathbf{dim}\, X = (\dim_k X(s))_{s \in Q_0} \in \mathbb{N}^{Q_0}$ and the *support* of X is $\mathrm{supp}\ X = \{s \in Q_0 \colon X(s) \neq 0\}$.

The *duality* $\mathrm{mod}_A \to \mathrm{mod}_{A^{op}}$ defined as $D = \mathrm{Hom}_k(-, k)$, where A^{op} is the opposite algebra of A.

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We consider the affine space $V = k^n$ with the *Zariski topology*, that is, closed sets are of the form

$$Z(p_1, \dots, p_s) = \{v \in V : p_i(v) = 0, \text{ for all } i = 1, \dots, s\},$$

where the $p_i \in k[t_1, \dots, t_n]$ are polynomials in n indeterminates.

Fundamental facts:

- $S \subset k[t_1, \dots, t_n]$, then $Z(S)$ is the zero set of S.
- $Z(S) = Z(\langle S \rangle) = Z(\sqrt{\langle S \rangle})$, where

$$\begin{aligned} \langle S \rangle &= \text{ideal of } k[t_1, \dots, t_n] \text{ generated by } S \\ \sqrt{I} &= (\text{radical of } I) = \{p \in k[t_1, \dots, t_n] : p^i \in I \text{ for some } i \in \mathbb{N}\} \end{aligned}$$

- $Z\left(\bigcup_{i \in I} S_i\right) = \bigcap_{i \in I} Z(S_i)$ and $Z(S \cdot S') = Z(S) \cup Z(S')$
- *Hilbert's basis theorem*: $\exists\, p_1, \dots, p_s \in S$ with $Z(S) = Z(p_1, \dots, p_s)$
- *Hilbert's Nullstellensatz*: $\{p \in k[t_1, \dots, t_n] : p \equiv 0 \text{ on } Z(S)\} = \sqrt{\langle S \rangle}$

We say that $Z = Z(S)$ is an *affine variety* and $k[Z] = k[t_1, \dots, t_n]/\sqrt{\langle S \rangle}$ is its *coordinate ring*.

Degenerations of algebras.
Lecture 1. Basic geometric concepts.

Examples

Let k be the field of complex numbers C. Let $\mathbf{A}^2$ be a two dimensional affine space over C. The polynomials f in the ring $k[x, y]$ can be viewed as complex valued functions on $\mathbf{A}^2$ by evaluating f at the points in $\mathbf{A}^2$. Let S be the subset of $k[x, y]$ containing a single element $f(x, y)$:

$$f(x,y) = x + y - 1 = 0.$$

The zero-locus of $f(x, y)$ the set of points in $\mathbf{A}^2$ on which this function vanishes: it is the set of all pairs of complex numbers (x,y) such that $y = 1 - x$, known as a line. This is the set $Z(f)$:

$$Z(f) = \{(x, 1-x) \in \mathbb{C}^2\}.$$

Thus the subset $V = Z(f)$ of $\mathbf{A}^2$ is an algebraic set. The set V is not an empty set. It is irreducible, that is, it cannot be written as the union of two proper algebraic subsets.

Let subset S of $k[x, y]$ contain a single element $g(x, y)$:

$$g(x,y) = x^2 + y^2 - 1 = 0.$$

The zero-locus of g(x, y), that is the set of points (x,y) such that $x^2 + y^2 = 1$, is a circle.

The variety defined by $\{(x, y, z) : x^2 + y^2 - z^2 = 0\}$ is a cone. Hence the variety

.

$$\{(x, y, z) : x^2 + y^2 - z^2 = 0,\ a\,x + b\,y + c\,z = 0\}$$

.

is the intersection of a cone and a plane, therefore a conic section.

Degenerations of algebras.
Lecture 1. Geometric concepts: irreducibility.

An affine variety $Z = Z(p_1, \ldots, p_s)$ is *reducible* if $Z = Z_1 \cup Z_2$ with proper closed subsets $Z_i \subset Z$. Otherwise Z is *irreducible*.

- There is a finite decomposition of any affine variety $Z = \bigcup_{i=1}^{s} Z_i$ into irreducible subsets $Z_i \subset Z$. If the decomposition is irredundant, we say that $Z_1, \ldots, Z_s$ are the *irreducible components* of Z.
- If Z is an irreducible variety, then the maximal length of a chain

$$\emptyset \neq Z_0 \subset Z_1 \subset \cdots \subset Z_s = Z$$

is called the *dimension* of Z $(=: \dim Z)$.

If $Z = \bigcup_{i=1}^{s} Z_i$ is an irreducible decomposition

$$\dim Z = \max_i \dim Z_i.$$

Degenerations of algebras.
Lecture 1. Example: commuting matrices.

The fact that commuting matrices have a common eigenvector – and hence by induction stabilize a common flag and are simultaneously triangularizable -- can be interpreted as a result of the Nullstellensatz, as follows:

commuting matrices $A_1,\dots,A_s$ form a commutative algebra

$k[A_1,\dots,A_s]$ over the polynomial ring $k[x_1,\dots,x_s]$

the matrices satisfy various polynomials such as their minimal polynomials, which form a proper ideal (because they are not all zero, in which case the result is trivial); one might call this the **characteristic ideal, by analogy with the characteristic polynomial.**

Define an eigenvector for a commutative algebra as a vector v *such that* *$x(v)=\lambda(v)x$ for a linear functional* $\lambda\colon A \to K.$ *and for all x in A.*

Observe that a common eigenvector, as if v is a common eigenvector, meaning $A_i(v) = \lambda_i v$, then the functional is defined as $\lambda(c_0 I + c_1 A_1 + \cdots c_k A_k) := c_0 + \sum c_i \lambda_i$

(treating scalars as multiples of the identity matrix A_0: = I, which has eigenvalue 1 for all vectors), and conversely an eigenvector for such a functional λ is a common eigenvector. Geometrically, the eigenvalue corresponds to the point in affine k-space with coordinates $(\lambda_1,\dots,\lambda_k)$ *with respect to the basis given by A_i.*

Then the existence of an eigenvalue λ is equivalent to the ideal generated by the A_i *being non-empty.*

Observe this proof generalizes the usual proof of existence of eigenvalues.

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Example: $A = kQ/I$ where $Q: \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$ and $I = \langle \alpha\beta \rangle$

We consider the set of representatations of vector dimension (2,2,2). They satisfy the system of (polynomial!) equations:

$$\begin{pmatrix} x_{\alpha 11} & x_{\alpha 12} \\ x_{\alpha 21} & x_{\alpha 22} \end{pmatrix} \begin{pmatrix} x_{\beta 11} & x_{\beta 12} \\ x_{\beta 21} & x_{\beta 22} \end{pmatrix} = \begin{pmatrix} x_{\alpha 11}x_{\beta 11} + x_{\alpha 12}x_{\beta 21} & x_{\alpha 11}x_{\beta 12} + x_{\alpha 12}x_{\beta 22} \\ x_{\alpha 21}x_{\beta 11} + x_{\alpha 22}x_{\beta 21} & x_{\alpha 21}x_{\beta 12} + x_{\alpha 22}x_{\beta 22} \end{pmatrix} = 0$$

$\mathrm{mod}_A(2,2,2) \subset k^{2\times 2} \times k^{2\times 2} = k^8$ defined by 4 equations.

This is the intersection of 4 quadrics and dim mod_A (2,2,2)=4.
But this variety is not irreducible. Observe it contains two 4 dimensional affine spaces A^4, which are irreducible components.

The map f: k → mod_A (2,2,2), such that $f(t) = \left[\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}, \begin{bmatrix} 0 & t^2 \\ t & 0 \end{bmatrix} \right]$ is polynomial in the coordinates, hence continuous in the Zariski topology.
Observe that the $\lim_{t\to 0} f(t)$ is a semisimple module.

A map $\mu\colon Y \to Z$ between affine varieties is a *morphism* (= a *regular map*), if $\mu^*\colon k[Z] \to k[Y]$, $p \mapsto p \circ \mu$ is well-defined. In fact, μ^* is a k-algebra homomorphism.

- Any morphism $\mu\colon Y \to Z$ is continuous in the Zariski topology.
- A map $\mu\colon Y \to Z$ is a morphism if and only if $\exists\, \mu_1, \ldots, \mu_m \in k[t_1, \ldots, t_n]$ such that $\mu(y) = (\mu_1(y), \ldots, \mu_m(y))$, $\forall\, y = (y_1, \ldots, y_n) \in Y \subset k^n$.

Proposition. *Let $\mu\colon Y \to Z$ be a morphism between irreducible affine varieties and assume μ is dominant (i.e. $\overline{\mu(Y)} = Z$). Then for every $z \in Z$ and every irreducible component C of $\mu^{-1}(Z)$ we have*

$$\dim C \geq \dim Y - \dim Z$$

with equality on a dense open set of Z.

In particular, if C is an irreducible component of $Z(p_1, \ldots, p_t) \subset k^n$, we have

$$\dim C \geq n - t$$

- $\mathrm{Bil}(n) = \{$ bilinear maps $k^n \times k^n \to k^n\}$ with the structure of k^{n^3}.
- $\mathrm{Ass}(n) = \{$ associative bilinear maps $\}$, it is a closed subset of $\mathrm{Bil}(n)$, so an affine variety.
- $\mathrm{Alg}(n) = \{$ associative algebra structures on k^n which have a 1. $\}$

Lemma. *(1) $Alg(n)$ is an open subset of $Ass(n)$.*
(2) The map $Alg(n) \to k^n$, $A \to 1_A$ is a regular map.
(3) $Alg(n)$ is an affine variety.

Proof. For a finite dimensional k-algebra A corresponding to the bilinear map m (not necessarily with 1), denote by $L_a^m, R_a^m : A \to A$ the left and right multiplication by $a \in A$.

Then A has a 1 exactly when for some $a \in A$ with both L_a and R_a invertible, in this case $1 = L_a^{-1}(a)$.

(1): $D(a) := \{m \in \mathrm{Ass}(n) : \det L_a^m \det R_a^m \neq 0\}$ is open in $\mathrm{Ass}(n)$. Then $\mathrm{Alg}(n) = \cup_a D(a)$ is open.

(2) On $D(a)$ the map is equal to $m \to (L_a^m)^{-1}(a)$ which is a quotient of polynomial functions on $\mathrm{Bil}(n)$; the denominator is det (L_a^m) which does not vanish on $D(a)$.

(3): In fact $\mathrm{Alg}(n) = \{(m, a) \in \mathrm{Ass}(n) \times k^n : a \text{ is } 1 \text{ for } m\}$.

Let G be an algebraic group.

An *action* of G on the affine variety Z is a morphism $\mu : G \times Z \to Z$ satisfying:
(1) $\mu(1_G, z) = z$;

(2) $\mu(g, \mu(h, z)) = \mu(gh, z)$.
We shall write $\mu(g, z) =: gz$.

A an algebra and g in GL(n)
A^g has multiplication: $a.b=gabg^{-1}$

The general linear group GL(n) acts on Alg(n) by conjugation and the orbits are the isomorphism classes of algebras. The orbit of A is denoted $o(A) := GL(n)A$.

The stabilizer $\mathrm{Stab}_{GL(n)}(A)$ of an algebra A is the automorphism group $\mathrm{Aut}(A)$. Then

$$\dim o(A) = \dim GL(n) - \dim \mathrm{Aut}(A)$$

Lecture 1. Geometric concepts: degenerations.

