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Homological and Geometrical Methods in Representation Theory

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Algebras of small homological dimension

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ALGEBRAS OF SMALL HOMOLOGICAL DIMENSION

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Contents

1. Module Category

K a field

algebra = finite dimensional K-algebra
(associative, with identity)

A algebra

$\operatorname{mod} A$ category of finite dimensional (over K) right A-modules

$\operatorname{ind} A$ full subcategory of $\operatorname{mod} A$ formed by all indecomposable modules

A^{op} opposite algebra of A

$\operatorname{mod} A^{\text{op}}$ category of finite dimensional (over K) left A-modules

$$\operatorname{mod} A \underset{D}{\overset{D}{\rightleftarrows}} \operatorname{mod} A^{\text{op}}$$

$D = \operatorname{Hom}_K(-, K)$ **standard duality** of $\operatorname{mod} A$

1_A identity of A

$$1_A = \sum_{i=1}^{n_A} \sum_{j=1}^{m_A(i)} e_{ij}$$

e_{ij} pairwise orthogonal primitive idempotents of A such that

$$\begin{aligned} e_{ij}A \cong e_{il}A \text{ for } & j, l \in \{1, \ldots, m_A(i)\}, \\ & i \in \{1, \ldots, n_A\}. \\ e_{ij}A \ncong e_{kl}A \text{ for } & i, k \in \{1, \ldots, n_A\} \text{ with } i \neq k \\ & j \in \{1, \ldots, m_A(i)\}, \\ & l \in \{1, \ldots, m_A(k)\}. \end{aligned}$$

canonical decomposition of 1_A

$e_i = e_{i1}$, $i \in \{1, \ldots, n_A\}$, **basic primitive idempotents** of A

$e_A = \sum_{i=1}^{n_A} e_i$ **basic idempotent** of A

A **basic algebra** if $e_A = 1_A$
(equivalently, $m_A(i) = 1$ for $i \in \{1, \ldots, n_A\}$)

In general, $A^b = e_A A e_A$ **basic algebra of** A

$$\operatorname{mod} A \underset{-\otimes_{A^b} e_A A}{\overset{(-)e_A}{\rightleftarrows}} \operatorname{mod} A^b$$

equivalence of categories (A and A^b are **Morita equivalent**)

- $P_i = e_i A$, $i \in \{1, \ldots, n_A\}$, complete set of pairwise nonisomorphic indecomposable projective right A-modules
- $I_i = D(Ae_i)$, $i \in \{1, \ldots, n_A\}$, complete set of pairwise nonisomorphic indecomposable injective right A-modules
- $S_i = \operatorname{top}(P_i) = e_i A / e_i \operatorname{rad} A$, $i \in \{1, \ldots, n_A\}$, complete set of pairwise nonisomorphic simple right A-modules
- $S_i \cong \operatorname{soc}(I_i)$, $i \in \{1, \ldots, n_A\}$.

$\operatorname{rad} A$ **Jacobson radical** of A

$\operatorname{rad} A =$ intersection of all maximal right ideals of A
$=$ intersection of all maximal left ideals of A

$\operatorname{rad} A$ two-sided ideal of A

$(\operatorname{rad} A)^m = 0$ for some $m \geq 1$

$\dim_K(e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j) = \dim_K \operatorname{Ext}^1_A(S_i, S_j)$
for $i, j \in \{1, \ldots, n_A\}$

Q_A **valued quiver** of A

$1, 2, \ldots, n = n_A$ vertices of Q_A

there is an arrow $i \longrightarrow j$ in Q_A if $\dim_K \operatorname{Ext}^1_A(S_i, S_j) \neq 0$, and has the valuation

$$(\dim_{\operatorname{End}_A(S_j)} \operatorname{Ext}^1_A(S_i, S_j), \dim_{\operatorname{End}_A(S_i)} \operatorname{Ext}^1_A(S_i, S_j))$$

$\operatorname{End}_A(S_1), \operatorname{End}_A(S_2), \ldots, \operatorname{End}_A(S_n)$ are **division** K-algebras

$G_A = \bar{Q}_A$ (underlying graph of Q_A)
valued graph of A

$K_0(A) = K_0(\text{mod}\, A)$ **Grothendieck group** of A

$K_0(A) = \mathcal{F}_A / \mathcal{F}'_A$

$\mathcal{F}_A$ free abelian group with $\mathbb{Z}$-basis given by the isoclasses $\{M\}$ of modules M in mod A

$\mathcal{F}'_A$ subgroup of $\mathcal{F}_A$ generated by

$$\{M\} - \{L\} - \{N\}$$

for all exact sequences

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in mod A

$[M]$ the class of a module M from mod A in $K_0(A)$

$K_0(A)$ free abelian group generated by $[S_1], [S_2], \ldots, [S_n]$
$S_1, S_2, \ldots, S_n$ complete set of pairwise nonisomorphic simple right A-modules

M module in mod A

$[M] = \sum\limits_{i=1}^{n} c_i(M)[S_i]$

$c_i(M)$ **multiplicity** of S_i as composition factor of M (**Jordan-Hölder theorem**)

Jacobson radical of $\operatorname{mod} A$

A algebra over K

X, Y modules in $\operatorname{mod} A$

$$\operatorname{rad}_A(X,Y) = \left\{ f \in \operatorname{Hom}_A(X,Y) \;\middle|\; \begin{array}{c} \mathrm{id}_X - gf \text{ invertible} \\ \text{in } \operatorname{End}_A(X) \text{ for any} \\ g \in \operatorname{Hom}_A(Y,X) \end{array} \right\}$$
$$= \left\{ f \in \operatorname{Hom}_A(X,Y) \;\middle|\; \begin{array}{c} \mathrm{id}_Y - fg \text{ invertible} \\ \text{in } \operatorname{End}_A(Y) \text{ for any} \\ g \in \operatorname{Hom}_A(X,Y) \end{array} \right\}$$

Jacobson radical of $\operatorname{Hom}_A(X,Y)$

$\operatorname{rad}_A(X,Y)$ subspace of $\operatorname{Hom}_A(X,Y)$ formed by all nonisomorphisms

$\operatorname{rad}_A(X,X) = \operatorname{rad}\operatorname{End}_A(X)$ Jacobson radical of $\operatorname{End}_A(X)$

Lemma (**Bautista**). *Let X and Y be indecomposable modules in* $\operatorname{mod} A$ *and $f \in \operatorname{Hom}_A(X,Y)$. Then $f \in \operatorname{rad}_A(X,Y) \setminus \operatorname{rad}_A^2(X,Y)$ if and only if f is an irreducible homomorphism (f is neither section nor retraction and, for any factorization in* $\operatorname{mod} A$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

g is a section or h is a retraction)

$\operatorname{rad} A$ ideal of the category $\operatorname{mod} A$

$\operatorname{rad}^m A$ m-th power of $\operatorname{rad} A$, $m \geq 1$

$\operatorname{rad}_A^\infty = \bigcap\limits_{m=1}^{\infty} \operatorname{rad}_A^m$

infinite (Jacobson) radical of $\operatorname{mod} A$

A is of **finite representation type** if $\operatorname{ind} A$ admits only a finite number of modules (up to isomorphism)

Theorem (Auslander). *An algebra A is of finite representation type if and only if* $\operatorname{rad}_A^\infty = 0$. *(⇒ **Harada-Sai lemma**)*

Theorem (Coelho-Marcos-Merklen-Skowroński). *Let A be an algebra of infinite representation type. Then* $\left(\operatorname{rad}_A^\infty\right)^2 \neq 0$.

2. Auslander-Reiten quiver

A finite dimensional K-algebra over a field K

Z module in $\operatorname{ind} A$

$\operatorname{End}_A(Z)$ local K-algebra

$$\begin{aligned} F_Z &= \operatorname{End}_A(Z)/\operatorname{rad}\operatorname{End}_A(Z) \\ &= \operatorname{End}_A(Z)/\operatorname{rad}_A(Z,Z) \\ &\quad \text{division } K\text{-algebra} \end{aligned}$$

X, Y modules in $\operatorname{ind} A$

$$\operatorname{irr}_A(X,Y) = \operatorname{rad}_A(X,Y)/\operatorname{rad}_A^2(X,Y)$$

the **space of irreducible homomorphisms** from X to Y

$\operatorname{irr}_A(X,Y)$ is an F_Y-F_X-bimodule

$$(h+\operatorname{rad}_A(Y,Y))(f+\operatorname{rad}_A^2(X,Y)) = hf+\operatorname{rad}_A^2(X,Y)$$

$$(f+\operatorname{rad}_A^2(X,Y))(g+\operatorname{rad}_A(X,X)) = fg+\operatorname{rad}_A^2(X,Y)$$

for $f \in \operatorname{rad}_A(X,Y)$, $g \in \operatorname{End}_A(X)$, $h \in \operatorname{End}_A(Y)$

$$d_{XY} = \dim_{F_Y} \operatorname{irr}_A(X,Y)$$

$$d'_{XY} = \dim_{F_X} \operatorname{irr}_A(X,Y)$$

Γ_A **Auslander Reiten quiver** of A

valued translation quiver defined as follows:

- The vertices of Γ_A are the isoclasses $\{X\}$ of modules X in $\operatorname{ind} A$

- For two vertices $\{X\}$ and $\{Y\}$, there is an arrow $\{X\} \longrightarrow \{Y\}$ provided $\operatorname{irr}_A(X,Y) \neq 0$. Then we have in Γ_A the valued arrow

$$\{X\} \xrightarrow{(d_{XY}, d'_{XY})} \{Y\}$$

- τ_A translation of Γ_A defined on each nonprojective vertex $\{X\}$ of Γ_A by

$$\tau_A\{X\} = \{\tau_A X\} = \{D \operatorname{Tr} X\}$$

- τ_A^{-1} translation of Γ_A defined on each noninjective vertex $\{X\}$ of Γ_A by

$$\tau_A^{-1}\{X\} = \{\tau_A^{-1} X\} = \{\operatorname{Tr} DX\}$$

Tr the **transpose operator**

D the **standard duality**

We identify a vertex $\{X\}$ of Γ_A with the indecomposable module X and write

$X \xrightarrow{(d_{XY},d'_{XY})} Y$ instead of $\{X\} \xrightarrow{(d_{XY},d'_{XY})} \{Y\}$

and $X \longrightarrow Y$ instead of $X \xrightarrow{(1,1)} Y$

X, Y modules in $\operatorname{ind} A$ (vertices of Γ_A)

$d_{XY} =$ multiplicity of Y in the codomain of a minimal left almost split homomorphism in $\operatorname{mod} A$ with the domain X

$$X \xrightarrow{f} M = Y^{d_{XY}} \oplus M'$$

M' without direct summand isomorphic to Y

$d'_{XY} =$ multiplicity of X in the domain of a minimal right almost split homomorphism in $\operatorname{mod} A$ with the codomain Y

$$N' \oplus X^{d'_{XY}} = N \xrightarrow{g} Y$$

N' without direct summand isomorphic to X

- X noninjective then there is in $\operatorname{mod} A$ an almost split sequence (Auslander-Reiten sequence)

$$0 \longrightarrow X \xrightarrow{f} M \xrightarrow{f'} \tau_A^{-1} X \longrightarrow 0$$

f a minimal left almost split homomorphism, f' a minimal right almost split homomorphism

- Y nonprojective, then there is in $\operatorname{mod} A$ an almost split sequence (Auslander-Reiten sequence)

$$0 \longrightarrow \tau_A Y \xrightarrow{g'} N \xrightarrow{g} Y \longrightarrow 0$$

g a minimal right almost split homomorphism, g' a minimal left almost split homomorphism

- P indecomposable projective, then the embedding

$$\operatorname{rad} P \hookrightarrow P$$

is a minimal right almost split homomorphism in $\operatorname{mod} A$

- I indecomposable injective, then the canonical epimorphism

$$I \longrightarrow I/\operatorname{soc} I$$

is a minimal left almost split homomorphism in $\operatorname{mod} A$

Assume $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ is an arrow in Γ_A

- X noninjective, then Γ_A admits a valued arrow
$$Y \xrightarrow{\left(d_{Y\,\tau_A^{-1}X}, d'_{Y\,\tau_A^{-1}X}\right)} \tau_A^{-1}X$$
with $d_{Y\,\tau_A^{-1}X} = d'_{XY}$ and $d'_{Y\,\tau_A^{-1}X} = d_{XY}$, so we have the arrows
$$X \xrightarrow{(d_{XY}, d'_{XY})} Y \xrightarrow{(d'_{XY}, d_{XY})} \tau_A^{-1}X$$

- Y nonprojective, then Γ_A admits a valued arrow
$$\tau_A Y \xrightarrow{\left(d_{\tau_A Y\,X}, d'_{\tau_A Y\,X}\right)} X$$
with $d_{\tau_A Y\,X} = d'_{XY}$ and $d'_{\tau_A Y\,X} = d_{XY}$, so we have the arrows
$$\tau_A Y \xrightarrow{(d'_{XY}, d_{XY})} X \xrightarrow{(d_{XY}, d'_{XY})} Y$$

- X simple projective, then Y is projective

- Y simple injective, then X is injective

- For each nonprojective indecomposable module Y in $\operatorname{mod} A$, the quiver Γ_A admits a valued mesh

$$\begin{tikzcd}[column sep=huge]
 & \{V_1\} \arrow[ddr, "{(d_{V_1Y},d'_{V_1Y})}"] & \\
 & \{V_2\} \arrow[dr, "{(d_{V_2Y},d'_{V_2Y})}"'] & \\
\tau_A\{Y\} = \{\tau_A Y\} \arrow[uur, "{(d'_{V_1Y},d_{V_1Y})}"] \arrow[ur, "{(d'_{V_2Y},d_{V_2Y})}"'] \arrow[dr, "{(d'_{V_rY},d_{V_rY})}"'] & \vdots & \{Y\} \\
 & \{V_r\} \arrow[ur, "{(d_{V_rY},d'_{V_rY})}"'] &
\end{tikzcd}$$

  such that there is in $\operatorname{mod} A$ an almost split sequence

$$0 \longrightarrow \tau_A Y \longrightarrow \bigoplus_{i=1}^{r} V_i^{d'_{V_iY}} \longrightarrow Y \longrightarrow 0.$$

- For each noninjective indecomposable module X in $\operatorname{mod} A$, the quiver Γ_A admits a valued mesh

$$\begin{tikzcd}[column sep=huge]
 & \{U_1\} \arrow[ddr, "{(d'_{XU_1},d_{XU_1})}"] & \\
 & \{U_2\} \arrow[dr, "{(d'_{XU_2},d_{XU_2})}"'] & \\
\{X\} \arrow[uur, "{(d_{XU_1},d'_{XU_1})}"] \arrow[ur, "{(d_{XU_2},d'_{XU_2})}"'] \arrow[dr, "{(d_{XU_s},d'_{XU_s})}"'] & \vdots & \{\tau_A^{-1}X\} = \tau_A^{-1}\{X\} \\
 & \{U_s\} \arrow[ur, "{(d'_{XU_s},d_{XU_s})}"'] &
\end{tikzcd}$$

  such that there is in $\operatorname{mod} A$ an almost split sequence

$$0 \longrightarrow X \longrightarrow \bigoplus_{j=1}^{s} U_j^{d_{XU_j}} \longrightarrow \tau_A^{-1} X \longrightarrow 0.$$

- For each nonsimple projective indecomposable module P in $\operatorname{mod} A$, the quiver Γ_A admits a valued subquiver

$$\begin{tikzcd}
\{R_1\} \arrow[drr, "(d_{R_1P}, d'_{R_1P})"] & & \\
\{R_2\} \arrow[rr, "(d_{R_2P}, d'_{R_2P})"'] & & \{P\} \\
\vdots & & \\
\{R_t\} \arrow[urr, "(d_{R_tP}, d'_{R_tP})"'] & &
\end{tikzcd}$$

  such that

$$\operatorname{rad} P \cong \bigoplus_{i=1}^{t} R_i^{d'_{R_iP}}.$$

- For each nonsimple injective indecomposable module I in $\operatorname{mod} A$, the quiver Γ_A admits a valued subquiver

$$\begin{tikzcd}
& & \{T_1\} \\
& & \{T_2\} \\
\{I\} \arrow[uurr, "(d_{IT_1}, d'_{IT_1})"] \arrow[urr, "(d_{IT_2}, d'_{IT_2})"'] \arrow[drr, "(d_{ITm}, d'_{ITm})"'] & & \vdots \\
& & \{T_m\}
\end{tikzcd}$$

  such that

$$I/\operatorname{soc} I \cong \bigoplus_{j=1}^{m} T_j^{d_{IT_j}}.$$

- Assume A is an algebra of finite representation type and $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ is an arrow of Γ_A. Then

$$d_{XY} = 1 \text{ or } d'_{XY} = 1.$$

- Assume A is an algebra over an algebraically closed field K and $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ is an arrow of Γ_A. Then

$$d_{XY} = d'_{XY}.$$

  In particular, $d_{XY} = d'_{XY} = 1$ if A is of finite representation type.

In representation theory of finite dimensional algebras over an algebraically closed field K, instead of a valued arrow

$$X \xrightarrow{(m,m)} Y$$

of an Auslander-Reiten quiver Γ_A, usually one writes a multiple arrow

$$X \; \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \vdots \\ \longrightarrow \end{array} \; Y$$

consisting of m arrows from X to Y.

Component of Γ_A = connected component of the quiver Γ_A

Shapes of components of Γ_A give important information on A and $\operatorname{mod} A$

Δ locally finite valued quiver without loops and multiple arrows

Δ_0 set of vertices of Δ

Δ_1 set of arrows of Δ

$d, d' : \Delta_1 \to \Delta_0$ the valuation maps

$$x \xrightarrow{(d_{xy}, d'_{xy})} y$$

$\mathbb{Z}\Delta$ valued translation quiver

$(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0 = \left\{(i,x) \,\middle|\, i \in \mathbb{Z}, x \in \Delta_0\right\}$ set of vertices of $\mathbb{Z}\Delta$.

$(\mathbb{Z}\Delta)_1$ set of arrows of $\mathbb{Z}\Delta$ consists of the valued arrows

$$(i,x) \xrightarrow{(d_{xy}, d'_{xy})} (i,y), \quad (i+1,y) \xrightarrow{(d'_{xy}, d_{xy})} (i,x),$$

$i \in \mathbb{Z}$, for all arrows $x \xrightarrow{(d_{xy}, d'_{xy})} y$ in Δ_1.

The translation $\tau : \mathbb{Z}\Delta_0 \to \mathbb{Z}\Delta_0$ is defined by

$\tau(i,x) = (i+1,x)$ for all $i \in \mathbb{Z}$, $x \in \Delta_0$.

$\mathbb{Z}\Delta$ **stable valued translation quiver**

For a subset I of $\mathbb{Z}$, $I\Delta$ is the full translation subquiver of $\mathbb{Z}\Delta$ given by the set of vertices $(I\Delta)_0 = I \times \Delta_0$.

In particular, we have the valued translation subquivers $\mathbb{N}\Delta$ and $(-\mathbb{N})\Delta$ of $\mathbb{Z}\Delta$.

Examples

$$\Delta : \qquad 1 \xrightarrow{(1,2)} 2 \xleftarrow{(4,3)} 3$$

$\mathbb{Z}\Delta$ of the form

$$\begin{tikzcd}
 & (1,1) \arrow[dr, "{(1,2)}"] & & (0,1) \arrow[dr, "{(1,2)}"] & & (-1,1) \arrow[dr, "{(1,2)}"] & & (-2,1) \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & (1,2) \arrow[ur, "{(2,1)}"] \arrow[dr, "{(4,3)}"] & & (0,2) \arrow[ur, "{(2,1)}"] \arrow[dr, "{(4,3)}"] & & (-1,2) \arrow[ur, "{(2,1)}"] \arrow[dr, "{(4,3)}"] & & \cdots \\
 & (1,3) \arrow[ur, "{(3,4)}"] & & (0,3) \arrow[ur, "{(3,4)}"] & & (-1,3) \arrow[ur, "{(3,4)}"] & & (-2,3) \arrow[ur] &
\end{tikzcd}$$

$\mathbb{N}\Delta$ of the form

$$\begin{tikzcd}
 & (3,1) \arrow[dr, "{(1,2)}"] & & (2,1) \arrow[dr, "{(1,2)}"] & & (1,1) \arrow[dr, "{(1,2)}"] & & (0,1) \arrow[dr, "{(1,2)}"] & \\
\cdots \arrow[ur] \arrow[dr] & & (3,2) \arrow[ur, "{(2,1)}"] \arrow[dr, "{(4,3)}"] & & (2,2) \arrow[ur, "{(2,1)}"] \arrow[dr, "{(4,3)}"] & & (1,2) \arrow[ur, "{(2,1)}"] \arrow[dr, "{(4,3)}"] & & (0,2) \\
 & (3,3) \arrow[ur, "{(3,4)}"] & & (2,3) \arrow[ur, "{(3,4)}"] & & (1,3) \arrow[ur, "{(3,4)}"] & & (0,3) \arrow[ur, "{(3,4)}"] &
\end{tikzcd}$$

$(-\mathbb{N})\Delta$ of the form

$$\begin{tikzcd}
(0,1) \arrow[dr, "{(1,2)}"] & & (-1,1) \arrow[dr, "{(1,2)}"] & & (-2,1) \arrow[dr, "{(1,2)}"] & & (-3,1) \arrow[dr] & \\
 & (0,2) \arrow[ur, "{(2,1)}"] \arrow[dr, "{(4,3)}"] & & (-1,2) \arrow[ur, "{(2,1)}"] \arrow[dr, "{(4,3)}"] & & (-2,2) \arrow[ur, "{(2,1)}"] \arrow[dr, "{(4,3)}"] & & \cdots \\
(0,3) \arrow[ur, "{(3,4)}"] & & (-1,3) \arrow[ur, "{(3,4)}"] & & (-2,3) \arrow[ur, "{(3,4)}"] & & (-3,3) \arrow[ur] &
\end{tikzcd}$$

$$\mathbb{A}_\infty : \qquad 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

$\mathbb{Z}\mathbb{A}_\infty$ is the translation quiver

$$\begin{tikzcd}[column sep=small]
 & (i+1,0) \arrow[dr] & & (i,0) \arrow[dr] & & (i-1,0) \arrow[dr] & & (i-2,0) \arrow[dr] & \\
\cdot^{\cdot^{\cdot}} \arrow[ur] & & (i+1,1) \arrow[ur] \arrow[dr] & & (i,1) \arrow[ur] \arrow[dr] & & (i-1,1) \arrow[ur] \arrow[dr] & & \ddots \\
 & \cdot^{\cdot^{\cdot}} \arrow[ur] & & (i+1,2) \arrow[ur] \arrow[dr] & & (i,2) \arrow[ur] \arrow[dr] & & \ddots & \\
 & & \cdot^{\cdot^{\cdot}} \arrow[ur] & & \vdots \arrow[ur] & & \ddots & &
\end{tikzcd}$$

$\tau(i,j) = (i+1, j)$ for all $i \in \mathbb{Z}$, $j \in \mathbb{N}$.

For $r \geq 1$, we may consider the translation quiver

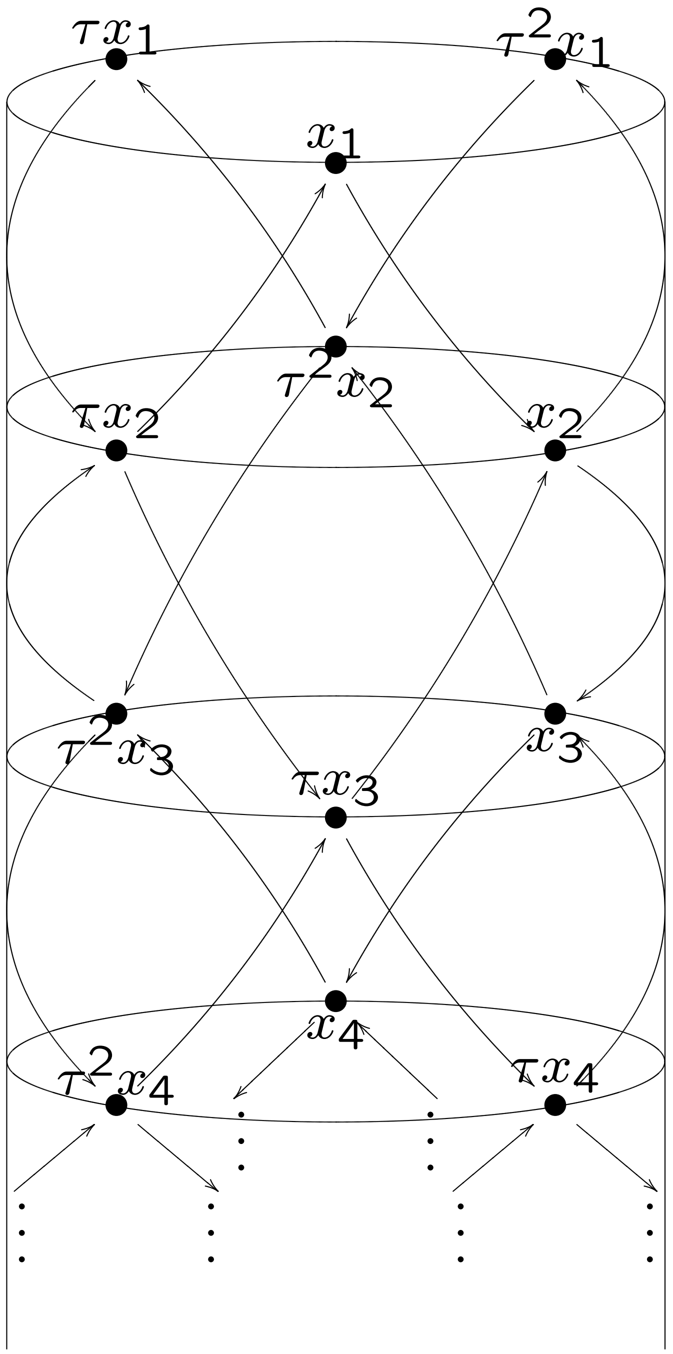
$$\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$$

obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each vertex x with $\tau^r x$ and each arrow $x \to y$ with $\tau^r x \to \tau^r y$.

$\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ **stable tube of rank** r.

All vertices of $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ are τ-**periodic** of period r

A stable tube of rank 3 is of the form



A algebra

$\mathscr{C}$ component of Γ_A is **regular** if $\mathscr{C}$ contains neither a projective module nor an injective module (equivalently, τ_A and τ_A^{-1} are defined on all vertices of $\mathscr{C}$)

Theorem (**Liu**, **Zhang**). *Let A be an algebra and $\mathscr{C}$ be a regular component of Γ_A. The following equivalences hold.*

(1) $\mathscr{C}$ contains an oriented cycle if and only if $\mathscr{C}$ is a stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for some $r \geq 1$.

(2) $\mathscr{C}$ is acyclic if and only if $\mathscr{C}$ is of the form $\mathbb{Z}\Delta$ for a connected, locally finite, acyclic, valued quiver Δ.

A component $\mathscr{C}$ of Γ_A is **postprojective** (**preprojective**) if $\mathscr{C}$ is acyclic and each module in $\mathscr{C}$ is of the form $\tau_A^{-m}P$ for a projective module P in $\mathscr{C}$ and some $m \geq 0$.

A component $\mathscr{C}$ of Γ_A is **preinjective** if $\mathscr{C}$ is acyclic and each module in $\mathscr{C}$ is of the form $\tau_A^{m}I$ for an injective module I in $\mathscr{C}$ and some $m \geq 0$.