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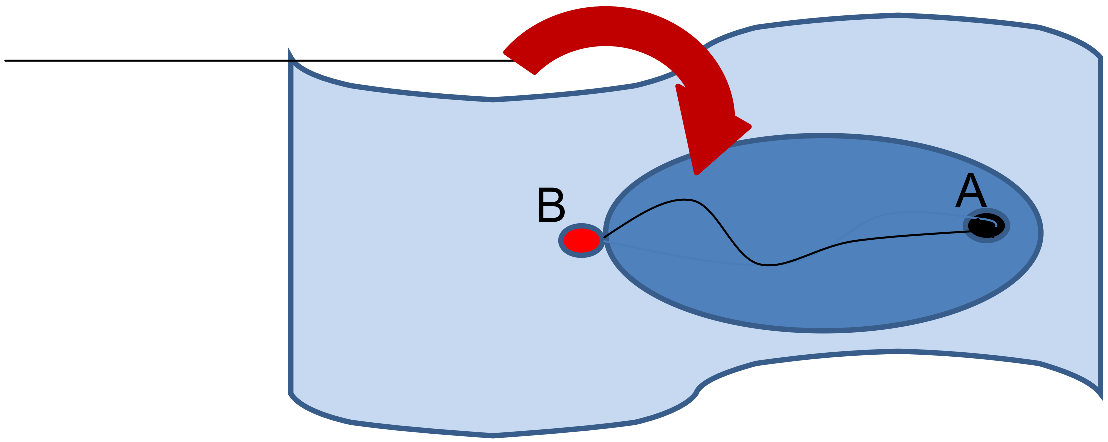
Given a variety Z and a morphism $\mu : Z \to \mathrm{Alg}(n)$, we say that $(\mu(z))_{z\in Z}$ is an *algebraic family.*

Given two algebras A and B in $\mathrm{Alg}(n)$ we say that B is a **degeneration** of A if there is an algebraic family $(A_z)_{z\in Z}$ such that $A_z \tilde{=} A$ for z in an open and dense subset of Z and and $A_{z_0} \tilde{=} B$ for some $z_0 \in Z$.

If B is a degeneration of A then $B \in \overline{o}(A)$ the closure of the orbit of A.

Non-trivial fact: Each degeneration can be obtained along the affine line $\mathbb{C}$.



Proposition. $\mathrm{Alg}(n)$ *is connected and contains exactly one closed orbit, namely that of the commutative algebra* $B_0 = \mathbb{C}[t_2, \ldots, t_n]/(t_i t_j : 2 \leq i, j \leq n)$.

Proof. Let A be any algebra in $\mathrm{Alg}(n)$ with a basis $1 = a_1, a_2, \ldots, a_n$ such that $a_i a_j = \sum_{s=1}^{n} \gamma_{ij}^s a_s$, the γ_{ij}^s are called the *structure constants*.

For $t \in \mathbb{C}$ we define the algebra A_t with basis $1 = a_1, a_2, \ldots, a_n$ and structure constants

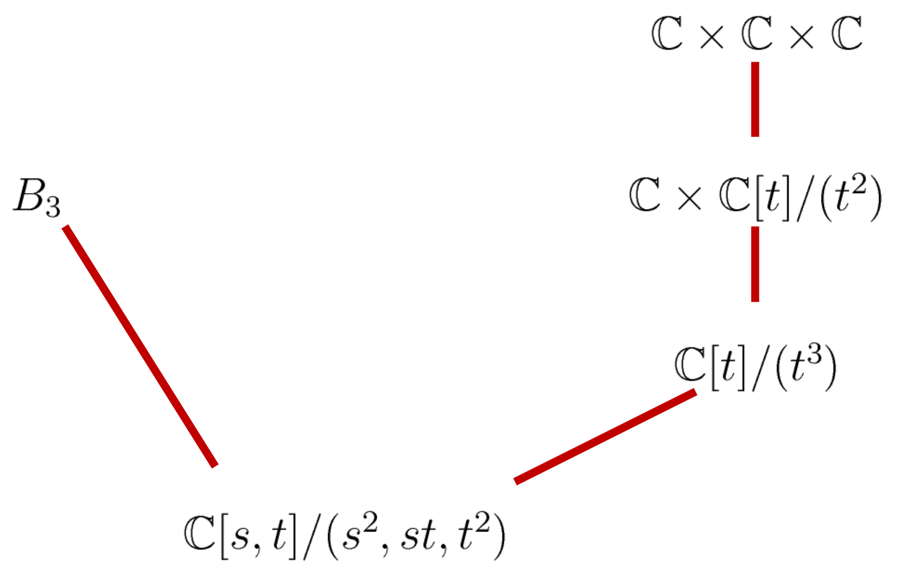
$$\gamma_{ij}^s(t) = \begin{cases} t\gamma_{ij}^s & \text{for } i, j, s \neq 1, \\ t^2 \gamma_{ij}^s & \text{for } i, j \neq 1, s = 1, \\ \gamma_{ij} & \text{otherwise} \end{cases}$$

We get $A_t \simeq A$ for $t \neq 0$ and A_0 is the given commutative algebra B_0. □

Degenerations of algebras.
Lecture 1. Geometric concepts: generic structures.

It is an interesting and difficult problem to determine the number of irreducible components of $\mathrm{Alg}(n)$ and the *generic structures* of those components, that is, those algebras which are not degenerations of other algebras.

Example: for $n = 3$ consider the non-commutative algebra $B_3 = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$, then the generic structures in $\mathrm{Alg}(3)$ are:



$\mathrm{Alg}(3)$ has two components, one of dimension 9, the closure of the orbit of $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$, and one of dimension 7, the closure of the orbit of $\begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$.

Remarks:

Few varieties $\mathrm{Alg}(n)$ have been described (Gabriel, Mazzola, Happel). Shafarevich proved in 1990 that $\mathrm{Alg}(n)$ has at least $n - \sqrt{7n}$ irreducible components.

One component of $\mathrm{Alg}(n)$ is the closure of the orbit of the semi-simple algebra $\mathbb{C}^n$. Mazzola showed that for $n \leq 7$ this set is formed by the commutative n-dimensional algebras. But in dimension 10 there are commutative algebras which are not degenerations of $\mathbb{C}^n$.

A fundamental result is the following

Theorem (Chevalley) *Let $\mu\colon Y \to Z$ be a morphism between affine varieties. Then the function*

$$y \mapsto \dim_y \mu^{-1}(\mu(y)) = \max\{\dim C \colon y \in C \text{ irreducible component of } \mu^{-1}(\mu(y))\}$$

is upper semicontinuous (that is, $d\colon Y \to \mathbb{N}$ has $\{y \in Y \colon d(y) < n\}$ open in Y, for all $n \in \mathbb{N}$).

As illustration consider $\mu\colon \mathbb{C}^3 \to \mathbb{C}^2$ with $\mu(x, y, z) = (x, xy)$. Then

$$\mu^{-1}(\mu(x_0, y_0, z_0)) = \mu^{-1}(x_0, x_0y_0) = \begin{cases} (x_0, y_0, x) & \text{if } x_0 \neq 0,\ \dim = 1 \\ (0, y, z) & \text{if } x_0 = 0,\ \dim = 2 \end{cases}$$

A general morphism $\mu\colon Y \to Z$ is neither open nor closed, but $\mu(Y)$ is a finite union of locally closed subsets of Z. A finite union of locally closed subsets of a variety Z is called a *constructible* subset.

Proposition. *If $\mu\colon Y \to Z$ is a morphism and $Y' \subset Y$ a constructible subset, then $\mu(Y')$ is also constructible.*

Few consequences:

(1) The function $A \to \dim_k Z(A)$, where $Z(A)$ is the center of an algebra A, is upper semicontinous. In particular, the commutative algebras form a closed set in $\mathrm{Alg}(n)$.

Proof. The set $Z := \{(a, A) : a \in Z(A)\}$ is closed in $k^n \times \mathrm{Alg}(n)$. Consider the maps: $\pi : Z \to \mathrm{Alg}(n)$ induced by the projection and the section $\sigma : \mathrm{Alg}(n) \to Z$, defined by $A \to (0, A)$. Clearly, $\pi^{-1}(A) = Z(A) \times \{A\}$ and $dim_k Z(A) = \dim \pi^{-1}(A) = \dim_{\sigma(A)} \pi^{-1}(\pi(\sigma(A)))$. □

(2) The function $A \to \dim \mathrm{Aut}(A)$ is upper semicontinous. In particular:

- the set $\{A \in \mathrm{Alg}(n) : \dim o(A) \leq s\}$ is closed for each s;
- the set $\{A \in \mathrm{Alg}(n) : \dim o(A) = s\}$ is locally closed for each s.

In particular, if B is a degeneration of A in $\mathrm{Alg}(n)$ then:

- $\dim_k Z(B) \geq \dim_k Z(A)$;
- $\dim \mathrm{Aut}(B) \geq \dim \mathrm{Aut}(A)$ (which we already knew by a dimension of orbits argument).

Degenerations of algebras.
Lecture 1. Geometric concepts.

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Let V be a n-dimensional k-vector space. Recall that the Zariski closed set of associative maps in the affine space $\mathrm{Hom}_k(V \otimes_k V, V)$.
Moreover, the associative algebra structures with 1 form an affine open subvariety $\mathrm{Alg}(n)$ of the associative structures.
On $\mathrm{Alg}(n)$ operates the algebraic group $\mathrm{GL}(n)$ by transport of structure. Thus the orbits of the points of this variety are in one to one correspondence with the isoclasses of the n-dimensional associative k-algebras with 1.
An algebra A defines the isotropy group $\mathrm{Aut}(A)$ which is a closed subscheme of $\mathrm{GL}(n)$.
In general, we use small greek letters $(\alpha, \beta \ldots)$ for the points of $\mathrm{Alg}(n)$ and the respective capital roman letters $(A, B \ldots)$ for the corresponding k-algebras.

Let $\alpha \in \mathrm{Alg}(n)$. We recall the construction of the standard **Hochschild complex** for A.

Let $C^{\cdot} = (C^i, d^i_\alpha)_{i\in\mathbb{Z}}$ be the complex with: $C^i = 0$, $d^i_\alpha = 0$ for $i < 0$, $C^0 = V$, $C^i = \mathrm{Hom}_k(V^{\otimes i}, V)$ for $i > 0$, where $V^{\otimes i}$ denotes the i-fold tensor product of V over k; $d^0_\alpha v(x) = \alpha(x \otimes v) - \alpha(v \otimes x)$ for $v, x \in V$, $d^i_\alpha \colon C^i \to C^{i+1}$ with

$$(d^i_\alpha f)(x_1 \otimes \cdots \otimes x_{i+1}) = \alpha(x_1 \otimes f(x_2 \otimes \cdots \otimes x_{i+1}))$$

$$+ \quad \sum_{j=1}^{i} (-1)^j f(x_1 \otimes \cdots \otimes \alpha(x_j \otimes x_{j+1}) \otimes \cdots \otimes x_{i+1})$$

$$+ \quad (-1)^{i+1} \alpha(f(x_1 \otimes \cdots \otimes x_i) \otimes x_{i+1}),$$

for $f \in C^i$, $x_1 \otimes \cdots x_{i+1} \in V^{\otimes(i+1)}$.
That is, we get a complex:

$$0 \longrightarrow V \xrightarrow{d^0} \mathrm{Hom}_k(V,V) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_k(V^{\otimes i}, V) \xrightarrow{d^i} \mathrm{Hom}_k(V^{\otimes i+1}, V) \longrightarrow \cdots$$

Then $H^i(A) := H^i(C^{\cdot})$ is called the i-th *Hochschild cohomology* of A.

Interpretation of the Hochschild cohomology in low degrees:

The small Hoschschild cohomology groups $H^i(A)$ $(0 \leq i \leq 3)$ have important interpretations. We briefly recall some facts.

$H^0(A) = Z(A)$;

the first group $H^1(A) \cong \mathrm{Der}A/\mathrm{Der}^0 A$, where $\mathrm{Der}A$ (resp. $\mathrm{Der}^0 A$) is the set of derivations (resp. inner derivations) of A.