A finite dimensional K-algebra over a field K

$\mathscr{C}, \mathscr{D}$ components of Γ_A

We write $\mathrm{Hom}_A(\mathscr{C}, \mathscr{D}) = 0$ if $\mathrm{Hom}_A(X, Y) = 0$ for all modules X in $\mathscr{C}$ and Y in $\mathscr{D}$

$\mathscr{C}$ and $\mathscr{D}$ are **orthogonal** if $\mathrm{Hom}_A(\mathscr{C}, \mathscr{D}) = 0$ and $\mathrm{Hom}_A(\mathscr{D}, \mathscr{C}) = 0$

In general, if $\mathscr{C} \neq \mathscr{D}$, then $\mathrm{Hom}_A(X, Y) = \mathrm{rad}_A^\infty(X, Y)$ for all modules X in $\mathscr{C}$ and Y in $\mathscr{D}$.

A component $\mathscr{C}$ of Γ_A is called **generalized standard** if $\mathrm{rad}_A^\infty(X, Y) = 0$ for all modules X and Y in $\mathscr{C}$.

- $\mathscr{C}$ postprojective or preinjective component of Γ_A, then $\mathscr{C}$ is generalized standard
- A of finite representation type, $\mathscr{C}$ component of Γ_A, then $\mathscr{C}$ is generalized standard
- $\mathscr{C}$ is generalized standard component of Γ_A, X and Y modules in $\mathscr{C}$, then every nonzero homomorphism $f \in \mathrm{rad}_A(X, Y)$ is a sum of compositions of irreducible homomorphisms between indecomposable modules from $\mathscr{C}$.

A component $\mathscr{C}$ of Γ_A is called **almost periodic** if all but finitely many τ_A-orbits in $\mathscr{C}$ are periodic.

Theorem (Skowroński). *Let A be an algebra and $\mathscr{C}$ be an almost periodic component of Γ_A. Then, for each natural number $d \geq 1$, $\mathscr{C}$ contains at most finitely many modules of dimension d.*

Theorem (Skowroński). *Let A be an algebra and $\mathscr{C}$ be a generalized standard component of Γ_A. Then $\mathscr{C}$ is almost periodic.*

Theorem (Skowroński). *Let A be an algebra. Then all but finitely many generalized standard components of Γ_A are stable tubes.*

$\mathscr{C}$ regular, generalized standard component of Γ_A, then

- $\mathscr{C}$ a stable tube, or
- $\mathscr{C} = \mathbb{Z}\Delta$, for a connected, **finite**, acyclic, valued quiver Δ.

A prominent role is played by the following

Lemma (**Skowroński**). *Let A be a finite dimensional K-algebra and n be the rank of $K_0(A)$. Assume*

$$M = M_1 \oplus \cdots \oplus M_r$$

is a module in $\operatorname{mod} A$ *such that*

- $M_1, \dots, M_r$ *are pairwise nonisomorphic and indecomposable*
- $\operatorname{Hom}_A(M, \tau_A M) = 0$.

Then $r \leq n$.

A finite dimensional K-algebra

$\mathscr{C}$ component of Γ_A

$\operatorname{ann}_A \mathscr{C} = \bigcap_{X \in \mathscr{C}} \operatorname{ann}_A X$ **annihilator** of $\mathscr{C}$ (two-sided ideal of A)

$\operatorname{ann}_A(X) = \{a \in A \mid Xa = 0\}$ annihilator of A-module X

$\mathscr{C}$ a **faithful component** of Γ_A if $\operatorname{ann}_A \mathscr{C} = 0$

In general, $\mathscr{C}$ is a faithful component of $\Gamma_{A/\operatorname{ann}_A \mathscr{C}}$

$\mathscr{C}$ faithful $\Rightarrow$ Γ_A is **sincere** (for any indecomposable projective A-module P there exists a module X in $\mathscr{C}$ with $\operatorname{Hom}_A(P, X) \neq 0$)

3. Homological dimensions

A finite dimensional K-algebra over a field K

M a module in $\operatorname{mod} A$

$\operatorname{pd}_A M$ **projective dimension** of M in $\operatorname{mod} A$

$\operatorname{pd}_A M = m \in \mathbb{N}$ if there exists a projective resolution

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

of M in $\operatorname{mod} A$ and M has no projective resolution in $\operatorname{mod} A$ of length $< m$.

$\operatorname{pd}_A M = \infty$ if M does not admit a finite projective resolution in $\operatorname{mod} A$

$\operatorname{id}_A M$ **injective dimension** of M in $\operatorname{mod} A$

$\operatorname{id}_A M = m \in \mathbb{N}$ if there exists an injective resolution

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_{m-1} \to I_m \to 0$$

of M in $\operatorname{mod} A$ and M has no injective resolution in $\operatorname{mod} A$ of length $< m$.

$\operatorname{id}_A M = \infty$ if M does not admit a finite injective resolution in $\operatorname{mod} A$

- $\operatorname{pd}_A M = m \in \mathbb{N}$ if and only if $\operatorname{Ext}_A^{m+1}(M, -) = 0$ and $\operatorname{Ext}_A^m(M, -) \neq 0$.
- $\operatorname{pd}_A M = \infty$ if and only if $\operatorname{Ext}_A^n(M, -) \neq 0$ for all $n \in \mathbb{N}$.
- $\operatorname{id}_A M = m \in \mathbb{N}$ if and only if $\operatorname{Ext}_A^{m+1}(-, M) = 0$ and $\operatorname{Ext}_A^m(-, M) \neq 0$.
- $\operatorname{id}_A M = \infty$ if and only if $\operatorname{Ext}_A^n(-, M) \neq 0$ for all $n \in \mathbb{N}$.

Moreover, we have the following useful facts

- $\operatorname{pd}_A M \leq 1$ if and only if $\operatorname{Hom}_A(D({}_AA), \tau_A M) = 0$.
- $\operatorname{id}_A M \leq 1$ if and only if $\operatorname{Hom}_A(\tau_A^{-1} M, A_A) = 0$.

For modules M and N in $\operatorname{mod} A$, we have

- If $\operatorname{pd}_A M \leq 1$, then
$$\operatorname{Ext}_A^1(M, N) \cong D \operatorname{Hom}_A(N, \tau_A M)$$
as K-vector spaces.
- If $\operatorname{id}_A M \leq 1$, then
$$\operatorname{Ext}_A^1(M, N) \cong D \operatorname{Hom}_A(\tau_A^{-1} N, M)$$
as K-vector spaces.

For a faithful module M in $\operatorname{mod} A$, we have

- If $\operatorname{Hom}_A(M, \tau_A M) = 0$, then $\operatorname{pd}_A M \leq 1$.
- If $\operatorname{Hom}_A(\tau_A^{-1} M, M) = 0$, then $\operatorname{id}_A M \leq 1$.

r. gl. dim $A = \max\{\mathrm{pd}_A M \mid M$ modules in $\mathrm{mod}\, A\}$
right global dimension of A

l. gl. dim $A = \max\{\mathrm{pd}_{A^{\mathrm{op}}} N \mid N$ modules in $\mathrm{mod}\, A^{\mathrm{op}}\}$
left global dimension of A

$$\mathrm{mod}\, A \underset{D}{\overset{D}{\rightleftarrows}} \mathrm{mod}\, A^{\mathrm{op}}$$

D standard duality of $\mathrm{mod}\, A$

$$\mathrm{pd}_A M = \mathrm{id}_{A^{\mathrm{op}}} D(M)$$

$$\mathrm{id}_A M = \mathrm{pd}_{A^{\mathrm{op}}} D(M)$$

for all modules M in $\mathrm{mod}\, A$

Hence,

l. gl. dim $A = \max\{\mathrm{id}_A M \mid M$ modules in $\mathrm{mod}\, A\}$

r. gl. dim $A = \max\left\{\mathrm{id}_{A^{\mathrm{op}}} N \mid N \text{ modules in } \mathrm{mod}\, A^{\mathrm{op}}\right\}$

Theorem (**Auslander**). *A finite dimensional K-algebra over a field K. Then*

r. gl. dim $= \{\mathrm{pd}_A S \mid S$ *simple right A-modules*$\}$.

Hence

- $\mathrm{r.gl.dim}\, A$ minimal $m \in \mathbb{N} \cup \{\infty\}$ such that $\mathrm{Ext}_A^{m+1}(M, N) = 0$ for all modules M, N in $\mathrm{mod}\, A$

- $$\mathrm{r.gl.dim}\, A = \max \left\{ \mathrm{id}_A M \;\middle|\; \begin{array}{c} M \text{ injective mo-} \\ \text{dules in } \mathrm{mod}\, A \end{array} \right\}$$
  $$= \mathrm{l.gl.dim}\, A$$

  $\mathrm{gl.dim}\, A = \mathrm{r.gl.dim}\, A = \mathrm{l.gl.dim}\, A$
  global dimension of A

- A algebra with acyclic valued quiver Q_A, then $\mathrm{gl.dim}\, A \leq \infty$
  ($\mathrm{gl.dim}\, A \leq$ length of longest path in Q_A)

Theorem (**Skowroński-Smalø-Zacharia**). *Let A be a finite dimensional K-algebra with* $\mathrm{gl.dim}\, A = \infty$. *Then there exists an indecomposable module M in* $\mathrm{mod}\, A$ *such that*

$$\mathrm{pd}_A M = \infty \text{ and } \mathrm{id}_A M = \infty.$$

A finite dimensional K-algebra

$\mathrm{gl.dim}\, A < \infty$

$\langle -, - \rangle_A : K_0(A) \times K_0(A) \longrightarrow \mathbb{Z}$

Euler nonsymmetric $\mathbb{Z}$-bilinear form

$$\langle [M], [N] \rangle_A = \sum_{i=0}^{\infty} (-1)^i \dim_K \mathrm{Ext}_A^i(M, N)$$

for modules M, N in $\mathrm{mod}\, A$

$q_A : K_0(A) \longrightarrow \mathbb{Z}$

Euler quadratic form

$$q_A([M]) = \sum_{i=0}^{\infty} (-1)^i \dim_K \mathrm{Ext}_A^i(M, M)$$

for a module M in $\mathrm{mod}\, A$

Semisimple algebras

A finite dimensional K-algebra over a field K

M a module in $\operatorname{mod} A$ is **semisimple** if M is a direct sum of simple right A-modules.

- M semisimple if and only if $M \operatorname{rad} A = 0$

Theorem. *A finite dimensional K-algebra. The following conditions are equivalent:*

(1) A_A is semisimple.

(2) Every module in $\operatorname{mod} A$ *is semisimple.*

(3) $\operatorname{rad} A = 0$.

(4) Every module in $\operatorname{mod} A^{\mathrm{op}}$ *is semisimple.*

(5) ${}_A A$ is semisimple.

A **semisimple algebra** if A_A and ${}_A A$ are semisimple modules

Theorem (Wadderburn). *A finite dimensional K-algebra over a field K. The following conditions are equivalent:*

(1) A is a semisimple algebra.

(2) $\text{gl.dim}\, A = 0$.

(3) There exist positive integers $n_1, \dots, n_r$ and division K-algebras $F_1, \dots, F_r$ such that

$$A \cong M_{n_1}(F_1) \times \cdots \times M_{n_1}(F_1).$$

Observe that

- A is a semisimple algebra if and only if the Auslander-Reiten quiver Γ_A consists of the isolated vertices

$$\{S_1\} \quad \{S_2\} \quad \dots \quad \{S_r\}$$

corresponding to a complete set $S_1, S_2, \dots, S_r$ of pairwise nonisomorphic simple (equivalently, indecomposable) modules in $\text{mod}\, A$.

4. Hereditary algebras

A finite dimensional K-algebra over a field K

A is **right hereditary** if any right ideal of A is a projective right A-module

A is **left hereditary** if any left ideal of A is a projective left A-module

Theorem. *Let A be a finite dimensional K-algebra over a field K. The following conditions are equivalent:*

(1) A is right hereditary.

(2) Every right A-submodule of a projective module in $\operatorname{mod} A$ is projective.

(3) The radical $\operatorname{rad} P$ of any indecomposable projective module P in $\operatorname{mod} A$ is projective.

(4) $\operatorname{gl.dim} A \leq 1$.

(5) The socle factor $I/\operatorname{soc} I$ of any indecomposable injective module I in $\operatorname{mod} A$ is injective.

(6) Every factor module of an injective module in $\operatorname{mod} A$ is injective.

(7) A is left hereditary.

A is **hereditary** if A is left and right hereditary

Examples. K a field

(1) Q finite acyclic quiver
(arrows with trivial valuation)

$A = KQ$ the path algebra of Q over K

A finite dimensional hereditary K-algebra

$Q_A = Q$

(2) F, G finite dimensional division K-algebras

${}_F M_G$ F-G-bimodule

K acts centrally on ${}_F M_G$

$\dim_K({}_F M_G) < \infty$

$$A = \begin{bmatrix} F & 0 \\ {}_F M_G & G \end{bmatrix} = \left\{ \begin{bmatrix} f & 0 \\ m & g \end{bmatrix} ; \begin{array}{c} f \in F,\ g \in G, \\ m \in {}_F M_G \end{array} \right\}$$

finite dimensional hereditary K-algebra

Q_A the valued quiver

$$2 \xrightarrow{(\dim_F({}_F M_G), \dim_G({}_F M_G))} 1$$

For example,

$$\begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{C} & \mathbb{C} \end{bmatrix}, \begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{C} & \mathbb{R} \end{bmatrix}, \begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{H} & \mathbb{H} \end{bmatrix}, \begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{H} & \mathbb{R} \end{bmatrix}$$

$\mathbb{R}$ real numbers, $\mathbb{C}$ complex numbers, $\mathbb{H}$ quaternions, are hereditary $\mathbb{R}$-algebras

(3) $F_1, F_2, \ldots, F_n$ family of finite dimensional division K-algebras

${}_iM_j$ F_i-F_j-bimodules, $i, j \in \{1, \ldots, n\}$

K acts centrally on ${}_iM_j$, $\dim_K({}_iM_j) < \infty$

Consider the valued quiver Q:

$1, 2, \ldots, n$ vertices of Q

There is an arrow $j \to i$ in $Q \iff {}_iM_j \neq 0$

Then we have the valued arrow $j \xrightarrow{(d_{ij}, d'_{ij})} i$

$d_{ij} = \dim_{F_i}({}_iM_j), \quad d'_{ij} = \dim_{F_j}({}_iM_j)$

Assume that the valued quiver Q is **acyclic**

$$F = \prod_{i=1}^{n} F_i, \qquad M = \bigoplus_{i,j=1}^{n} {}_iM_j,$$

M is an F-F-bimodule, $\dim_K M < \infty$

$A = T_F(M) = \bigoplus_{n=0}^{\infty} M^{(n)}$ **tensor algebra** of M over F

$M^{(0)} = F, \quad M^{(1)} = M,$
$M^{(n)} = M \otimes_F \cdots \otimes_F M$ n-times, for $n \geq 2$

Q acyclic $\Rightarrow M^{(r)} = 0$ for large r

A finite dimensional hereditary K-algebra

$Q_A = Q$

Theorem. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K. The following conditions are equivalent:*

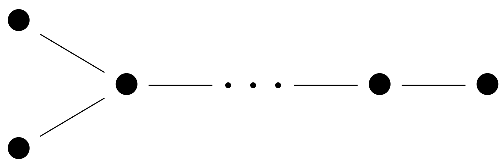
(1) The Euler form q_A is positive definite.

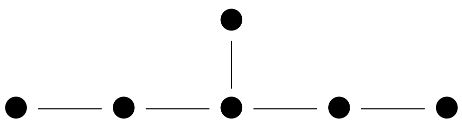
(2) The valued graph G_A of A is one of the following Dynkin graphs

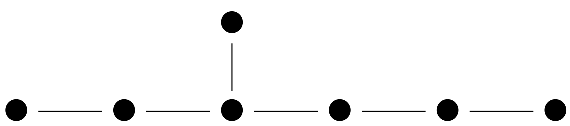
$\mathbb{A}_m$: $\bullet \,\text{—}\, \bullet \,\text{—}\, \cdots \,\text{—}\, \bullet \,\text{—}\, \bullet$ *(m vertices), $m \geq 1$*

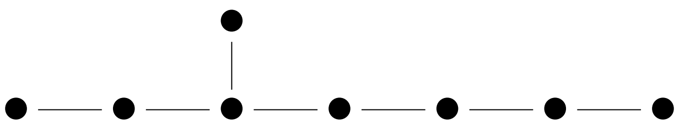
$\mathbb{B}_m$: $\bullet \overset{(1,2)}{\text{—}} \bullet \,\text{—}\, \cdots \,\text{—}\, \bullet \,\text{—}\, \bullet$ *(m vertices), $m \geq 2$*

$\mathbb{C}_m$: $\bullet \overset{(2,1)}{\text{—}} \bullet \,\text{—}\, \cdots \,\text{—}\, \bullet \,\text{—}\, \bullet$ *(m vertices), $m \geq 3$*

$\mathbb{D}_m$:  *(m vertices), $m \geq 4$*

$\mathbb{E}_6$: 

$\mathbb{E}_7$: 

$\mathbb{E}_8$: 

$\mathbb{F}_4$: $\bullet \,\text{—}\, \bullet \overset{(1,2)}{\text{—}} \bullet \,\text{—}\, \bullet$

$\mathbb{G}_2$: $\bullet \overset{(1,3)}{\text{—}} \bullet$

Theorem. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K. The following conditions are equivalent:*

(1) The Euler form q_A is positive semidefinite but not positive definite.

(2) The valued graph G_A of A is one of the Euclidean graphs

$$\widetilde{\mathbb{A}}_{11} : \bullet \overset{(1,4)}{\text{——}} \bullet$$

$$\widetilde{\mathbb{A}}_{12} : \bullet \overset{(2,2)}{\text{——}} \bullet$$

$$\widetilde{\mathbb{A}}_m : \begin{tikzcd}[column sep=small, row sep=small] & \bullet \arrow[r, dash] & \cdots \arrow[r, dash] & \bullet \arrow[dr, dash] & \\ \bullet \arrow[ur, dash] \arrow[dr, dash] & & & & \bullet \\ & \bullet \arrow[r, dash] & \cdots \arrow[r, dash] & \bullet \arrow[ur, dash] & \end{tikzcd}$$

($m + 1$ vertices), $m \geq 4$

$$\widetilde{\mathbb{B}}_m : \bullet \overset{(1,2)}{\text{——}} \bullet \text{——} \cdots \text{——} \bullet \overset{(2,1)}{\text{——}} \bullet$$

($m + 1$ vertices), $m \geq 2$

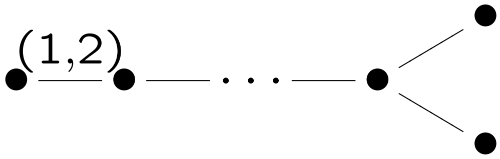
$$\widetilde{\mathbb{C}}_m : \bullet \overset{(2,1)}{\text{——}} \bullet \text{——} \cdots \text{——} \bullet \overset{(1,2)}{\text{——}} \bullet$$

($m + 1$ vertices), $m \geq 2$

$$\widetilde{\mathbb{BC}}_m : \bullet \overset{(1,2)}{\text{——}} \bullet \text{——} \cdots \text{——} \bullet \overset{(1,2)}{\text{——}} \bullet$$

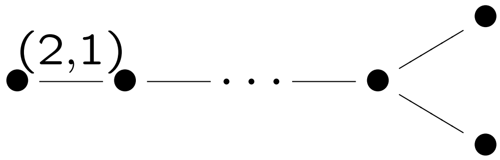
($m + 1$ vertices), $m \geq 2$

$\widetilde{\mathbb{BD}}_m$:



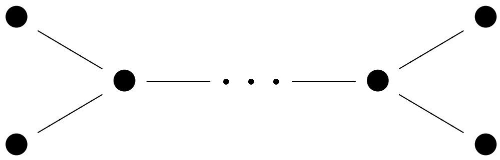
($m+1$ vertices), $m \geq 3$

$\widetilde{\mathbb{CD}}_m$:



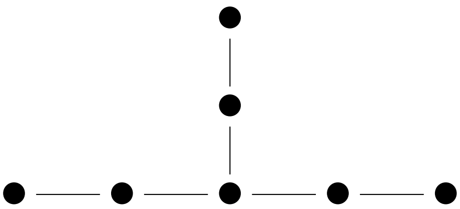
($m+1$ vertices), $m \geq 3$

$\widetilde{\mathbb{D}}_m$:

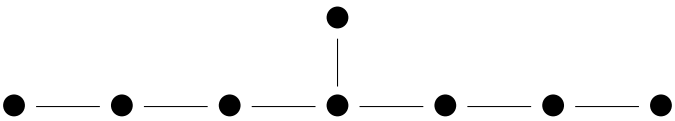


($m+1$ vertices), $m \geq 4$

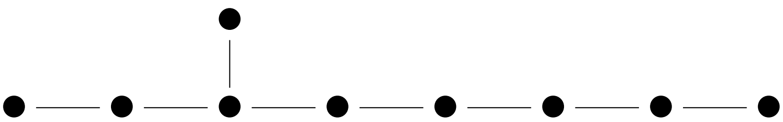
$\widetilde{\mathbb{E}}_6$:



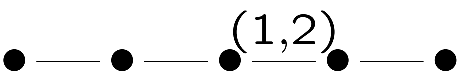
$\widetilde{\mathbb{E}}_7$:



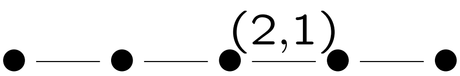
$\widetilde{\mathbb{E}}_8$:



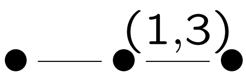
$\widetilde{\mathbb{F}}_{41}$:



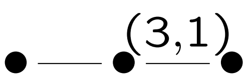
$\widetilde{\mathbb{F}}_{42}$:



$\widetilde{\mathbb{G}}_{21}$:



$\widetilde{\mathbb{G}}_{22}$:



A hereditary K-algebra

- A is of **Dynkin type** if G_A is a Dynkin graph

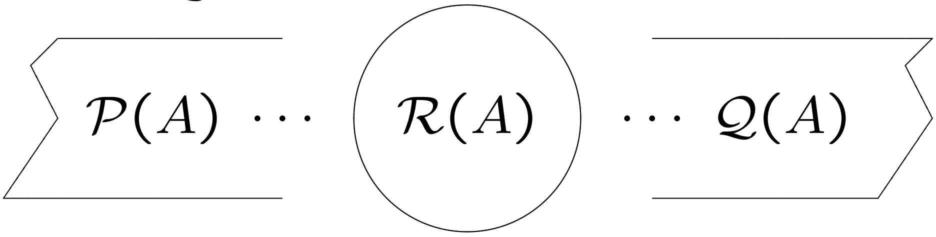
- A is of **Euclidean type** if G_A is an Euclidean graph

- A is of **wild type** if G_A is neither a Dynkin nor Euclidean graph

- A wild type, then there exists an indecomposable module M in $\operatorname{mod} A$ such that

$$q_A([M]) = \dim_K \operatorname{End}_A(M) - \dim_K \operatorname{Ext}^1_A(M, M) < 0$$

Theorem. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K, and $Q = Q_A$ the valued quiver of A. Then the Auslander-Reiten quiver Γ_A has the following shape*



- *$\mathcal{P}(A)$ is the postprojective component containing all indecomposable projective A-modules*
- *$Q(A)$ is the preinjective component containing all indecomposable injective A-modules*
- *$\mathcal{R}(A)$ is the family of all regular components*

Moreover

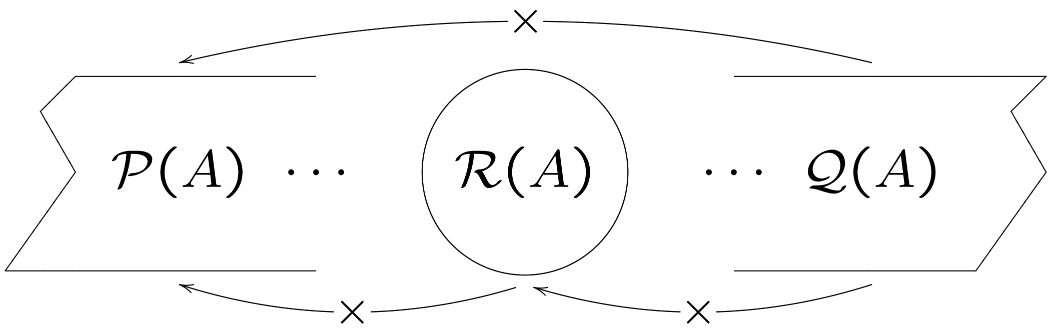
(1) *If A is of Dynkin type, then $\mathcal{P}(A) = \mathcal{Q}(A)$ is finite and $\mathcal{R}(A)$ is empty.*

(2) *If A is of Euclidean type, then $\mathcal{P}(A) \cong (-\mathbb{N})Q^{\mathrm{op}}$, $\mathcal{Q}(A) \cong \mathbb{N}Q^{\mathrm{op}}$ and $\mathcal{R}(A)$ is an infinite family of stable tubes, all but finitely many of them of rank one.*

(3) *If A is of wild type, then $\mathcal{P}(A) \cong (-\mathbb{N})Q^{\mathrm{op}}$, $\mathcal{Q}(A) \cong \mathbb{N}Q^{\mathrm{op}}$, and $\mathcal{R}(A)$ is an infinite family of components of type $\mathbb{Z}\mathbb{A}_\infty$.*

A indecomposable hereditary not of Dynkin type, then

- $\mathrm{Hom}_A(\mathcal{P}(A), \mathcal{R}(A)) \neq 0$,
  $\mathrm{Hom}_A(\mathcal{R}(A), \mathcal{P}(A)) = 0$,
- $\mathrm{Hom}_A(\mathcal{R}(A), \mathcal{Q}(A)) \neq 0$,
  $\mathrm{Hom}_A(\mathcal{Q}(A), \mathcal{R}(A)) = 0$,
- $\mathrm{Hom}_A(\mathcal{P}(A), \mathcal{Q}(A)) \neq 0$,
  $\mathrm{Hom}_A(\mathcal{Q}(A), \mathcal{P}(A)) = 0$,



A hereditary of Euclidean type, then $\mathcal{R}(A)$ is an infinite family $(\mathcal{T}_\lambda^A)_{\lambda\in\Lambda}$ of pairwise orthogonal generalized standard stable tubes separating $\mathcal{P}(A)$ form $\mathcal{Q}(A)$: for any homomorphism $f : X \to Y$ with X in $\mathcal{P}(A)$ and Y in $Q(A)$ there exists a module Z in $\mathcal{R}(A)$ and a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

A hereditary of Euclidean type, then

$$(\mathrm{rad}_A^\infty)^3 = 0$$

A hereditary of wild type, then $(\mathrm{rad}_A^\infty)^m \neq 0$ for all $m \geq 1$

5. Tilted algebras

A finite dimensional K-algebra over a field K

A module T in $\operatorname{mod} A$ is a **tilting module** if the following conditions are satisfied:

(T1) $\operatorname{pd}_A T \leq 1$;

(T2) $\operatorname{Ext}^1_A(T, T) = 0$;

(T3) T is a direct sum of n pairwise nonisomorphic indecomposable modules, where $n =$ rank of $K_0(A)$.