For $f \in \ker d^2$, there is an associative algebra $A \ltimes_f A$ defined on $A \oplus A$ wiht multiplication $(a,b)(a',b') = (aa', ab' + ba' + f(a \otimes a'))$. Two structures $A \ltimes_f A$, $A \ltimes_g A$ are isomorphic if and only if f and g represent the same element in $H^2(A)$.

semidirect product

gl dim A=s, then for t>s we have H^t (A)=0.

Lemma. a) *For given $d, i \in \mathbb{N}$, the function*

$$\delta^i \colon \mathrm{Alg}(n) \to \mathbb{N}, \alpha \mapsto \dim H^i(A)$$

is upper semicontinuous.
b) *If $H^n(A) = 0$, there exists an open neighborhood $\mathcal{U}$ of α in $\mathrm{Alg}(n)$ and integers c_α, $\tilde{c}_\alpha$ such that for all $\beta \in \mathcal{U}$ we have,*

(i) $\dim_k \ker d_\beta^{n-1} = c_\alpha$,

(ii) $\sum_{i=0}^{n-1} (-1)^i \dim_k H^i(B) = \tilde{c}_\alpha$ *and*

(iii) $H^n(B) = 0$.

Proof. Observe that $\dim_k H^0(A) = \dim_k \ker d_\alpha^0$ and for $i > 0$,

$$\dim_k H^i(A) = \dim_k \ker d_\alpha^i + \dim_k \ker d_\alpha^{i-1} - \dim_k C^{i-1}.$$

Since $d^i \colon \mathrm{Alg}(n) \to \mathrm{Hom}_k(C^i, C^{i+1})$ is a regular map, then $\alpha \mapsto \dim_k \ker d_\alpha^i$ is an upper semicontinuous function by a simple subdeterminant argument. This shows (a).

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Assume that $H^n(A) = 0$. The upper semicontinuity of δ^n implies the existence of an open connected neighborhood $\mathcal{U}$ of α in $\mathrm{Alg}(n)$ where $\delta^n(\beta) = 0$ for $\beta \in \mathcal{U}$. In particular, for $\beta \in \mathcal{U}$ we have

$$\dim_k \ker d_\beta^{n-1} = -\dim_k \ker d_\beta^n + \dim_k C^{n-1}$$

which is a constant function in $\mathcal{U}$ (the left side and the additive inverse of the right side being upper semicontinuous). Finally, observe that

$$\sum_{i=0}^{n-1} (-1)^i \dim_k H^i(B) = \sum_{i=0}^{n-2} (-1)^i \dim_k C^i + (-1)^{n-1} \dim_k \ker d_\beta^{n-1}$$

is a constant for $\beta \in \mathcal{U}$. □

In particular, if B is a degeneration of A, for every $n \in \mathbb{N}$, we have

$$\dim_k H^n(B) \geq \dim_k H^n(A)$$

Degenerations of algebras.
Lecture 2. Tangent space.

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In algebraic geometry, the **Zariski tangent space** is a construction that defines a tangent space, at a point *P* on an algebraic variety *V* . It does not use differential calculus, being based directly on abstract algebra.

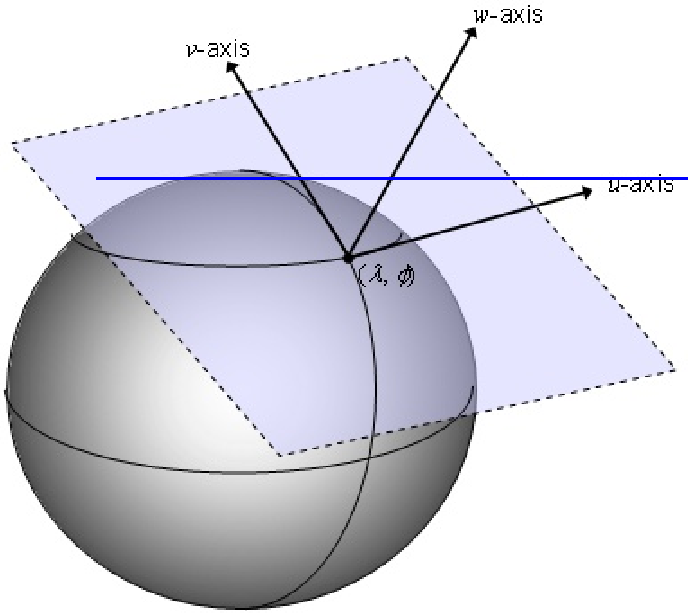
For example, suppose given a plane curve *C* defined by a polynomial equation

$F(X,Y) = 0$

and take *P* to be the origin (0,0). When *F* is considered only in terms of its first-degree terms, we get a 'linearised' equation reading

$L(X,Y) = 0$

in which all terms $X^a Y^b$ have been discarded if $a + b > 1$.



We have two cases: *L* may be 0, or it may be the equation of a line. In the first case the (Zariski) tangent space to *C* at (0,0) is the whole plane, considered as a two-dimensional affine space. In the second case, the tangent space is that line, considered as affine space.

José-Antonio de la Peña

Suppose $V \subset k^n$ is defined by certain polynomials $f(T_1, \ldots, T_n)$. For $x \in V$, define

$$d_x f = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i}(x)(T_i - x_i)$$

the *derivative of f at the point x*. Then the *tangent space* of V at x is the linear variety $T_x(V)$ in the k^n defined by the vanishing of all $d_x f$ as $f(T)$ ranges over the polynomials in the radical ideal $\mathcal{I}(V)$ defining V.

There are more algebraic ways to define tangent spaces: let $R = k[V]$ be the affine algebra associated with V and M_x be the maximal ideal of R vanishing at x. Since R/M_x can be identified with k and M_x is a finitely generated R-module, then then R/M_x-module M_x/M_x^2 is a finite dimensional k-vector space. Then $(M_x/M_x^2)^*$ the dual space over k may be identified with $T_x(V)$.

For $\alpha \in \mathrm{Alg}(n)$ we have canonical inclusions:

$$T_{\mathrm{Alg}(n),\alpha} \to \ker d_\alpha^2 \qquad \text{and} \qquad T_{\mathrm{Aut}(A),\mathrm{id}} \to \ker d_\alpha^1.$$

Moreover, we denote $T^0_{\mathrm{Alg}(n),\alpha}$ the tangent space to the orbit of α at α, then the image of d_α^1 is included in $T^0_{\mathrm{Alg}(n),\alpha}$.

Degenerations of algebras.
Lecture 2. Hochschild coh. and rigidity.

Proposition. *Assume* $H^1(A) = 0$ *then the following holds:*
(1) the maps $T_{\mathrm{Aut}(A),\mathrm{id}} \to \ker d^1_\alpha$ *and Im* $d^1_\alpha \to T^0_{\mathrm{Alg}(n),\alpha}$ *are isomorphisms;*
(2) there is a canonical inclusion

$$T_{\mathrm{Alg}(n),\alpha}/T^0_{\mathrm{Alg}(n),\alpha} \to H^2(A).$$

(3) In case $H^3(A) = 0$*, then the above inclusion is an isomorphism. Moreover, the point* α *is* smooth *in the variety* $\mathrm{Alg}(n)$.

Corollary. *(a) If* $H^2(A) = 0$ *then the orbit* $o(A)$ *is open in* $\mathrm{Alg}(n)$.
(b) There are (up to isomorphism) only finitely many algebras A *with dimension* n *and* $H^2(A) = 0$.
(c) If $H^1(A) = 0 = H^3(A)$*, then* $o(A)$ *is open in* $\mathrm{Alg}(n)$ *if and only if* $H^2(A) = 0$.

Proposition. *Let A be a n-dimensional k-algebra. If $H^1(A) = 0$, there is an open neighborhood $\mathcal{U}$ of α in $\mathrm{Alg}(n)$ such that the dimension of the $GL(n)$-orbits of points in $\mathcal{U}$ is constant (in fact, equal to $n^2 - n + \dim_k H^0(A)$). In particular, if A is a degeneration of B, then A and B are isomorphic.*

Proof. There is an open neighborhood $\mathcal{U}$ of α such that $H^1(B) = 0$ and $\dim_k H^0(B) = \dim_k H^0(A)$, for every $\beta \in \mathcal{U}$. Moreover, all algebras in $\mathcal{U}$ have smooth automorphism groups of constant dimension $n - \dim_k H^0(A)$. Therefore for $\alpha \in \mathrm{Alg}(n)$, the $\mathrm{GL}(n)$-orbit of α in $\mathrm{Alg}(n)$ has dimension $n^2 - n + \dim_k H^0(A)$.

Let A be a degeneration of B, that is, α belongs to the closure of the $\mathrm{GL}(n)$-orbit of β. Therefore $\mathcal{U}$ contains a point corresponding to an algebra B' isomorphic to B. Since the orbits of A and B' have the same dimension (and are irreducible), they coincide. Hence B is isomorphic to A. □

Degenerations of algebras.
Lecture 2. A tool for calculation.

José-Antonio de la Peña

We recall that A is a *one-point extension* of B by M if

$$A = \begin{pmatrix} B & M \\ 0 & k \end{pmatrix}$$

with the usual matrix operations.

For $A = B[M]$ with M a B-module, *Happel's long exact sequence* relates the Hochschild cohomology groups $H^i(A)$ and $H^i(B)$ in the following way:

$$0 \to H^0(A) \to H^0(B) \to \mathrm{End}_B(M)/k \to H^1(A) \to H^1(B) \to \mathrm{Ext}^1_B(M,M) \to H^2(A) \to \cdots$$

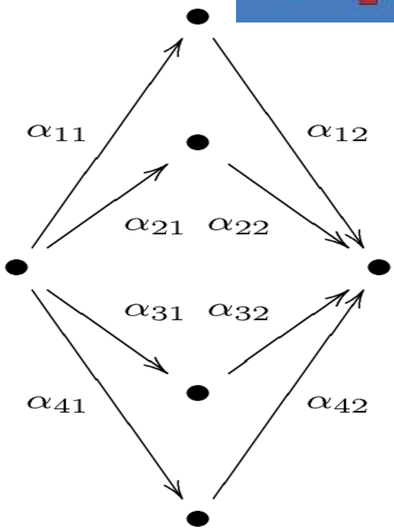
Many applications arise:

- Inductive calculation of $\dim_k H^i(A)$;
- if M is exceptional, that is $\mathrm{Ext}^1_A(M,M) = 0$, we get $H^i(A) = H^i(B)$, for $i \geq 0$, and moreover, the cohomology rings $H^*(A)$ and $H^*(B)$ are isomorphic;
- If A is representation finite with a preprojective component and $H^1(A) = 0$, then for all $n \geq 1$ we have $H^n(A) = 0$.

Degenerations of algebras.
Lecture 2. An example.

José-Antonio de la Peña

Let $C_\lambda = kQ/I_\lambda$, for $\lambda \in k^*$, be the algebra given by the following quiver:



and ideal I_λ generated by the relations $\sum_{i=1}^{3} \alpha_{i2}\alpha_{i1}$ and $\alpha_{12}\alpha_{11} + \lambda\alpha_{22}\alpha_{21} + \alpha_{42}\alpha_{41}$

The following holds:

- The algebras C_λ are isomorphic in $\mathrm{Alg}(16)$.
- $\dim \mathrm{Aut}(C_1) = 15$, hence $\dim\, o(C_1) = \dim\ \mathrm{GL}(16) - 14 = 242$
- we have $C_1 = H[M]$ for a hereditary algebra of type $\tilde{\mathbb{D}}_4$ and M an indecomposabl module with $\dim_k \mathrm{End}_{C_1}(M) = 1 = \dim_k \mathrm{Ext}^1_{C_1}(M, M)$, then

$$H^i(C_1) = 0, \text{ for } i \neq 0, 2$$

and $H^0(C_1) = k = H^2(C_1)$.

- The algebra C_0 is a degeneration of C_1, that is, $C_0 = \lim_{\lambda\to 0} C_\lambda$. In this case $C_0 = H[N_1 \oplus N_2]$ and there exists an exact sequence

$$0 \to N_1 \to M \to N_2 \to 0$$

- $H^0(C_0) = H^1(C_0) = H^2(C_0) = k$, all others $H^n(C_0) = 0$.

Degenerations of algebras.
Lecture 2. Formal deformations.

José-Antonio de la Peña

Let $k[[T]]$ be the algebra of formal power series and $\tilde{p}\colon k[[T]] \to k$ be the canonical projection. A *formal deformation* of $\alpha \in \mathrm{Alg}(n)$ is an element

$$\tilde{\alpha} \in \mathrm{Alg}_{k[[T]]}(n) \text{ such that } \mathrm{Alg}(n)(\tilde{p})(\tilde{\alpha}) = \alpha.$$

Two formal deformations α_1 and α_2 of α are *equivalent* if they are conjugate in $\mathrm{Alg}_{k[[T]]}(n)$ under some $g \in \mathrm{GL}_{k[[T]]}(n)$ of the form $g = E_n + Tg_1 + T^2 g_2 + \ldots$, where for all i, g_i is a $n \times n$ matrix over k.
Moreover, a deformation of α is *trivial* if it is equivalent to α.

An *infinitesimal deformation* of $\alpha \in \mathrm{Alg}(n)$ is an element $\tau \in \mathrm{Alg}_{k[\epsilon]}(n)$ such that $\mathrm{Alg}(p)(\tau) = \alpha$.
Thus the infinitesimal deformations of α may be identified with the tangent space $T_{\mathrm{Alg}(n),\alpha}$. The equivalence classes of the infinitesimal deformations of α may be identified with $H^2(A)$.

An infinitesimal deformation τ of α is *integrable* if there exists a formal deformation $\tilde{\alpha}$ such that the projection $\mathrm{Alg}_{k[[T]]}(n) \to \mathrm{Alg}_{k[\epsilon]}(n)$ send $\tilde{\alpha}$ to τ.

Let $\tilde{\alpha} \in \mathrm{Alg}_{k[[T]]}(n)$ be a formal deformation of $\alpha \in \mathrm{Alg}(n)$. We can write

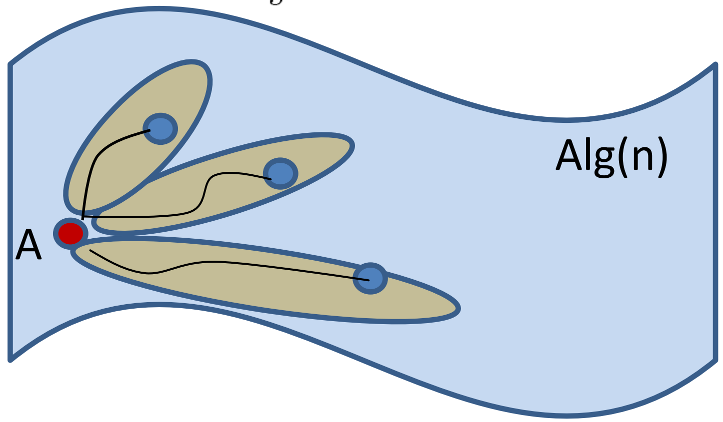
$$\tilde{\alpha} = \alpha + \alpha_1 T + \alpha_2 T^2 + \ldots$$

for k-linear maps $\alpha_i \in \mathrm{Hom}_k(k^n \otimes k^n, k^n)$, for $i \geq 1$.

Let $s \in \mathbb{N}$ and let J_s be the canonical nilpotent $s \times s$-Jordan block. Consider the ring of truncated polynomials $R_s = k[T]/(T^s)$ and let $p_s : \mathrm{Alg}_{k[[T]]}(n) \to \mathrm{Alg}_{R_s}(n)$ be the map induced by the canonical quotient. Then

$$A_s^{\alpha} := p_s(\tilde{\alpha})(J_s) = \alpha + \alpha_1 J_s + \ldots + \alpha_{s-1} J_s^{s-1}$$

is an algebra in $\mathrm{Alg}(n)$.



The following holds:
(1) Equivalent deformations α and β yield isomorphic algebras A_s^{α} and A_s^{β}.

(2) A is a degeneration of A_s^{α}.
For (2) consider the algebraic family $B_\lambda := \alpha + \alpha_1 \lambda J_s + \ldots + \alpha_{s-1}(\lambda J_s)^{s-1}$ which lies in $o(A^{\alpha_s})$ for all $\lambda \neq 0$ and $B_0 = A$.

Proposition. (a) *if* $H^3(A) = 0$, *every* infinitesimal deformation $\tau \in T_{\mathrm{Alg}(n),\alpha}$ *can be lifted to a formal deformation* $\tilde{\alpha}_\tau$ *of* α.
(b) *if* $H^2(A) = 0$ *every formal deformation is trivial.*

Proof. (b): Let $\alpha_t = \alpha + \alpha_1 \cdot t + \alpha_2 \cdot t^2 + \cdots$ be a formal deformation with $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0$ and $\alpha_n \neq 0$. Then $\alpha_n \in \ker d^2_\alpha = \operatorname{Im} d^2_\alpha$, thus there exists $g_n \in C^1 = \mathrm{Hom}_k(k^n, k^n)$ such that $\alpha_n = d_\alpha(g_n)$ and with $g = \mathrm{id} + g_n \cdot t^n$ we get $\alpha_t^g = \alpha_0 + \alpha'_{n+1} \cdot t^{n+1} + \cdots$. □

The algebra A is said to be *absolutely rigid* if $H^2(A) = 0$. It is said to be *analytically rigid* if every formal deformation of α is trivial. Finally, A is said to be *geometrically rigid* if the orbit of α in $\mathrm{Alg}(n)$ is open.

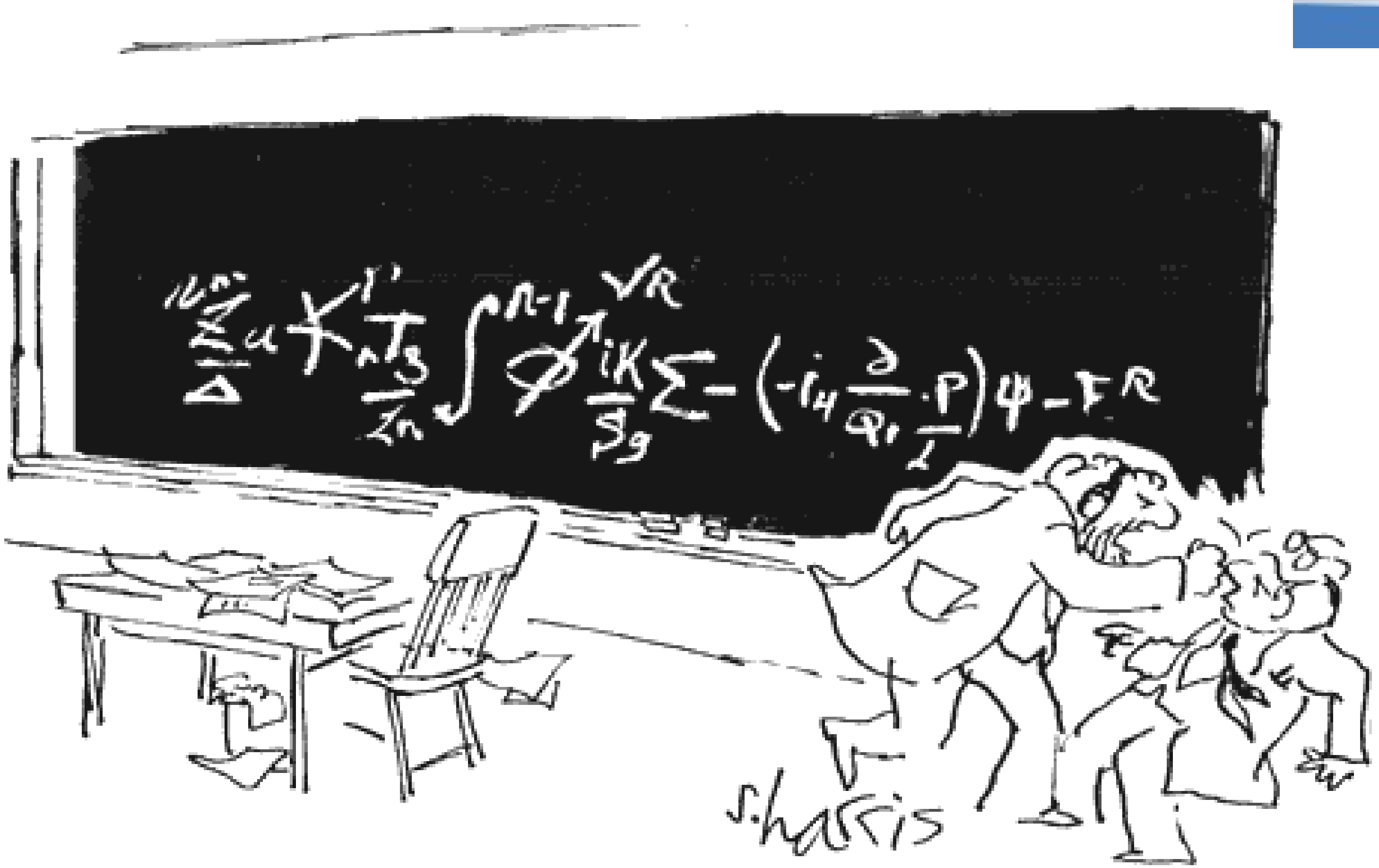
We have the chain of implications:

absolutely rigid $\Rightarrow$ analytically rigid $\Rightarrow$ geometrically rigid.

The converse of the first implication is known to be false for positive characteristic, while it is true if $H^3(A) = 0$. If $\mathrm{char}\, k = 0$, the converse of the second implication holds.

Degenerations of algebras.
Lecture 2. Methodology of proofs.

José-Antonio de la Peña



"You want proof? I'll give you proof!"

José-Antonio de la Peña

A fundamental problem in the *representation theory of algebras* is the classification of all indecomposable A-modules (up to isomorphism). We say that A is of *finite representation type* if there are only finitely many indecomposable A-modules up to isomorphism.

One of the first successes of modern representation theory was the identification by Gabriel of the Dynkin diagrams as the underlying graphs of quivers Q such that kQ is representation-finite. But representation-infinite algebras are common.

Already in the 19^{th} century, Kronecker completed work of Weierstrass to classify all indecomposable 'pencils' by means of infinite families of pairwise non-isomorphic normal forms, which in modern terminology corresponds to the classification of the indecomposable modules over the Kronecker algebra.

The first explicit recognition that infinite representation type splits in two different classes arises in representations of groups: in 1954, Highman showed that the Klein group has infinitely many representations in characteristic 2 and Heller and Reiner classified them; in contrast, Krugljak showed in 1963 that solving the classification problem of groups of type (p, p) with $p \geq 3$ implies the classification of the representations of any group of the same characteristic, a task that was recognized as 'wild'.

Degenerations of algebras.
Lecture 3. Tame and wild definitions.

José-Antonio de la Peña

Donovan and Freislich conjectured at the middle of the 1970's that algebras split in *tame* or *wild* types. This was finally showed by Yuri Drozd in 1980.

The algebra A is *tame* if for every number n, almost every indecomposable A-module of dimension n is isomorphic to a module belonging to a finite number of 1-parameter families of modules.
Formally, an algebra A is *tame* if for every $n \in \mathbb{N}$ there is a finite family of $A - k[t]$-bimodules $M_1, \dots, M_{t(n)}$ with the following properties:

(i) M_i is finitely generated free as a right $k[t]$-module;
(ii) almost every indecomposable left A-module X with $\dim_k X = n$ is isomorphic to a module of the form $M_i \otimes_{k[t]} S_\lambda$ for some $\lambda \in k$.