(**Brenner-Butler**, **Happel-Ringel**, **Bongartz**)

$B = \operatorname{End}_A(T)$ **tilted algebra** of A

We have the **torsion pairs**

$$(\mathcal{F}(T), \mathcal{T}(T)) \text{ in } \operatorname{mod} A$$

with **torsion-free part**

$$\begin{aligned}\mathcal{F}(T) &= \{X \in \operatorname{mod} A \mid \operatorname{Hom}(T, X) = 0\} \\ &= \operatorname{Cogen} \tau_A T\end{aligned}$$

torsion part

$$\begin{aligned}\mathcal{T}(T) &= \left\{X \in \operatorname{mod} A \mid \operatorname{Ext}^1_A(T, X) = 0\right\} \\ &= \operatorname{Gen} T\end{aligned}$$

and

$$(\mathcal{Y}(T), \mathcal{X}(T)) \text{ in } \operatorname{mod} B$$

with **torsion-free part**

$$\begin{aligned}\mathcal{Y}(T) &= \left\{Y \in \operatorname{mod} B \mid \operatorname{Tor}^B_1(T, Y) = 0\right\} \\ &= \operatorname{Gen} \tau_B^{-1} D({}_B T)\end{aligned}$$

torsion part

$$\begin{aligned}\mathcal{X}(T) &= \{Y \in \operatorname{mod} B \mid Y \otimes_B T = 0\} \\ &= \operatorname{Cogen} D({}_B T)\end{aligned}$$

Theorem (**Brenner-Butler**). *Let A be a finite dimensional K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$, *and* $B = \operatorname{End}_A(T)$. *Then*

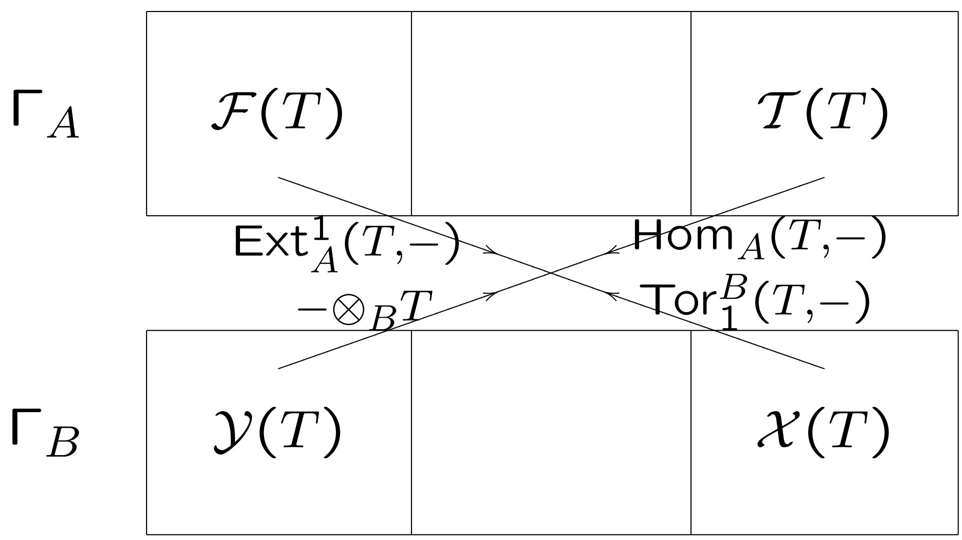
(1) ${}_BT$ *is a tilting module in* $\operatorname{mod} B^{\mathrm{op}}$ *and there is a canonical isomorphism of K-algebras* $A \to \operatorname{End}_{B^{\mathrm{op}}}({}_BT)^{\mathrm{op}}$.

(2) *The functors* $\operatorname{Hom}_A(T,-) : \operatorname{mod} A \to \operatorname{mod} B$ *and* $-\otimes_B T : \operatorname{mod} B \to \operatorname{mod} A$ *induce mutually inverse equivalences*

$$\mathcal{T}(T) \xrightarrow{\sim} \mathcal{Y}(T)$$

(3) *The functors* $\operatorname{Ext}^1_A(T,-) : \operatorname{mod} A \to \operatorname{mod} B$ *and* $\operatorname{Tor}_1^B(T,-) : \operatorname{mod} B \to \operatorname{mod} A$ *induce mutually inverse equivalences*

$$\mathcal{F}(T) \xrightarrow{\sim} \mathcal{X}(T)$$



$\operatorname{inj} A \subseteq \mathcal{T}(T)$, $\operatorname{proj} B \subseteq \mathcal{Y}(T)$,

A finite dimensional K-algebra, T a tilting module in $\operatorname{mod} A$, and $B = \operatorname{End}_A(T)$. Then

- $|\operatorname{gl.dim} A - \operatorname{gl.dim} B| \leq 1$.
- There is a canonical isomorphism $f : K_0(A) \to K_0(B)$ of Grothendieck groups such that

  $$f([M]) = [\operatorname{Hom}_A(T, M)] - [\operatorname{Ext}^1_A(T, M)]$$

  for any module M in $\operatorname{mod} A$.
  Moreover, if $\operatorname{gl.dim} A < \infty$, then

  $$\langle [M], [N] \rangle_A = \langle f([M]), f([N]) \rangle_B$$

  for all modules M, N in $\operatorname{mod} A$.
- If $\operatorname{gl.dim} A < \infty$ then the Euler forms q_A of A and q_B of B are $\mathbb{Z}$-equivalent.

A hereditary finite dimensional K-algebra

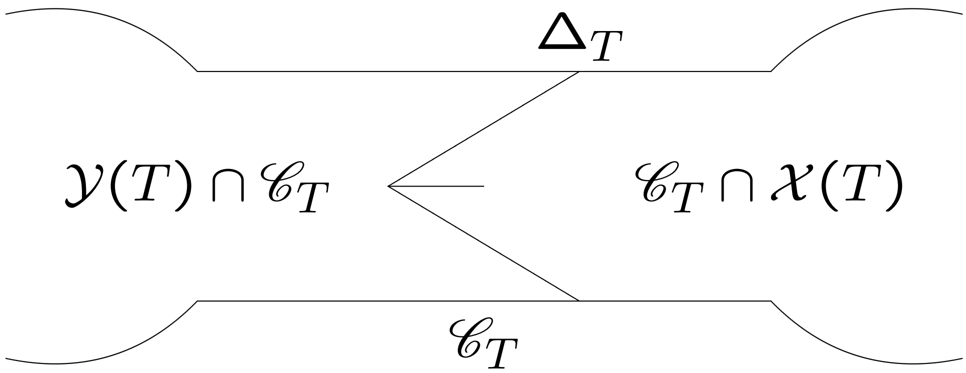
T tilting module in $\operatorname{mod} A$

$B = \operatorname{End}_A(T)$ **tilted algebra (of type** G_A (valued graph of A))

Then

- $\operatorname{gl.dim} B \leq 2$;
- For every indecomposable module Y in $\operatorname{mod} B$, we have $\operatorname{pd}_B Y \leq 1$ or $\operatorname{id}_B Y \leq 1$;
- The torsion pair $(\mathcal{Y}(T), \mathcal{X}(T))$ in $\operatorname{mod} B$ is **splitting**: every module from $\operatorname{ind} B$ belongs to $\mathcal{Y}(T)$ or $\mathcal{X}(T)$.

Moreover, the images $\operatorname{Hom}_A(T, I)$ of the indecomposable injective modules I in $\operatorname{mod} A$ via the functor $\operatorname{Hom}_A(T, -) : \operatorname{mod} A \to \operatorname{mod} B$ belong to one component $\mathscr{C}_T$ of Γ_B, and form a faithful section $\Delta_T \cong Q_A^{\mathrm{op}}$ of $\mathscr{C}$



$\mathscr{C}_T$ **connecting component** of Γ_B determined by T (connects the torsion-free part with the torsion part of Γ_B: every predecessor of a module $\operatorname{Hom}_A(T, I)$ from Δ_T in $\operatorname{ind} B$ lies in $\mathcal{Y}(T)$ and every successor of a module $\tau_B^{-1} \operatorname{Hom}_A(T, I)$ in $\operatorname{ind} B$ lies in $\mathcal{X}(T)$)

Δ_T **section**: acyclic, convex in $\mathscr{C}$, and intersects each τ_Λ-orbit of $\mathscr{C}$ exactly once

Δ_T **faithful**: the direct sum of all modules lying on Δ is a faithful B-module (has zero annihilator in B)

$\mathscr{C}_T$ **faithful generalized standard component** of Γ_A with a section Δ_T

Theorem (**Ringel**). *Let A be a hereditary algebra, T a tilting module in* $\operatorname{mod} A$, $B = \operatorname{End}_A(T)$ *and $\mathscr{C}_T$ the connecting component of Γ_B determined by T. Then*

(1) $\mathscr{C}_T$ contains a projective B-module if and only if T admits a preinjective indecomposable direct summand.

(2) $\mathscr{C}_T$ contains an injective B-module if and only if T admits a postprojective indecomposable direct summand.

(3) $\mathscr{C}_T$ is regular if and only if T is regular (belongs to $\operatorname{add} \mathcal{R}(A)$*).*

Theorem (**Ringel**). *Let A be a hereditary algebra. Then there is a regular tilting module in* $\operatorname{mod} A$ *if and only A is of wild type and $K_0(A)$ is of rank ≥ 3.*

Handy criterion for a tilted algebra

Theorem (**Liu**, **Skowroński**). *Let B be a finite dimensional K-algebra over a field K. Then B is a tilted algebra if and only if Γ_B admits a component $\mathscr{C}$ with a faithful section Δ such that $\mathrm{Hom}_B(X, \tau_B Y) = 0$ for all modules X, Y from Δ.*

Moreover, in this case, if T^* is the direct sum of all modules lying on Δ, then

- T^* *is a tilting module in* $\mathrm{mod}\, B$.
- $A = \mathrm{End}_B(T^*)$ *is a hereditary K-algebra of type Δ^{op}.*
- $T = D({}_A T^*)$ *is a tilting module in* $\mathrm{mod}\, A$.
- $B \cong \mathrm{End}_A(T)$.

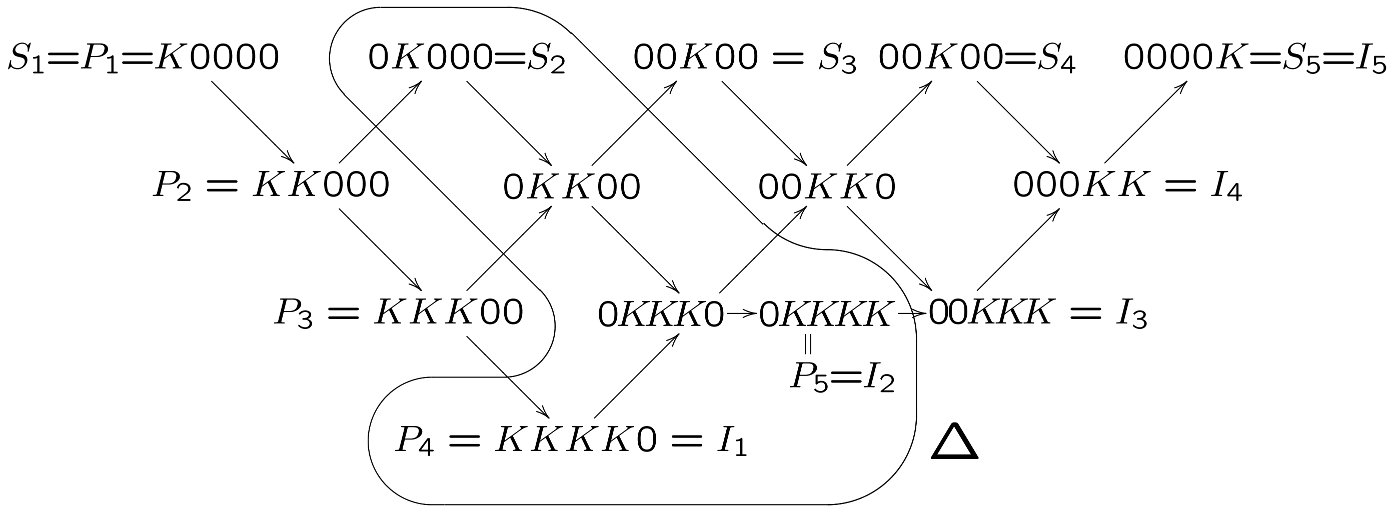
Theorem (**Liu**, **Skowroński**). *Let B be a finite dimensional K-algebra over a field K. Then B is a tilted algebra if and only if Γ_B admits a faithful generalized standard component $\mathscr{C}$ with a section Δ.*

Example. Let $B = KQ/I$ where Q is the quiver

$$1 \xleftarrow{\sigma} 2 \xleftarrow{\gamma} 3 \xleftarrow{\beta} 4 \xleftarrow{\alpha} 5$$

and I is the ideal of KQ generated by $\alpha\beta\gamma\sigma$

Γ_B is of the form



Δ faithful section of $\mathscr{C} = \Gamma_B$

$T_1^* = S_2$, $\quad T_2^* = 0KK00$, $\quad T_3^* = 0KKK0$, $T_4^* = P_5$, $\quad T_5^* = P_4$

$T^* = T_1^* \oplus T_2^* \oplus T_3^* \oplus T_4^* \oplus T_5^*$,

T^* faithful tilting B-module,

$\mathrm{Hom}_B(T^*, \tau_B T^*) = 0$

$A = \mathrm{End}_B(T^*)$ hereditary K-algebra $K\Delta^{\mathrm{op}}$, where Δ^{op} is of the form

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & 4 \arrow[dl] \\
1 & 2 \arrow[l] & 3 \arrow[l] \arrow[dr] & \\
 & & & 5
\end{tikzcd}$$