The algebra A is *wild* if the classification of the indecomposable A-modules implies the classification of the indecomposable modules over the associative algebra $k\langle x, y\rangle$ in two indeterminates.
More formally, A is wild if there exists a functor $F : \text{mod}_{k<x,y>} \to \text{mod}_A$ such that

- F preserves indecomposability of modules;
- if $F(X)$ and $F(Y)$ are isomorphic, then X and Y are isomorphic.

The indecomposable modules over the quiver algebra A:

$$\bullet \rightrightarrows \bullet$$

were classified by Weierstrass and Kronecker in the following families:

$$k^n \underset{\left[\begin{array}{c|c} 0 & \\ \vdots & I_n \\ 0 & \end{array}\right]}{\overset{\left[\begin{array}{c|c} & 0 \\ I_n & \vdots \\ & 0 \end{array}\right]}{\rightrightarrows}} k^{n+1} \qquad\qquad k^{n+1} \underset{\left[\begin{array}{ccc} 0 & \cdots & 0 \\ \hline & I_n & \end{array}\right]}{\overset{\left[\begin{array}{ccc} & I_n & \\ \hline 0 & \cdots & 0 \end{array}\right]}{\rightrightarrows}} k^n$$

(preprojective representation) (preinjective representation)

$$R_n(\lambda): \; k^n \underset{J_n(\lambda)}{\overset{I_n}{\rightrightarrows}} k^n \qquad\qquad R_n(\infty): \; k^n \underset{I_n}{\overset{J_n(0)}{\rightrightarrows}} k^n$$

Let M_n be the $A - k[t]$-bimodule

$$k[t]^n \underset{J_n(t)}{\overset{I_n}{\rightrightarrows}} k[t]^n$$

then $M_n \otimes_{k[t]} k[t]/(t-\lambda) \cong R_n(\lambda)$.

Proposition. *Let p be a prime number ≥ 3. Assume k has characteristic p. The group algebra $A = k[\mathbb{Z}_p \times \mathbb{Z}_p]$ is wild.*

Proof. Let $\varphi\colon k[u,v] \to A$, $x \mapsto g-1$, $y \mapsto h-1$, where $\mathbb{Z}_p \times \mathbb{Z}_p = \langle g \rangle \times \langle h \rangle$. Then $A \cong k[u,v]/\ker\varphi = k[u,v]/(u^p, v^p)$.

Moreover $k[u,v]/(u^p,v^p) \twoheadrightarrow k[u,v]/(u,v)^3 = k[u,v]/(u^3,v^3,uv^2,vu^2) =: B$. It is enough to show that B is wild.

Consider the $B-k\langle x,y\rangle$-bimodule M defined as $M_{k\langle x,y\rangle} = k\langle x,y\rangle^4$ and the structure as B-module defined by the matrices

$$ {}_uM = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & x & y & 0 \end{bmatrix} \qquad {}_vM = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & x & 0 \end{bmatrix} $$

Check that ${}_BM$ is well defined and

$$ M \otimes_{k\langle x,y\rangle} - \colon \mathrm{mod}_{k\langle x,y\rangle} \to \mathrm{mod}_B $$

insets indecomposable modules. $\square$

Proposition. *Let B be any finitely generated k-algebra, then there exists a fully faithful functor* $F\colon \mathrm{mod}_B \to \mathrm{mod}_{k\langle x,y\rangle}$.

Proof. Let $b_1, \dots, b_s$ be a system of generators of B. Define the $k\langle x, y\rangle - B$-bimodule M as $M_B = B^{s+2}$ and the structure of left $k\langle x, y\rangle$-module given by the $(s+2)\times(s+2)$-matrices

$$
{}_xM = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 0 & & & & 0 \end{bmatrix}
\qquad
{}_yM = \begin{bmatrix} 0 & & & & & 0 \\ 1 & 0 & & & & \\ b_1 & 1 & & & & \\ & \ddots & \ddots & & & \\ & & \ddots & 1 & 0 & \\ 0 & & & b_s & 1 & 0 \end{bmatrix}
$$

We set $F = M\otimes_B\colon \mathrm{mod}_B \to \mathrm{mod}_{k\langle x,y\rangle}$.

Check that F is full and faithful. □

Let A be a n-dimensional algebra with basis $1 = e_1, e_2, \ldots, e_n$ with structure constants a_{ij}^s, that is, $e_ie_j = \sum_{s=1}^{n} a_{ij}^s e_s$.
The *module variety* $\mathrm{mod}_A(r)$ is the closed subset of the affine space $\mathrm{mat}_k(r)^n$ formed by matrices $(1_r = E_1, E_2, \ldots, E_n)$ satisfying $E_iE_j = \sum_{s=1}^{n} a_{ij}^s E_s$.
On $\mathrm{mod}_A(r)$ acts by conjugation the algebraic group $\mathrm{GL}(r)$ in such a way that the orbit of a point μ identifies with the isomorphism class of modules M corresponding to μ. We write $o(\mu) = \mathrm{GL}(r)M$. We get:

- $\mathrm{Stab}_{\mathrm{GL}(r)}M = \mathrm{Aut}_A(M)$ which is open and dense in the variety $\mathrm{End}_A(M)$;
- $\dim\ o(\mu) = \dim\ \ \mathrm{GL}(r) - \dim\ \mathrm{Stab}_{\mathrm{GL}(r)}M = r^2 - \dim_k \mathrm{End}_A(M)$;
- the tangent space $T_{(\mathrm{mod}_A(r),\mu)}$ to $\mathrm{mod}_A(r)$ at μ has as subspace $T_{(o(\mu),\mu)}$ and the quotient $T_{(\mathrm{mod}_A(r),\mu)}/T_{(o(\mu),\mu)}$ is a subspace of $\mathrm{Ext}^1(M,M)$;
- $\dim\ \mathrm{mod}_A(r) \leq \dim\ T_{(\mathrm{mod}_A(r),\mu)} \leq \dim_k \mathrm{Ext}_A^1(M,M) + \dim\ T_{(o(\mu),\mu)}$

$$= \dim_k \mathrm{Ext}_A^1(M,M) + \dim\ o(\mu) = \dim_k \mathrm{Ext}_A^1(M,M) + r^2 - \dim_k \mathrm{End}_A(M);$$

- $\dim\ \ \mathrm{GL}(r) - \dim\ \mathrm{mod}_A(r) \geq \dim_k \mathrm{End}_A(M) - \dim_k \mathrm{Ext}_A^1(M,M)$;
- there are only finitely many modules M (up to isomorphism) of dimension r satisfying $\mathrm{Ext}^1(M,M) = 0$.

Degenerations of algebras.
Lecture 3. Degenerations of modules.

José-Antonio de la Peña

Let M be a r-dimensional A module corresponding to the point $\mu \in \text{mod}_A(r)$. The orbit $o(\mu) = \text{GL}(r)M$ is *locally closed*. In particular, $\overline{o}(\mu) \setminus o(\mu)$ is formed by the union of orbits of dimension strictly smaller than $o(\mu)$.

Let $X, Y \in \text{mod}_A$ be modules of dimension r. If the orbit $o(y)$ is contained in $\overline{o}(x)$, we say that Y is a *degeneration* of X.

Proposition. *Let $X \in \text{mod}_A$ of dimension r. We have the following.*

(a) *Let $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ be an exact sequence. Then $X' \oplus X''$ is a degeneration of X.*

(b) *Consider the semisimple module $gr\ X = \oplus_{i=1}^{s} S_i$, obtained as direct sum of the composition factors S_i of X. Then $gr\ X$ is a degeneration of X.*

Corollary. *The orbit* GL(r)X *is closed if and only if X is semisimple.*

Examples (a) Let $F = k\langle T_1, \dots, T_m\rangle$ be the free algebra in m indeterminates. Let M be a $A - F$-bimodule which is free as right F-module.

Then the functor $M \otimes_F - : \mathrm{mod}_F \longrightarrow \mathrm{mod}_A$ induces a family of regular maps $f_M^n : \mathrm{mod}_F(n) \to \mathrm{mod}_A(nr)$ for some number $r \in \mathbb{N}$ and every $n \in \mathbb{N}$.

Indeed, set $r = \mathrm{rk}_F M$. Since $M \otimes F^s = M^s$, then

$$M \otimes coker(\ F^s \xrightarrow{\nu} F^t\ ) = coker(\ M \otimes F^s \xrightarrow{M\otimes\nu} M \otimes F^t\ ) = coker(\ M^s \xrightarrow{M(\nu)} M^t\ ).$$

(b) The subset $\mathrm{ind}_A(r)$ of $\mathrm{mod}_A(r)$ is constructible.

Indeed, the set of pairs.

$$\{(X, f) : X \in\ \mathrm{mod}_A(r), f \in\ \mathrm{End}_A(X) \text{ with } 0 \neq f \neq 1_X \text{ and } f^2 = 1_X\}.$$

is a locally closed subset of $\mathrm{mod}_A(r) \times k^{r^2}$. The projection $\pi_1 : \mathrm{mod}_A(r) \times k^{r^2} \longrightarrow \mathrm{mod}_A(r)$ is a regular map with image

$$\mathrm{mod}_A(r)\backslash\mathrm{ind}_A(r).$$

Degenerations of algebras.
Lecture 3. Number of parameters.

José-Antonio de la Peña

Let $\text{mod}_A(r,s)$ be the set of all modules M with orbit dim $o(\mu) = s$. By upper semicontinuity, the set $\text{mod}_A(r,s)$ is locally closed in $\text{mod}_A(r)$.

Let Y be a constructible subset of $\text{mod}_A(r)$ which is closed under the action of $\text{GL}(r)$, we set $Y_{(s)} = Y \cap \text{mod}_A(r,s)$ which is constructible.
We define the *number of parameters* of $\text{GL}(r)$ on Y as

$$\mu(Y) = \max_s(\dim Y_{(s)} - s).$$

Observe that:
(1) If Z is a constructible subset of Y meeting each orbit, then $\mu(Y) \le \dim Z$.
(2) Let $f : \text{mod}_B(t) \to \text{mod}_A(r)$ be a regular map and Y be a constructible subset of $\text{mod}_A(r)$ which is closed under the action of $\text{GL}(r)$. Assume that Z is a constructible subset of $\text{mod}_B(t)$ restricting to $f : Z \to Y$ such that $\dim f^{-1}(o(y)) \le d$ for each $y \in Y$.
Then $\mu(Y) \ge \dim Z - d$.

Proposition. *1. If A is wild then there is some r such that $\mu(\text{mod}_A(sr)) \geq s^2$, for all s.*
2. If A is tame then $\mu(\text{mod}_A(r)) \leq r$, for all r.

Proof. (1): If A is wild we find a number r and regular maps $\text{mod}_{k<x,y>}(s) \to \text{mod}_A(sr)$ which has as inverse image of an $\text{GL}(sr)$-orbit a $\text{GL}(s)$-orbit. Then

$$\mu(\text{mod}_A(sr)) \geq \dim\, \text{mod}_{k<x,y>}(s) - \dim\, \text{GL}(s) = 2s^2 - s^2 = s^2.$$

(2): If A is tame for every $n \in \mathbb{N}$ there is a finite family of $A - k[t]$-bimodules $M_{n,1}, \ldots, M_{n,t(n)}$ with the following properties:

- $M_{n,i}$ is finitely generated free as a right $k[t]$-module;
- almost every indecomposable left A-module X with $\dim_k X = n$ is isomorphic to a module of the form $M_{n,i} \otimes_{k[t]} S_\lambda$ for some $\lambda \in k$.

Let r be a positive number and $1 \leq i_1, \ldots, i_s \leq r$ be a sequence with $\sum_{p=1}^{s} \text{rank}\, M_{i_p,j_p} = r$ for some selection of $1 \leq j_p \leq t(i_p)$ for each p. Then $\sum\limits_{p=1}^{s} M_{i_p,j_p} \otimes S_\lambda$ defines a constructible subset of $\text{mod}_A(r)$ of dimension $\leq s \leq r$.
Let Z be the union of all these constructible sets for all possible sequences. Since Z meets every orbit, we get $\mu(\text{mod}_A(r)) \leq \dim Z \leq r$. □

José-Antonio de la Peña

Corollary. *An algebra is not simultaneously tame and wild.*

Theorem. *A degeneration of a wild algebra is wild.*

Proof. The set $\{\alpha \in \mathrm{Alg}(n) : \text{A is wild}\} = \cup_r W_r$ where

$$W_r = \{\alpha \in \mathrm{Alg}(n) : \mu(\mathrm{mod}_A(r)) > r\}.$$

The sets W_r are closed and $\mathrm{GL}(r)$-stable.
Hence if $\beta \in \overline{o}(\alpha)$ and A is wild, then $\alpha \in W_r$ for some r which implies $\beta \in W_r$ and B is wild. □

Degenerations of algebras.
Lecture 3. An example.
José-Antonio de la Peña

$$A = k < x, y > /(x^2 - yxy, y^2 - xyx, (xy)^2, (yx)^2)$$

degenerates to $B = k < x, y > /(x^2, y^2, (xy)^2, (yx)^2)$.

Indeed, a point A_t in Alg(7) with basis $1, x, y, xy, yx, xyx, yxy$ and with the multiplication laws given by $x^2 = yxy, y^2 = xyx, (xy)^2 = 0 = (yx)^2$, satisfies:

$A_t \equiv A$ for $t \neq 0$ and $A_0 \equiv B$.

Since B is tame, then A is tame.

This is the only known proof of the tameness of A

Degenerations of algebras.
Lecture 3. Tameness and the structure of the AR quiver .

José-Antonio de la Peña

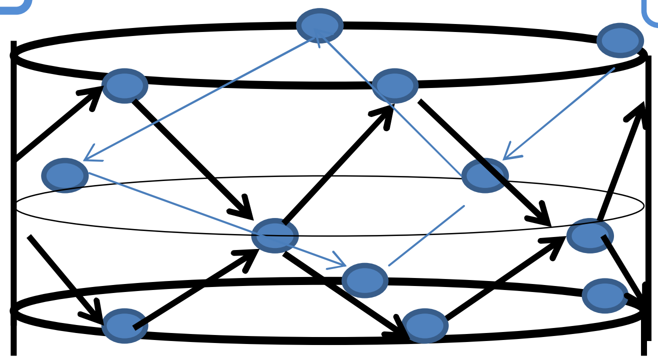
The following is a central fact about the structure of the Auslander-Reiten quiver Γ_A of a tame algebra A.

Theorem *Let A be a tame algebra. Then almost every indecomposable lies in a homogeneous tube. In particular, almost every indecomposable X satisfies $X \simeq \tau X$.*

Open problem: Is it true that an algebra is of tame type if and only if almost every indecomposable module belongs to a homogeneous tube?

Proposition. *Let A be an algebra such that almost every indecomposable lies in a standard tube. Then A is tame.*

Proof. Our hypothesis implies that almost every indecomposable X satisfies $\dim_k \mathrm{End}_A(X) \leq \dim_k X$. We show that this condition implies the tameness of A.



Degenerations of algebras. Proof of proposition.

Indeed, assume that A is wild and let M be a $A - k\langle u, v\rangle$-bimodule which is finitely generated free as right $k\langle u, v\rangle$-module and the functor $M \otimes_{k\langle u,v\rangle}$-insets indecomposables. Consider the algebra B given by the quiver with rad $^2 = 0$:

Then there is a $A - B$-bimodule N such that N_B is free and $N \otimes_B - : \text{mod } B \longrightarrow \text{mod } A$ is fully faithful.

Therefore the composition $F = M \otimes_A (N \otimes_B -)$ is faithful and insets indecomposables. Moreover, $\dim_k FX \leq m \dim_k X$ for any $X \in \text{mod } B$ if we set $m = \dim_k (M \otimes_A N)$.

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Consider also the functor $H : \text{mod } A \longrightarrow \text{mod } B$ sending X to the space $X' = X \oplus X$ with endomorphisms

$$X'(t_1) = \begin{bmatrix} 0 & X(w) \\ 0 & 0 \end{bmatrix}, X'(t_2) = \begin{bmatrix} 0 & X(v) \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad X'(t_3) = \begin{bmatrix} 0 & 1_X \\ 0 & 0 \end{bmatrix}$$

This functor insets indecomposables. For the simple A-modules X of dimension n, we get indecomposable A-modules $FH(X)$ with

$$\dim_k FH(X) \leq m \dim_k H(X) = 2mn$$

and

$$\dim_k \text{End}_A(FH(X)) \geq \dim_k \text{End}_B(H(X)) = n^2 + \dim_k \text{End}_A(X) = n^2 + 1. \qquad \square$$

Consider the (infinite dimensional!) algebra $A := k[x,y]/(y^2 - x^3 + x)$ with char $k \neq 2$

We claim that A is neither tame not wild.
A is not wild: if it were wild, then there is a functor

$$_AM_B \otimes - : \text{mod}_B \to \text{mod}_A$$

which **insets** indecomposable modules, where B is a finite dimensional wild algebra and M_B is free of finite rank.
Choose $m_1, \ldots, m_s$ a k-basis of M and define $f : A \to B^n$ the morphism such that $f(a) = (am_1, \ldots, am_s)$. Then $A' = A/kerf$ is a finite dimensional wild algebra.

But A' is a quotient of some algebra $A'' = k[x,y]/(y^2 - x^3 + x, p(x))$ for some polynomial $p(x) \in k[x]$.
Exercise: A'' is a quotient of $k[x]$ which is representation-finite.

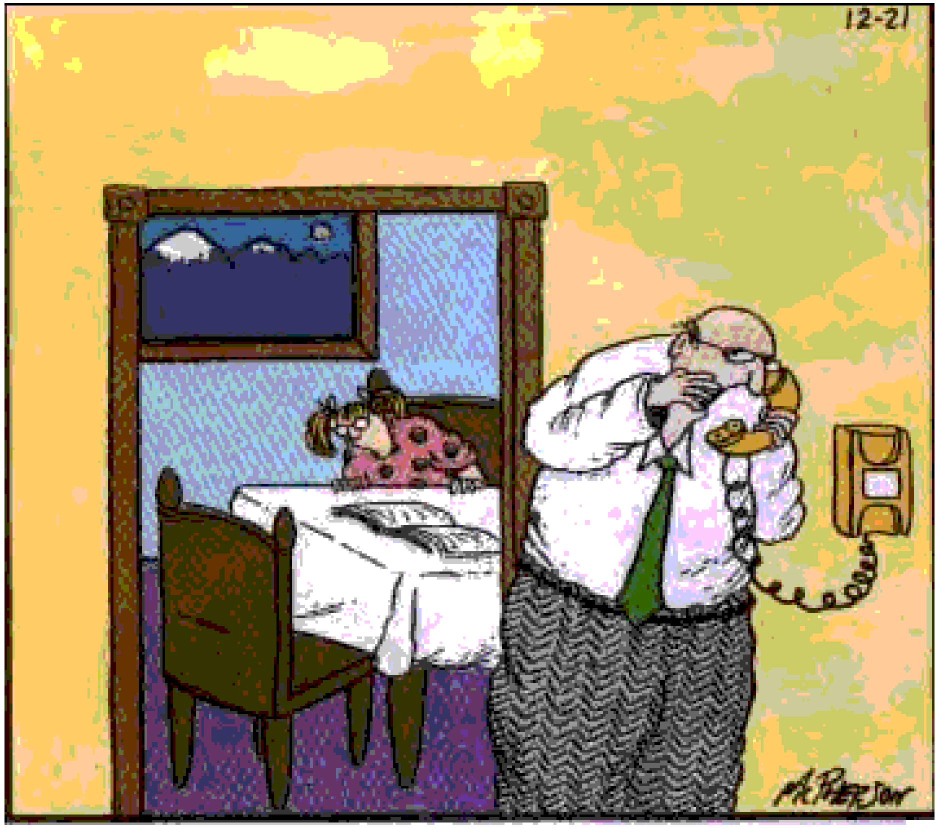
A is not tame: assume otherwise, and consider the module variety $\text{mod}_A(1)$. There should exists an open subset U of k and a regular map $U \to \text{mod}_A(1)$.

Since A is commutative without nilpotent elements, $A = k[\text{mod}_A(1)] \subset k(t)$. By Lüroth theorem we have $A = k(x)$ for a certain trascendental variable x.

This is not true for A, that is, the curve $y^2 - x^3 + x$ is not rational.

Degenerations of algebras.
Lecture 3. Next lecture: in 1 hour! .

José-Antonio de la Peña



"Uh, yeah, Homework Help Line? I need to have you explain the Quadratic Equation in roughly the amount of time it takes to get a cup of coffee."

Let $A = kQ/I$ be a finite dimensional k-algebra and fix a finite set $R \subset \bigcup_{x,y \in Q_0} I(x,y)$ of admissible generators of I. Let $z \in \mathbb{N}^{Q_0}$ be a dimension vector.

The *module variety* $\text{mod}_A(z)$ is the closed subset, with respect to the Zariski topology, of the affine space $k^z = \prod_{x \to y} k^{z(y)z(x)}$ defined by the polynomial equations given by the entries of the matrices

$$m_r = \sum_{i=1}^{t} \lambda_i m_{\alpha i1} \dots m_{\alpha i s_i}, \text{ where } r = \sum_{i=1}^{t} \lambda_i \alpha_{i1} \dots \alpha_{is_i} \in R$$

and for each arrow $x \xrightarrow{\alpha} y$, m_α is the matrix of size $z(y) \times z(x)$

$$m_\alpha = (X_{\alpha ij})_{ij}$$

where $X_{\alpha ij}$ are pairwise different indeterminates. We shall identify points in the variety $\text{mod}_A(z)$ with representations X of A with vector dimension $\mathbf{dim}\, X = z$.

Example: $A = kQ/I$ where $Q\colon \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$ and $I = \langle \alpha\beta \rangle$

$$\begin{pmatrix} x_{\alpha 11} & x_{\alpha 12} \\ x_{\alpha 21} & x_{\alpha 22} \end{pmatrix} \begin{pmatrix} x_{\beta 11} & x_{\beta 12} \\ x_{\beta 21} & x_{\beta 22} \end{pmatrix} = \begin{pmatrix} x_{\alpha 11}x_{\beta 11} + x_{\alpha 12}x_{\beta 21} & x_{\alpha 11}x_{\beta 12} + x_{\alpha 12}x_{\beta 22} \\ x_{\alpha 21}x_{\beta 11} + x_{\alpha 22}x_{\beta 21} & x_{\alpha 21}x_{\beta 12} + x_{\alpha 22}x_{\beta 22} \end{pmatrix}$$

$\mathrm{mod}_A(2,2,2) \subset k^{2\times 2} \times k^{2\times 2} = k^8$ defined by 4 equations.

The group $G(z) = \prod_{i \in Q_0} GL_{z(i)}(k)$ acts on k^z by conjugation, that is, for $X \in k^z$, $g \in G(z)$ and $x \xrightarrow{\alpha} y$, then $X^g(\alpha) = g_y X(\alpha) g_x^{-1}$.

By restriction of this action, $G(z)$ also acts on $\mathrm{mod}_A(z)$. Moreover, there is a bijection between the isoclasses of A-modules X with $\mathbf{dim}\, X = z$ and the $G(z)$-orbits in $\mathrm{mod}_A(z)$.

Given $X \in \mathrm{mod}_A(z)$, we denote by $G(z)X$ the $G(z)$-orbit of X. Then

$$\dim G(z)X = \ dim\ G(z) - \ \dim \mathrm{Stab}_{G(z)}(X),$$

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Theorem *Let* $X \in \operatorname{mod}_A(z)$.