$T = D({}_AT^*)$ tilting module in $\operatorname{mod} A$

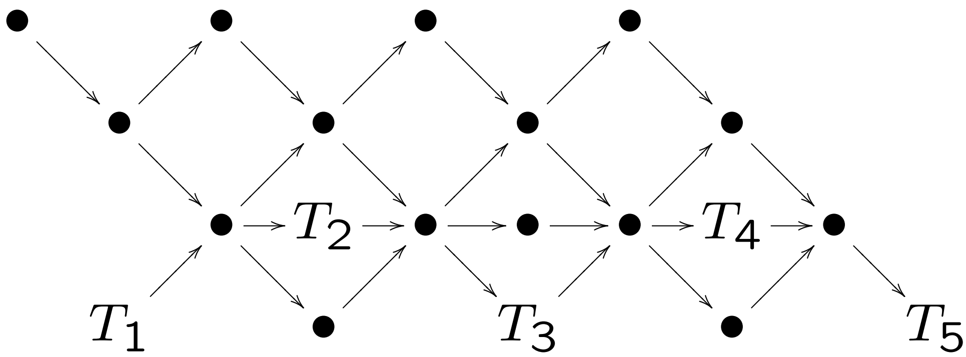
$T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5$

$T_i = D(T_i^*)$ for $i \in \{1, 2, 3, 4, 5\}$

$T_1 = 000\genfrac{}{}{0pt}{}{0}{K} \qquad T_2 = KKK\genfrac{}{}{0pt}{}{K}{K} \qquad T_3 = 0KK\genfrac{}{}{0pt}{}{K}{K}$

$T_4 = 00K\genfrac{}{}{0pt}{}{K}{K} \qquad T_5 = 000\genfrac{}{}{0pt}{}{K}{0}$

Γ_A



$\operatorname{Ext}^1_A(T, T) \cong D \operatorname{Hom}_A(T, \tau_A T) = 0$

$\operatorname{End}_A(T) \cong B = KQ/I$

A indecomposable hereditary finite dimensional K-algebra

T tilting module in $\operatorname{mod} A$

$B = \operatorname{End}_A(T)$

- A of Dynkin type
  $\Rightarrow$ A of finite representation type
  $\Rightarrow$ B of finite representation type

- B of finite representation type
  $\Rightarrow$ $\Gamma_B = \mathscr{C}_T$ and finite
  $\Rightarrow$ $\mathscr{C}_T$ contains all indecomposable projective modules and all indecomposable injective modules
  $\Rightarrow$ T has a postprojective and a preinjective direct summand

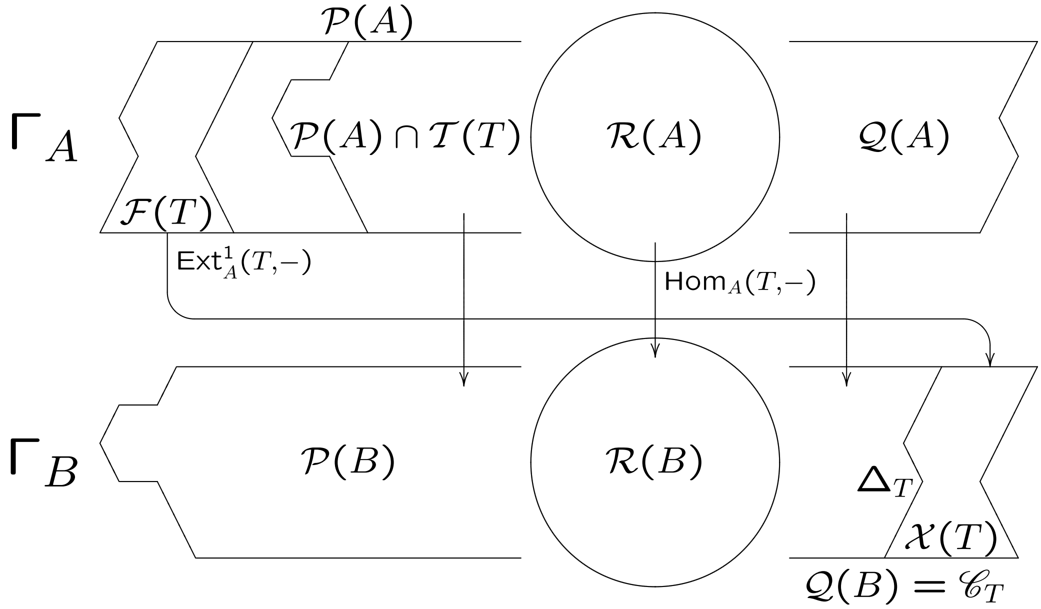
- A of Euclidean type, T has a postprojective and a preinjective direct summand
  $\Rightarrow$ B is of finite representation type

Concealed algebras

A indecomposable hereditary of infinite representation type

T postprojective tilting module in $\operatorname{mod} A$, $T \in \operatorname{add} \mathcal{P}(A)$

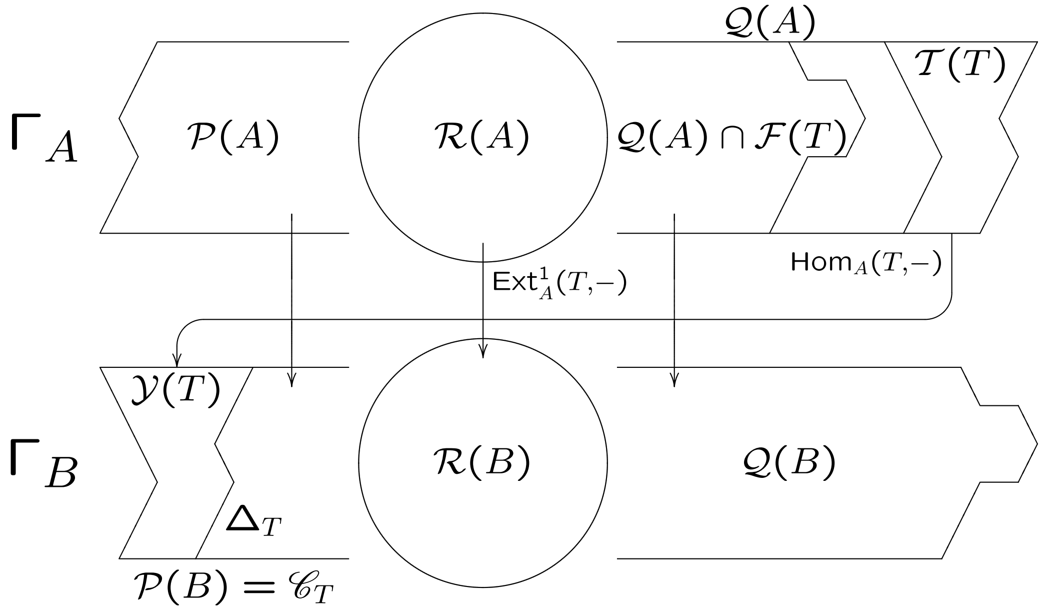
$B = \operatorname{End}_A(T)$ **concealed algebra of type** G_A



- $\mathcal{P}(B) = \operatorname{Hom}_A(T, \mathcal{P}(A) \cap \mathcal{T}(T))$ postprojective component of Γ_B containing all indecomposable projective B-modules
- $\mathcal{Q}(B) = \mathscr{C}_T = \operatorname{Hom}_A(T, \mathcal{Q}(A)) \cup \mathcal{X}(T)$ preinjective component of Γ_B containing all indecomposable injective B-modules
- $\mathcal{R}(B) = \operatorname{Hom}_A(T, \mathcal{R}(A))$ family of all regular components of Γ_B
- A of Euclidean type $\Rightarrow \mathcal{R}(B)$ infinite family of pairwise orthogonal generalized standard stable tubes
- A of wild type $\Rightarrow \mathcal{R}(B)$ infinite family of components of type $\mathbb{Z}\mathbb{A}_\infty$

T preinjective tilting module in $\operatorname{mod} A$, $T \in \operatorname{add} \mathcal{Q}(A)$

$B = \operatorname{End}_A(T)$



- $\mathcal{P}(B) = \mathscr{C}_T = \mathcal{Y}(T) \cup \operatorname{Ext}^1_A(T, \mathcal{P}(A))$ postprojective component of Γ_B containing all indecomposable projective B-modules
- $\mathcal{Q}(B) = \operatorname{Ext}^1_A(T, \mathcal{Q}(A) \cap \mathcal{F}(T))$ preinjective component of Γ_B containing all indecomposable injective B-modules
- $\mathcal{R}(B) = \operatorname{Ext}^1_A(T, \mathcal{R}(A))$ family of all regular components of Γ_B
- A of Euclidean type $\Rightarrow \mathcal{R}(B)$ infinite family of pairwise orthogonal generalized standard stable tubes
- A of wild type $\Rightarrow \mathcal{R}(B)$ infinite family of components of type $\mathbb{Z}\mathbb{A}_\infty$

$B \cong \operatorname{End}_A(T)$ for a postprojective tilting A-module $T \iff B \cong \operatorname{End}_A(T')$ for a preinjective tilting A-module T'

Representation-infinite tilted algebras of Euclidean type

A indecomposable hereditary of Euclidean type

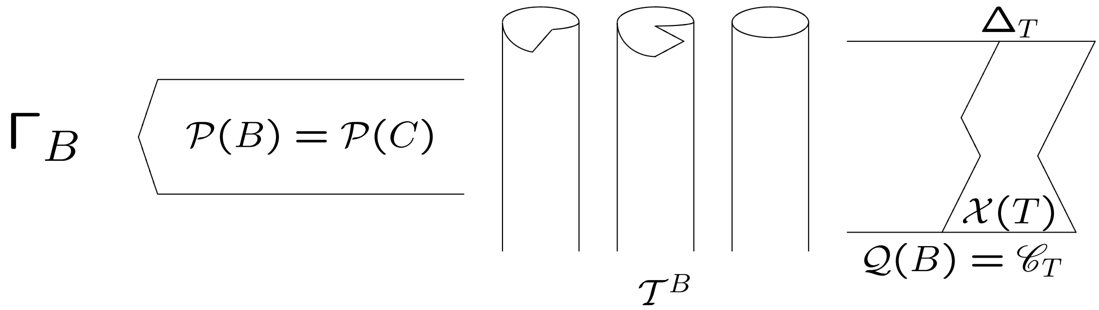
T tilting module in $\operatorname{mod} A$ without preinjective direct summands

$B = \operatorname{End}_A(T)$

$T = T^{pp} \oplus T^{rg}$, $T^{pp} \in \operatorname{add} \mathcal{P}(A)$, $T^{rg} \in \operatorname{add} \mathcal{R}(A)$
$\Rightarrow T^{pp} \neq 0$,

$C = \operatorname{End}_A(T^{pp})$ concealed algebra of Euclidean type

C factor algebra of B



- $\mathcal{P}(B) = \operatorname{Hom}_A(T, \mathcal{T}(T) \cap \mathcal{P}(A)) = \operatorname{Hom}_A(T^{pp}, \mathcal{T}(T) \cap \mathcal{P}(A)) = \mathcal{P}(C)$ postprojective component of Γ_B containing all indecomposable projective C-modules
- $\mathcal{Q}(B) = \mathscr{C}_T = \operatorname{Hom}_A(T, \mathcal{Q}(A)) \cup \mathcal{X}(T)$ preinjective component of Γ_B containing all indecomposable injective B-modules
- $\mathcal{T}^B = \operatorname{Hom}_A(T, \mathcal{R}(A) \cap \mathcal{T}(T))$ infinite family of pairwise orthogonal generalized standard ray tubes
- $\mathcal{T}^B$ contains a projective module $\iff$ $T^{rg} \neq 0$

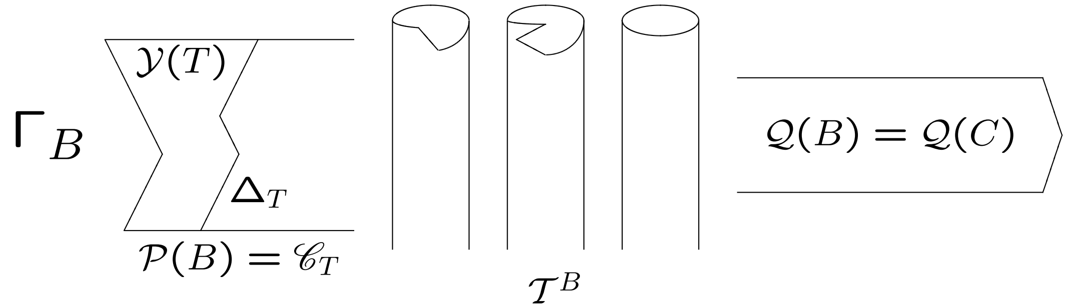
T tilting module in $\operatorname{mod} A$ without postprojective direct summands

$B = \mathrm{End}_A(T)$

$T = T^{rg} \oplus T^{pi}$, $T^{rg} \in \operatorname{add} \mathcal{R}(A)$, $T^{pi} \in \operatorname{add} \mathcal{Q}(A)$
$\Rightarrow T^{pi} \neq 0$,

$C = \mathrm{End}_A(T^{pi})$ concealed algebra of Euclidean type

C factor algebra of B



- $\mathcal{P}(B) = \mathscr{C}_T = \mathcal{Y}(T) \cup \mathrm{Ext}^1_A(T, \mathcal{P}(A))$ postprojective component of Γ_B containing all indecomposable projective B-modules
- $\mathcal{Q}(B) = \mathrm{Ext}^1_A(T, \mathcal{F}(T) \cap \mathcal{Q}(A)) = \mathrm{Ext}^1_A(T^{pi}, \mathcal{F}(T) \cap \mathcal{Q}(A)) = \mathcal{Q}(C)$ preinjective component of Γ_B containing all indecomposable injective C-modules
- $\mathcal{T}^B = \mathrm{Ext}^1_A(T, \mathcal{R}(A) \cap \mathcal{F}(T))$ infinite family of pairwise orthogonal generalized standard coray tubes
- $\mathcal{T}^B$ contains an injective module $\iff$ $T^{rg} \neq 0$