Consider $T_X(G(z)X)$ *as a linear subspace of* $T_X(\operatorname{mod}_A(X))$. *Then there exists a natural linear monomorphism*

$$T_X(\operatorname{mod}_A(X))/T_X(G(z)X) \hookrightarrow \operatorname{Ext}^1_A(X,X).$$

(b) *Assume that* X *satisfies* $\operatorname{Ext}^2_A(X,X) = 0$. *Then the linear morphism*

$$T_X(\operatorname{mod}_A(X))/T_X(G(z)X) \xrightarrow{\sim} \operatorname{Ext}^1_A(X,X).$$

is an isomorphism.

The inclusion above is not always an isomorphism, as the following simple example shows:

Let $A = k[T]/(T^2)$. Consider the simple module $S \in \operatorname{mod}_A(1)$. Then $\operatorname{mod}_A(1) = G(1)S = \{S\}$ and $T_S\ (\operatorname{mod}_A(1))$ is trivial. On the other hand $\operatorname{Ext}^1_A(S,S)$ has dimension 1.

Degenerations of algebras.
Lecture 4. Definition of the Tits form.

José-Antonio de la Peña

Let $A = kQ/I$ be a *triangular* algebra, that is, Q has no oriented cycles. Choose R a minimal set of generators of I, such that $R \subset \bigcup_{i,j \in Q_0} I(i,j)$. We have:

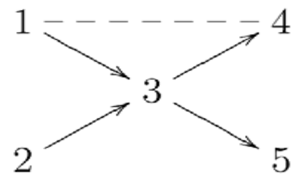
- $\dim_k \mathrm{Ext}^1_A(S_i, S_j) = \#$ arrows from i to j
- $r(i,j) = |R \cap I(i,j)|$ is independent of the choice of R
- $r(i,j) = \dim_k \mathrm{Ext}^2_A(S_i, S_j)$

The *Tits form* of A is the quadratic form

$$q_A \colon \mathbb{Z}^{Q_0} \to \mathbb{Z},$$

given by $q_A(v) = \sum_{i \in Q_0} v(i)^2 - \sum_{i \to j} v(i)v(j) + \sum_{i,j \in Q_0} r(i,j)v(i)v(j).$

$A:$



$$q_A(x_1, x_2, x_3, x_4, x_5) = \sum_{i=1}^{5} x_i^2 - x_1x_3 - x_2x_3 - x_3x_4 - x_3x_5 + x_1x_4 =$$

$$(x_1 - \frac{1}{2}x_3 + \frac{1}{2}x_4)^2 + (x_2 - \frac{1}{2}x_3)^2 + \frac{1}{2}(x_3 - \frac{1}{2}x_4 - \frac{1}{2}x_5)^2 + \frac{5}{8}(x_4 + \frac{1}{5}x_5)^2 + \frac{17}{20}x_5^2$$

Proposition. *Assume $A = kQ/I$ is triangular. Let $z \in N^{Q_0}$. Then for any $X \in mod_A(z)$.*

$$q_A(z) \geq \dim_k \operatorname{End}_A(X) - \dim_k \operatorname{Ext}^1_A(X, X).$$

Proof. Let $X \in \operatorname{mod}_A(z)$. The *local dimension* $\dim_X \operatorname{mod}_A(z)$ is the maximal dimension of the irreducible components of $\operatorname{mod}_A(z)$ containing X. By Krull's Hauptidealsatz, we have

$$\dim_X \operatorname{mod}_A(z) \geq \sum_{i \to j} z(i)z(j) - \sum_{ij \in Q_0} r(i,j)z(i)z(j).$$

Therefore, we get the following inequalities,

$$\begin{aligned} q_A(z) &\geq \dim G(z) - \dim_X \operatorname{mod}_A(z) \geq \dim G(z) - \dim T_X \geq \\ &\geq \dim_k \operatorname{End}_A(X) - \dim_k \operatorname{End}^1_A(X, X). \end{aligned}$$

In 1975, Brenner observed certain connections between properties of q_A and the representation type of A. She wrote about her observations: "...is written in the spirit of experimental science. It reports some regularities and suggests that there should be a theory to explain them".

Theorem. *Let $A = kQ/I$ be a triangular algebra.*

- *If A is representation-finite, then q_A is weakly positive.*
- *If A is tame, then q_A is weakly non-negative.*

Proof. In general, for $v \in \mathbb{N}^{Q_0}$ we have

$$\dim \operatorname{mod}_A(v) \geq \sum_{i \to j} v(i)v(j) - \sum_{i,j \in Q_0} r(i,j)v(i)v(j) rm \text{ and } \dim G(v) = \sum_{i \in Q_0} v(i)^2$$

$$q_A(v) \geq \dim G(v) - \dim \operatorname{mod}_A(v)$$

If A is tame, then $q_A(v) \geq 0$.

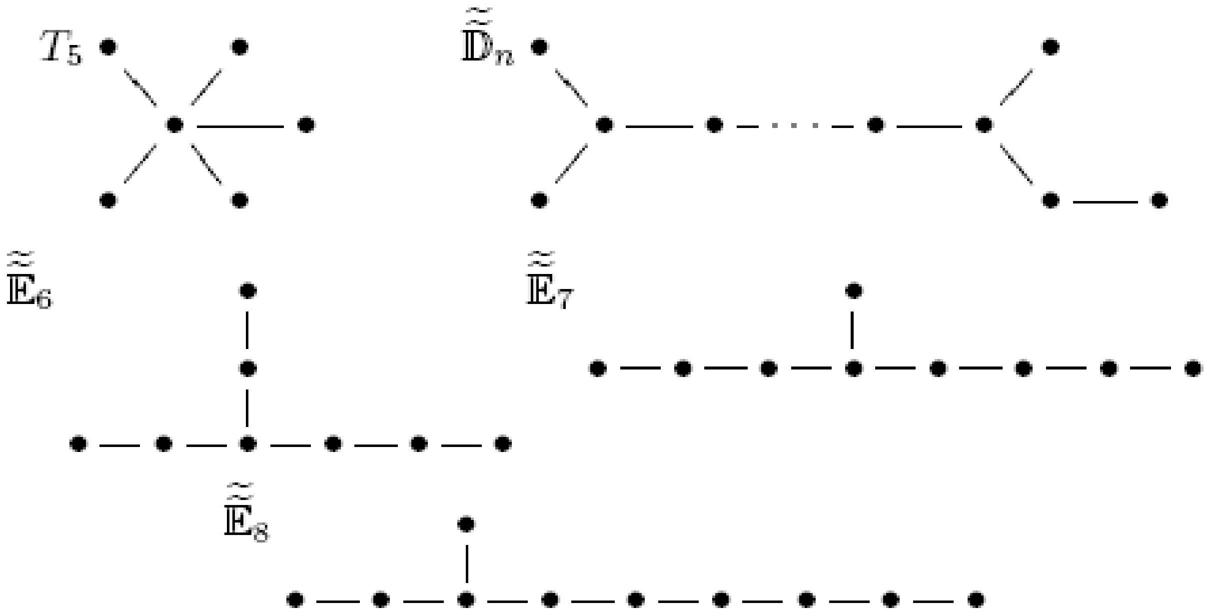
If A is representation-finite, $\operatorname{mod}_A(v) = \bigcup_{i=1}^{m} G(v)X_i$ where $X_1, \ldots, X_m$ are representatives of the isoclasses of A-modules of $\mathbf{dim} = v$. Hence $\dim \operatorname{mod}_A(v) = \dim G(v)X_j = \dim G(v) - \dim \operatorname{Stab}_{G(v)} X_j \leq \dim G(v) - 1$ and $q_A(v) \geq 1$. $\square$

Degenerations of algebras.
Lecture 4. Strongly simply connected algebras .

José-Antonio de la Peña

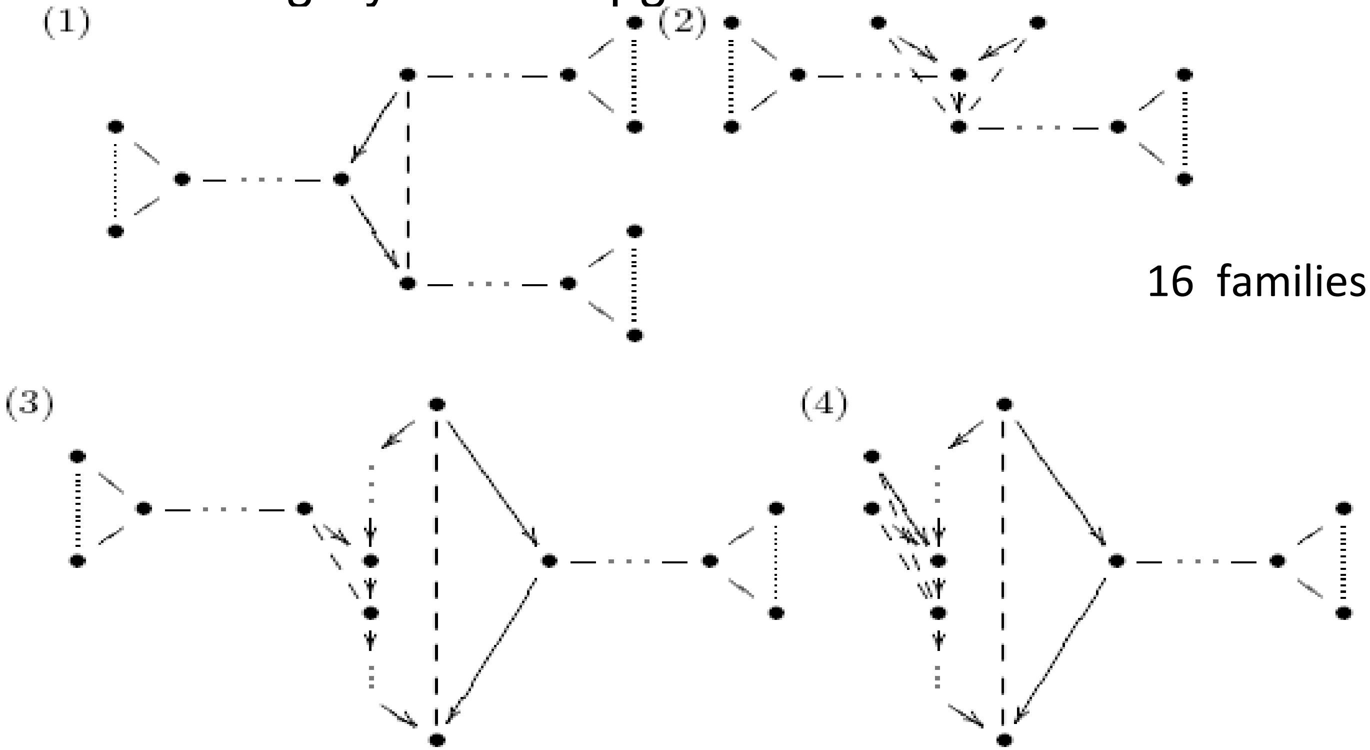
A is strongly simply connected if for every convex B in A we have that The first Hochschild cohomology H^1 (A)=0.

The Tits form of a strongly simply connected algebra A is weakly nonnegative if and only if A does not contain a convex subcategory (called a hypercritical algebra) which is a preprojective tilt of a wild hereditary algebra of one of the following tree types



Proposition . Let A be a strongly simply connected algebra. Tfae:

(i) A is of polynomial growth.
(ii) A does not contain a convex subcategory which is pg-critical or hypercritical.
(iii) The Tits form q_A of A is weakly nonnegative and A does not contain a convex subcategory which is pg-critical.



16 families.

Theorem. Let A be a strongly simply connected algebra. Then A is tame if and only if the Tits form q_A of A is weakly non-negative.

Corollary 1. Let A be a strongly simply connected algebra. Then A is tame if and only if A does not contain a convex hypercritical subcategory.

Since the quivers of hypercritical algebras have at most 10 vertices, we obtain also the following consequence.

Corollary 2. Let A be a strongly simply connected algebra. Then A is tame if and only if every convex subcategory of A with at most 10 objects is tame.

Degenerations of algebras.
Lecture 4. Biserial algebras.

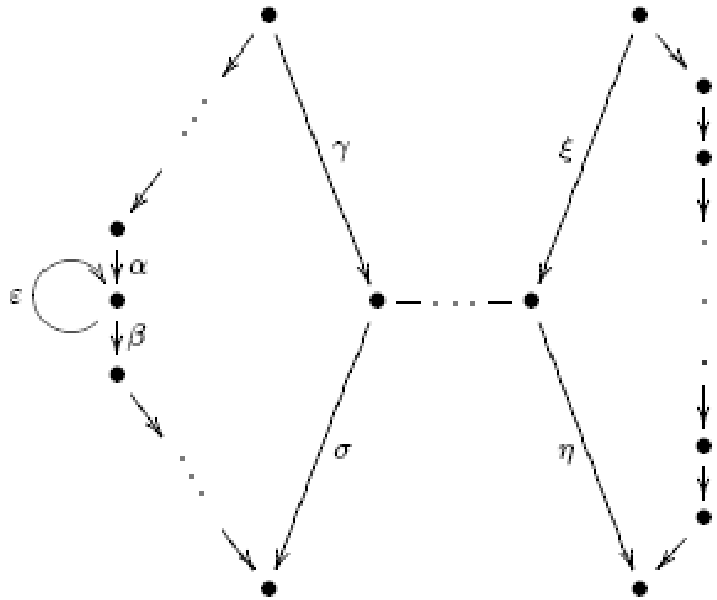
José-Antonio de la Peña

An algebra A is said to be *special biserial* if A is isomorphic to a bound quiver algebra KQ/I, where the bound quiver satisfies the conditions:

(a) each vertex of Q is a source and sink of at most two arrows,

(b) for any arrow α of Q there are at most one arrow β and at most one arrow γ with $\alpha\beta \notin I$ and $\gamma\alpha \notin I$.

Proposition. Every special biserial algebra is tame.



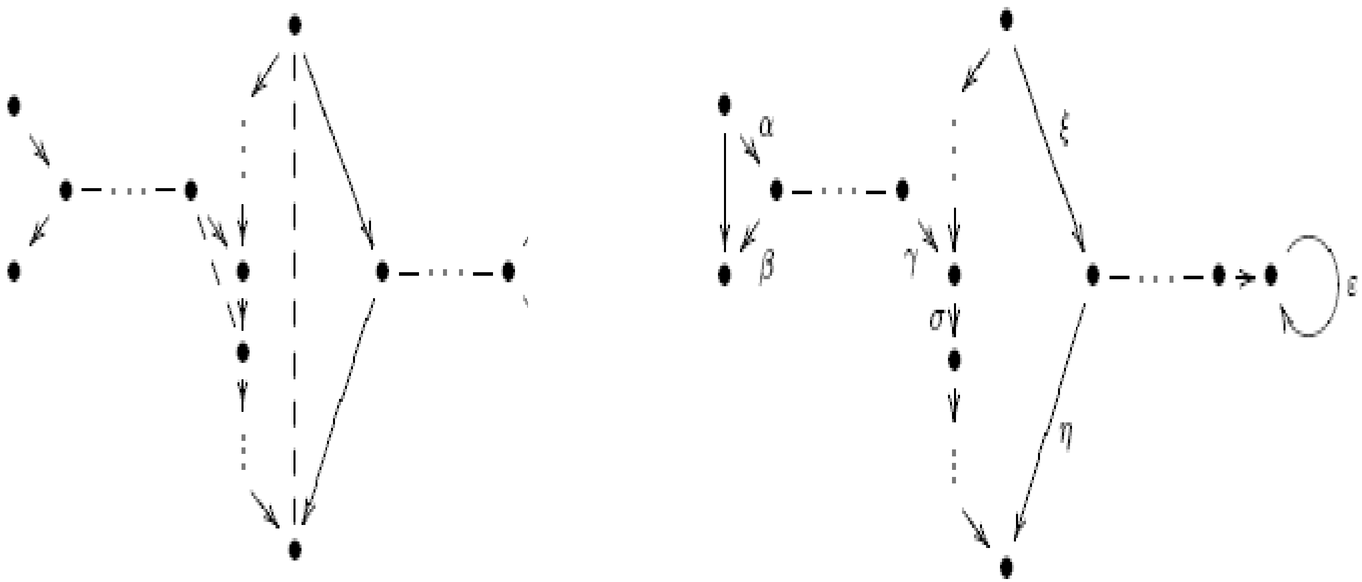
relations $\varepsilon^2 = 0$, $\alpha\beta = 0$, $\gamma\sigma = 0$ and $\xi\eta = 0$.

Degenerations of algebras.
Lecture 4. Degenerations of pg-critical algebras.

José-Antonio de la Peña

An essential role in the proof of our main result will be played by the following proposition.

Proposition. Every pg-critical algebra degenerates to a special biserial algebra



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Lemma *Let $A = KQ/I$ be a bound quiver algebra whose quiver Q contains a convex subquiver Q' of the form*

$$x_1 \xrightarrow{\alpha_1} y \xleftarrow{\alpha_2} x_2$$

where x_1, x_2 are sources of Q and α_1 and α_2 are unique arrows starting at x_1 and x_2, respectively. Assume that the ideal I admits a set R of generators of the form

$$R = \{\alpha_1 b_1, \ldots, \alpha_1 b_n, \alpha_2 b_1, \ldots, \alpha_2 b_n, c_1 \ldots, c_m\}$$

with certain elements $b_1, \ldots, b_n \in e_y(KQ)$ and $c_1, \ldots, c_m \in e_z(KQ)$ for $x_1 \neq z \neq x_2$.

Let $\bar{A} = K\bar{Q}/\bar{I}$ be the bound quiver algebra obtained from A as follows: the quiver $\bar{Q}$ is obtained from Q by replacing the subquiver Q' by the subquiver $\bar{Q}'$ of the form

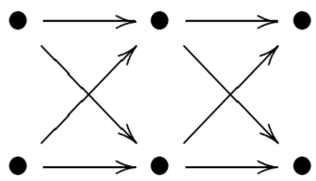
$$\varepsilon \circlearrowright x \xrightarrow{\alpha} y$$

and $\bar{I}$ is the ideal of $K\bar{Q}$ generated by the set

$$\bar{R} = \{\varepsilon^2, \alpha b_1, \ldots, \alpha b_n, c_1 \ldots, c_m\}.$$

Then A degenerates to $\bar{A}$.

Let A be the algebra given by the following quiver with all commutative relations:



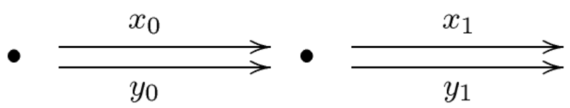
Observe that the group G with two elements acts on A without fixing vertices. Consider the Galois covering functor

$$F\colon A \to \bar{A} = A/G.$$

We show that $\bar{A}$ is a tame or a wild algebra depending whether or not char $k \neq 2$.

Assume first that char $k = 2$ and consider the following change of variables: $x_0 = \alpha_0 + \beta_0$, $y_0 = \beta_0$, $x_1 = \alpha_1 + \beta_1$, $y_1 = \beta_1$. Then $\bar{A}$ is isomorphic to the algebra A' given by the quiver with relations.

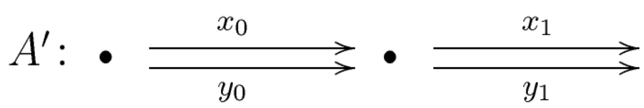
A':



$$\left.\begin{array}{rcl} x_1 x_0 &=& 0 \\ y_1 x_0 &=& x_1 y_0 \end{array}\right\}$$

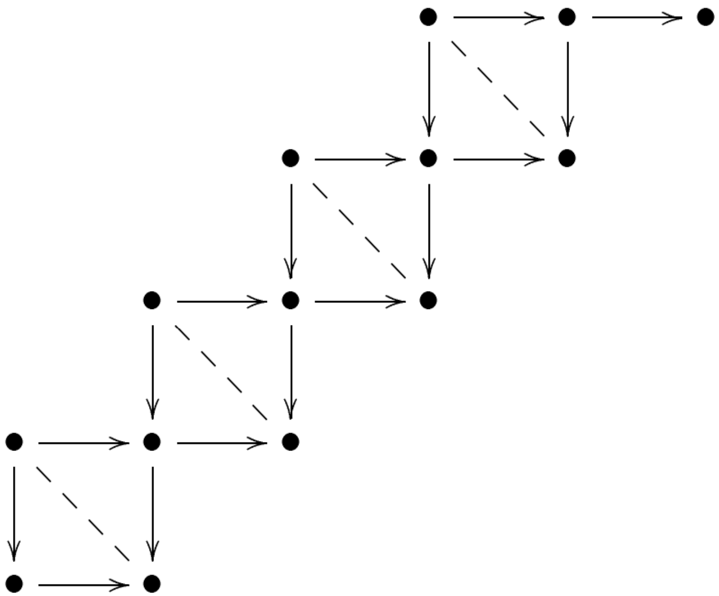
Degenerations of algebras.
Lecture 4. Deciding wildness.

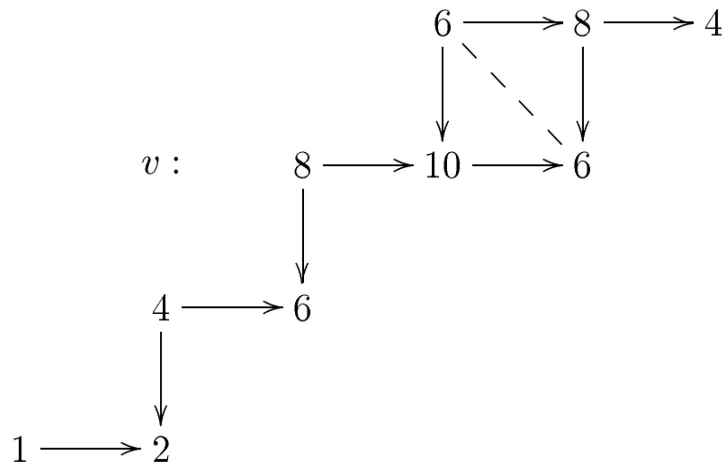
José-Antonio de la Peña



$$\left.\begin{aligned} x_1 x_0 &= 0 \\ y_1 x_0 &= x_1 y_0 \end{aligned}\right\}$$

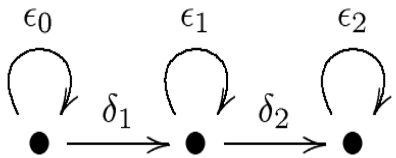
There is a covering $\tilde{A}' \to A'$, where $\tilde{A}'$ satisfies the commutativity relations marked by dashed lines and the vertical product of arrows equal zero, defined by the action of $\mathbb{Z}$ admiting a full convex subcategory B as follows:





Since $q_B(v) = -1$, then B (and thus $\tilde{A}'$) is wild. Then A is also wild.

Assume char $k \neq 2$, by Galois covering theory, if A is tame then so is $\bar{A}$.
There is an equivalence $F : \mathrm{mod}_A \to \mathrm{mod}_C$ where C is given by:



satisfying $\delta_1\delta_2 = 0$ and $\epsilon_i^2 = \epsilon_i$, for $i = 0, 1, 2$.

C is isomorphic to C_λ (for $\lambda \neq 0$) with $\epsilon_i^2 = \lambda\epsilon_i$.

Hence C deforms to C_0 which is a special biserial algebra and hence tame.

Degenerations of algebras.
Lecture 4. Deciding tameness.

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