Almost concealed algebras of wild type

A indecomposable hereditary of wild type

T tilting module in $\operatorname{mod} A$

$T = T^{pp} \oplus T^{rg} \oplus T^{pi}$,

$T^{pp} \in \operatorname{add} \mathcal{P}(A)$, $T^{rg} \in \operatorname{add} \mathcal{R}(A)$, $T^{pi} \in \operatorname{add} \mathcal{Q}(A)$

$B = \operatorname{End}_A(T)$

B **almost concealed** if $T^{pp} = 0$ or $T^{pi} = 0$

The cases

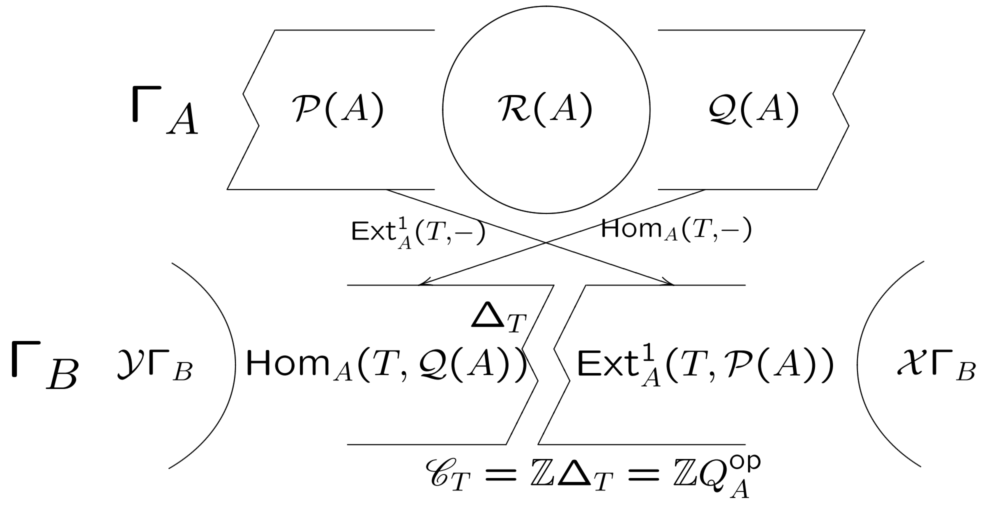
- $T = T^{pp}$
- $T = T^{pi}$

were considered above

It remains to consider the cases

- $T = T^{rg}$
- $T = T^{pp} \oplus T^{rg}$, $T^{pp} \neq 0$, $T^{rg} \neq 0$
- $T = T^{rg} \oplus T^{pi}$, $T^{rg} \neq 0$, $T^{pi} \neq 0$

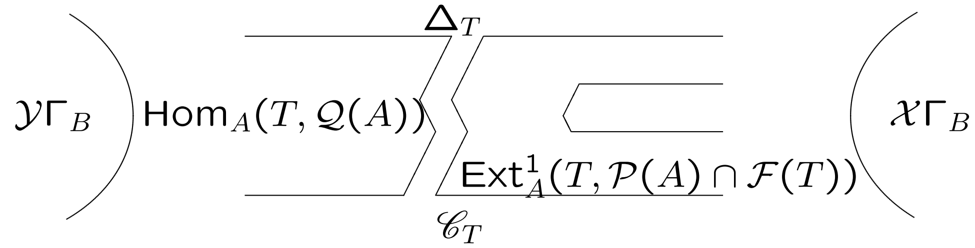
$T = T^{rg}$ regular tilting module, $B = \mathrm{End}_A(T)$



- $\mathscr{C}_T$ regular connecting component
- $\mathcal{Y}\Gamma_B = \mathrm{Hom}_A(T, \mathcal{T}(T) \cap \mathcal{R}(A))$ contains all indecomposable projective B-modules and consist of
  - one postprojective component $\mathcal{P}(B) = \mathcal{P}(C)$, for a wild concealed factor algebra C of B
  - an infinite family of components obtained from components of type $\mathbb{Z}\mathbb{A}_\infty$ by ray insertions, containing at least one projective B-module
- $\mathcal{X}\Gamma_B = \mathrm{Ext}^1_A(T, \mathcal{F}(T) \cap \mathcal{R}(A))$ contains all indecomposable injective B-modules and consist of
  - one preinjective component $\mathcal{Q}(B) = \mathcal{Q}(C')$, for a wild concealed factor algebra C' of B
  - an infinite family of components obtained from components of type $\mathbb{Z}\mathbb{A}_\infty$ by coray insertions, containing at least one injective B-module

$T = T^{pp} \oplus T^{rg}$, $T^{pp} \neq 0$, $T^{rg} \neq 0$

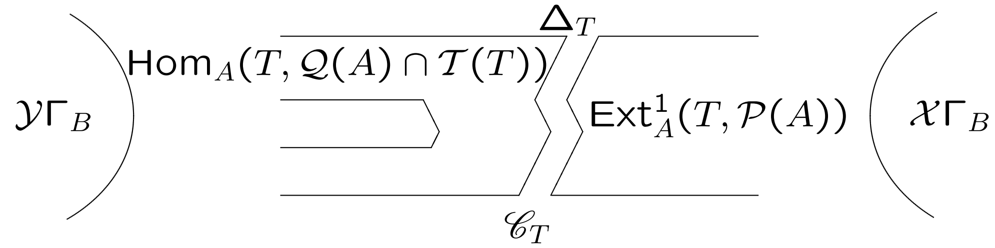
Γ_B is of the form



- $\mathscr{C}_T$ connecting component containing at least one injective module and no projective modules
- $\mathcal{Y}\Gamma_B = \mathrm{Hom}_A(T, \mathcal{T}(T) \cap (\mathcal{P}(A) \cup \mathcal{R}(A)))$ contains all indecomposable projective B-modules and consist of
  - one postprojective component $\mathcal{P}(B) = \mathcal{P}(C)$, for a wild concealed factor algebra C of B
  - an infinite family of components obtained from components of type $\mathbb{Z}\mathbb{A}_\infty$ by ray insertions, containing at least one projective B-module
- $\mathcal{X}\Gamma_B = \mathrm{Ext}_A^1(T, \mathcal{F}(T) \cap \mathcal{R}(A))$ consists of preinjective components and components obtained from stable tubes or components of type $\mathbb{Z}\mathbb{A}_\infty$ by coray insertions

$T = T^{rg} \oplus T^{pi}$, $T^{rg} \neq 0$, $T^{pi} \neq 0$

Γ_B of the form



- $\mathscr{C}_T$ connecting component containing at least one projective module and no injective modules
- $\mathcal{Y}\Gamma_B = \mathrm{Hom}_A(T, \mathcal{T}(T) \cap (\mathcal{R}(A) \cup \mathcal{Q}(A)))$ consists of preprojective components and components obtained from stable tubes or components of type $\mathbb{Z}\mathbb{A}_\infty$ by ray insertions
- $\mathcal{X}\Gamma_B = \mathrm{Ext}^1_A(T, \mathcal{F}(T) \cap (\mathcal{R}(A) \cup \mathcal{Q}(A)))$ contains all indecomposable injective B-modules and consist of
  - one preinjective component $\mathcal{Q}(B) = \mathcal{Q}(C')$, for a wild concealed factor algebra C' of B
  - an infinite family of components obtained from components of type $\mathbb{Z}\mathbb{A}_\infty$ by coray insertions, containing at least one injective B-module

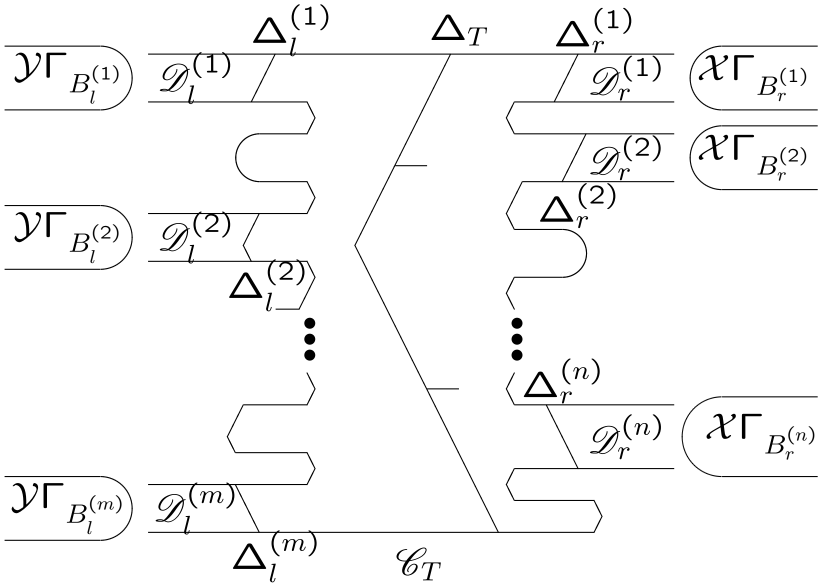
Tilted algebras of wild type – general case

A indecomposable hereditary algebra of wild type

T tilting module in $\operatorname{mod} A$

$B = \operatorname{End}_A(T)$

Γ_B is of the form



where

- $\mathscr{C}_T$ connecting component of Γ_B determinend by T, possibly $\mathscr{C}_T = \Gamma_B$ (if B is of finite representation type)
- For each $i \in \{1,\ldots,m\}$, $\Delta_l^{(i)}$ connected valued subquiver of Δ_T of Euclidean or wild type, $\mathscr{D}_l^{(i)} = \mathbb{N}\Delta_l^{(i)}$ full translation subquiver of $\mathscr{C}_T$ closed under predecessors
- For each $j \in \{1,\ldots,n\}$, $\Delta_r^{(j)}$ connected valued subquiver of Δ_T of Euclidean or wild type, $\mathscr{D}_r^{(j)} = (-\mathbb{N})\Delta_r^{(j)}$ full translation subquiver of $\mathscr{C}_T$ closed under successors

- For each $i \in \{1, \ldots, m\}$, there exists a tilted algebra

$$B_l^{(i)} = \mathrm{End}_{A_l^{(i)}}(T_l^{(i)})$$

where $A_l^{(i)}$ is a hereditary algebra of type $\Delta_l^{(i)}$ and $T_l^{(i)}$ is a tilting module in mod $A_l^{(i)}$ without preinjective direct summands such that

  - $B_l^{(i)}$ is a factor algebra of B
  - $\mathscr{D}_l^{(i)} = \mathcal{Y}(T_l^{(i)}) \cap \mathscr{C}_{T_l^{(i)}}$
  - $\mathcal{Y}\Gamma_{B_l^{(i)}}$ family of all connected components of $\Gamma_{B_l^{(i)}}$ contained entirely in the torsion-free part $\mathcal{Y}(T_l^{(i)})$ of mod $B_l^{(i)}$

- For each $j \in \{1, \ldots, n\}$, there exists a tilted algebra

$$B_r^{(j)} = \operatorname{End}_{A_r^{(j)}}(T_r^{(j)})$$

where $A_r^{(j)}$ is a hereditary algebra of type $\Delta_r^{(j)}$ and $T_r^{(j)}$ is a tilting module in mod $A_r^{(j)}$ without postprojective direct summands such that
  - $B_r^{(j)}$ is a factor algebra of B
  - $\mathscr{D}_r^{(j)} = \mathcal{X}(T_r^{(j)}) \cap \mathscr{C}_{T_r^{(j)}}$
  - $\mathcal{X}\Gamma_{B_r^{(j)}}$ family of all connected components of $\Gamma_{B_r^{(j)}}$ contained entirely in the torsion part $\mathcal{X}(T_r^{(j)})$ of mod $B_r^{(j)}$

- All but finitely many modules of $\mathscr{C}_T$ are in

$$\mathscr{D}_l^{(1)} \cup \cdots \cup \mathscr{D}_l^{(m)} \cup \mathscr{D}_r^{(1)} \cup \cdots \cup \mathscr{D}_r^{(n)}$$

We know from the facts described before that

- For each $i \in \{1, \ldots, m\}$, the translation quiver $\mathcal{Y}\Gamma_{B_l^{(i)}}$ consists of
  - one postprojective component $\mathcal{P}(B_l^{(i)})$
  - an infinite family of pairwise orthogonal generalized standard ray tubes, if $\Delta_l^{(i)}$ is an Euclidean quiver, or an infinite family of components obtained from components of type $\mathbb{Z}\mathbb{A}_\infty$ by ray insertions, if $\Delta_l^{(i)}$ is a wild quiver
- For each $j \in \{1, \ldots, n\}$, the translation quiver $\mathcal{X}\Gamma_{B_r^{(j)}}$ consists of
  - one preinjective component $\mathcal{Q}(B_r^{(j)})$
  - an infinite family of pairwise orthogonal generalized standard coray tubes, if $\Delta_r^{(j)}$ is an Euclidean quiver, or an infinite family of components obtained from components of type $\mathbb{Z}\mathbb{A}_\infty$ by coray insertions, if $\Delta_r^{(j)}$ is a wild quiver

Acyclic generalized standard Auslander-Reiten components

Theorem (**Skowroński**). *Let A be a finite dimensional K-algebra over a field K, $\mathscr{C}$ a component of Γ_A and $B = A/\operatorname{ann}_A \mathscr{C}$.*

(1) $\mathscr{C}$ is generalized standard, acyclic, without projective modules if and only if B is a tilted algebra of the form $\operatorname{End}_H(T)$, where H is a hereditary algebra, T is a tilting module in $\operatorname{mod} H$ without preinjective direct summands, and $\mathscr{C}$ is the connecting component $\mathscr{C}_T$ of Γ_B determined by T.

(2) $\mathscr{C}$ is generalized standard, acyclic, without injective modules if and only if B is a tilted algebra of the form $\operatorname{End}_H(T)$, where H is a hereditary algebra, T is a tilting module in $\operatorname{mod} H$ without postprojective direct summands, and $\mathscr{C}$ is the connecting component $\mathscr{C}_T$ of Γ_B determined by T.

(3) $\mathscr{C}$ is generalized standard, acyclic, regular if and only if B is a tilted algebra of the form $\operatorname{End}_H(T)$, where H is a hereditary algebra, T is a regular tilting module in $\operatorname{mod} H$, and $\mathscr{C}$ is the connecting component $\mathscr{C}_T$ of Γ_B determined by T.

In general, an arbitrary acyclic generalized standard component $\mathscr{C}$ of Γ_A is a glueing of

- torsion-free parts $\mathcal{Y}(T_l^{(i)}) \cap \mathscr{C}_{T_l^{(i)}}$ of the connecting components $\mathscr{C}_{T_l^{(i)}}$ of tilted algebras $B_l^{(i)} = \mathrm{End}_{A_l^{(i)}}(T_l^{(i)})$ of hereditary algebras $A_l^{(i)}$ by tilting $A_l^{(i)}$-modules $T_l^{(i)}$ without preinjective direct summands
- torsion parts $\mathcal{X}(T_r^{(j)}) \cap \mathscr{C}_{T_r^{(j)}}$ of the connecting components $\mathscr{C}_{T_r^{(j)}}$ of tilted algebras $B_r^{(j)} = \mathrm{End}_{A_r^{(j)}}(T_r^{(j)})$ of hereditary algebras $A_r^{(j)}$ by tilting $A_r^{(j)}$-modules $T_r^{(j)}$ without postprojective direct summands

along a finite acyclic part in the middle of $\mathscr{C}$ (and usually $\mathscr{C}$ does not admit a section)

6. Quasitilted algebras

Abelian K-category $\mathscr{H}$ over a field K is said to be **hereditary** if, for all objects X and Y of $\mathscr{H}$, the following conditions are satisfied

- $\mathrm{Ext}^2_{\mathscr{H}}(X,Y) = 0$
- $\mathrm{Hom}_{\mathscr{H}}(X,Y)$ and $\mathrm{Ext}^1_{\mathscr{H}}(X,Y)$ are finite dimensional K-vector spaces

An object T of a hereditary abelian K-category $\mathscr{H}$ is said a **tilting object** if the following conditions are satisfied

- $\mathrm{Ext}^1_{\mathscr{H}}(T,T) = 0$
- For an object X of $\mathscr{H}$, $\mathrm{Hom}_{\mathscr{H}}(T,X) = 0$ and $\mathrm{Ext}^1_{\mathscr{H}}(T,X) = 0$ force $X = 0$
- T direct sum of pairwise nonisomorphic indecomposable objects of $\mathscr{H}$

A finite dimensional hereditary K-algebra. Then

- $\mathscr{H} = \mathrm{mod}\, A$ hereditary abelian K-category
- A module T in $\mathrm{mod}\, A$ is a tilting object of $\mathrm{mod}\, A$ if and only if T is a tilting module

A **quasitilted algebra** is an algebra of the form $\mathrm{End}_{\mathscr{H}}(T)$, where T is a tilting object of an abelian hereditary K-category $\mathscr{H}$.

A finite dimensional K-algebra over a field K

A **path** in $\operatorname{ind} A$ is a sequence of homomorphisms

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \dots \longrightarrow M_{t-1} \xrightarrow{f_t} M_t$$

in $\operatorname{ind} A$ with $f_1, f_2, \dots, f_t$ nonzero and nonisomorphisms

M_0 **predecessor** of M_t in $\operatorname{ind} A$

M_t **successor** of M_0 in $\operatorname{ind} A$

Every module M in $\operatorname{ind} A$ is its own (trivial) predecessor and successor

$\mathcal{L}_A$ full subcategory of $\operatorname{ind} A$ formed by all modules X such that $\operatorname{pd}_A Y \leq 1$ for every predecessor Y of X in $\operatorname{ind} A$

$\mathcal{R}_A$ full subcategory of $\operatorname{ind} A$ formed by all modules X in $\operatorname{ind} A$ such that $\operatorname{id}_A Y \leq 1$ for every successor Y of X in $\operatorname{ind} A$

$\mathcal{L}_A$ closed under predecessors in $\operatorname{ind} A$

$\mathcal{R}_A$ closed under successors in $\operatorname{ind} A$

Theorem (Happel-Reiten-Smalø). *Let B be a finite dimensional K-algebra. The following conditions are equivalent:*

(1) B is a quasitilted algebra.

(2) $\operatorname{gl.dim} B \leq 2$ *and every module X in* $\operatorname{ind} B$ *satisfies* $\operatorname{pd}_B X \leq 1$ *or* $\operatorname{id}_B X \leq 1$.

(3) $\mathcal{L}_B$ contains all indecomposable projective B-modules.

(4) $\mathcal{R}_B$ contains all indecomposable injective B-modules.

Theorem (Happel-Reiten-Smalø). *Let B be a quasitilted K-algebra. Then*

(1) The quiver Q_B of B is acyclic.

(2) $\operatorname{ind} B = \mathcal{L}_B \cup \mathcal{R}_B$.

(3) If B is of finite representation type, then B is a tilted algebra.

Theorem (**Skowroński**). *Let B be an indecomposable finite dimensional K-algebra. The following conditions are equivalent:*

(1) B is a tilted algebra.

(2) $\operatorname{gl.dim} B \leq 2$, $\operatorname{ind} B = \mathcal{L}_B \cup \mathcal{R}_B$ *and* $\mathcal{L}_B \cap \mathcal{R}_B$ *contains a directing module.*

A module M in $\operatorname{ind} B$ is **directing** if M does not lie on an oriented cycle in $\operatorname{ind} B$.

Theorem (**Coelho-Skowroński**). *Let B be a quasitilted but not tilted algebra. Then every component of Γ_B is semiregular.*

A component $\mathscr{C}$ of Γ_B is **semiregular** if $\mathscr{C}$ does not contain simultaneously a projective module and an injective module.

Canonical algebras

Special case: K a field

$m \geq 2$ natural number

$\mathbf{p} = (p_1, \ldots, p_m)$ m-tuple of natural numbers

$\underline{\lambda} = (\lambda_1, \ldots, \lambda_m)$ m-tuple of pairwise different elements of $\mathbb{P}_1(K) = K \cup \{\infty\}$, normalised such that $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_3 = 1$

$$\Delta(\mathbf{p}):\quad \begin{tikzcd}
 & \circ \arrow[dl, "\alpha_{11}"'] & \circ \arrow[l, "\alpha_{12}"'] & \cdots \arrow[l, "\alpha_{13}"'] & \circ \arrow[l, "\alpha_{1p_1-1}"'] & \\
0\ \circ & \circ \arrow[l, "\alpha_{21}"'] & \circ \arrow[l, "\alpha_{22}"'] & \cdots \arrow[l, "\alpha_{23}"'] & \circ \arrow[l, "\alpha_{2p_2-1}"'] & \circ\ \omega \arrow[l, "\alpha_{2p_2}"'] \arrow[ul, "\alpha_{1p_1}"'] \arrow[ddl, "\alpha_{mp_m}"] \\
 & \vdots & \vdots & & \vdots & \\
 & \circ \arrow[uul, "\alpha_{m1}"] & \circ \arrow[l, "\alpha_{m2}"] & \cdots \arrow[l, "\alpha_{m3}"] & \circ \arrow[l, "\alpha_{mp_m-1}"] &
\end{tikzcd}$$

$C(\mathbf{p}, \underline{\lambda})$ defined as follows.

For $m = 2$, $C(\mathbf{p}, \underline{\lambda}) = K\Delta(\mathbf{p})$ path algebra of $\Delta(\mathbf{p})$

For $m \geq 3$, $C(\mathbf{p}, \underline{\lambda}) = K\Delta(\mathbf{p})/I(\mathbf{p}, \underline{\lambda})$
$I(\mathbf{p}, \underline{\lambda})$ ideal of $K\Delta(\mathbf{p})$ generated by $\alpha_{jp_j} \ldots \alpha_{j2}\alpha_{j1} + \alpha_{1p_1} \ldots \alpha_{12}\alpha_{11} + \lambda_j \alpha_{2p_2} \ldots \alpha_{22}\alpha_{21}$ for $j \in \{3, \ldots, m\}$

$C(\mathbf{p}, \underline{\lambda})$ **canonical algebra of type** $(\mathbf{p}, \underline{\lambda})$

$\mathbf{p}$ weight sequence, $\underline{\lambda}$ parameter sequence

For K algebraically closed, these are all canonical algebras (up to isomorphism)

General case (version of Crawley-Boevey)

Let F and G be finite dimensional division algebras over a field K, ${}_FM_G$ an F-G-bimodule with $(\dim {}_FM)(\dim M_G) = 4$, K acting centrally on ${}_FM_G$.

Denote

$$\chi = \sqrt{\frac{\dim {}_FM}{\dim M_G}},$$

hence $\chi = \frac{1}{2}$, 1, or 2.

An M-**triple** is a triple $({}_FN, \varphi, N'_G)$, where ${}_FN$ is a finite dimensional nonzero left F-module, N'_G a finite dimensional nonzero right G-module, and $\varphi : {}_FN \otimes_{\mathbb{Z}} N'_G \to {}_FM_G$ an F-G-homomorphism such that

- $\frac{\dim {}_FN}{\dim N'_G} = \chi$,
- whenever ${}_FX$ and X'_G are nonzero submodules of ${}_FN$ and N'_G, respectively, with $\varphi(X \otimes_{\mathbb{Z}} X') = 0$, then $\frac{\dim {}_FX}{\dim {}_FN} + \frac{\dim X'_G}{\dim N'_G} < 1$.

Two M-triples (N_1, φ_1, N_1') and (N_2, φ_2, N_2') are said to be **congruent** if there are isomorphisms of modules $\Theta : {}_F(N_1) \to {}_F(N_2)$ and $\Theta' : (N_1')_G \to (N_2')_G$ such that the following diagram is commutative

$$\begin{tikzcd}
N_1 \otimes_{\mathbb{Z}} N_1' \arrow[d, "\Theta\otimes\Theta'"'] \arrow[dr, "\varphi_1"] & \\
 & M \\
N_2 \otimes_{\mathbb{Z}} N_2' \arrow[ur, "\varphi_2"'] &
\end{tikzcd} .$$

The **middle** D of an M-triple $({}_FN, \varphi, N_G')$ is defined to be the set of pairs (d, d'), where d is an endomorphism of ${}_FN$ and d' is an endomorphism of N_G' such that $\varphi(d \otimes 1) = \varphi(1 \otimes d')$. Then D is a division K-algebra under componentwise addition and multiplication, N is an F-D-bimodule, N' a D-G-bimodule, and φ induces an F-G-homomorphism $\varphi : {}_FN \otimes_D N_G' \to {}_FM_G$.

Let $r \geq 0$ and $n_1, \ldots, n_r \geq 2$ be integers.
A canonical algebra Λ of type $(n_1, \ldots, n_r)$ over a field K is an algebra isomorphic to a matrix algebra of the form

$$
\begin{array}{r}
\\
n_1 - 1 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\
n_2 - 1 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\
\\
n_r - 1 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\
\\
\end{array}
\left[
\begin{array}{c|ccc|ccc|c|ccc|c}
F & N_1 & \cdots & N_1 & N_2 & \cdots & N_2 & \cdots & N_r & \cdots & N_r & M \\
\hline
 & D_1 & \cdots & D_1 & & & & & & & & N_1' \\
0 & & \ddots & \vdots & & 0 & & \cdots & & 0 & & \vdots \\
 & 0 & & D_1 & & & & & & & & N_1' \\
\hline
 & & & & D_2 & \cdots & D_2 & & & & & N_2' \\
0 & & 0 & & & \ddots & \vdots & \cdots & & 0 & & \vdots \\
 & & & & 0 & & D_2 & & & & & N_2' \\
\hline
\vdots & & \vdots & & & \vdots & & \vdots & & \vdots & & \vdots \\
\hline
 & & & & & & & & D_r & \cdots & D_r & N_r' \\
0 & & 0 & & & 0 & & \cdots & & \ddots & \vdots & \vdots \\
 & & & & & & & & 0 & & D_r & N_r' \\
\hline
0 & & 0 & & & 0 & & 0 & & 0 & & G
\end{array}
\right]
$$

where F and G are finite dimensional division algebras over K, $M = {}_F M_G$ an F-G-bimodule with $(\dim {}_F M)(\dim M_G) = 4$ and K acting centrally on ${}_F M_G$, $(N_1, \varphi_1, N_1'), \ldots, (N_r, \varphi_r, N_r')$ are mutually noncongruent M-triples with the middles $D_1, \ldots, D_r$, and the multiplication given by the actions of division algebras on bimodules and the appropriate homomorphisms $\varphi_1, \ldots, \varphi_r$.

The valued quiver Q_Λ of a canonical algebra Λ of type $(n_1, \ldots, n_r)$ is of the form

$$
\begin{tikzcd}
 & (1,1) \arrow[dl, "{(a_1,b_1)}"'] & (1,2) \arrow[l] & \cdots \arrow[l] & (1,n_1-1) \arrow[l] & \\
0 & (2,1) \arrow[l, "{(a_2,b_2)}"'] & (2,2) \arrow[l] & \cdots \arrow[l] & (2,n_2-1) \arrow[l] & \omega \arrow[ul, "{(c_1,d_1)}"'] \arrow[l, "{(c_2,d_2)}"'] \arrow[dl, "{(c_r,d_r)}"] \\
 & (r,1) \arrow[ul, "{(a_r,b_r)}"] & (r,2) \arrow[l] & \cdots \arrow[l] & (r,n_r-1) \arrow[l] &
\end{tikzcd}
$$

$$
\begin{aligned}
a_i &= \dim_F N_i, & b_i &= \dim (N_i)_{F_i}, \\
c_i &= \dim_{F_i} N_i', & d_i &= \dim (N_i')_G
\end{aligned}
$$

for $i \in \{1, \ldots, r\}$

Λ canonical algebra $\Rightarrow$ $\operatorname{gl.dim} \Lambda \leq 2$

Hence the Euler form q_Λ of Λ is defined

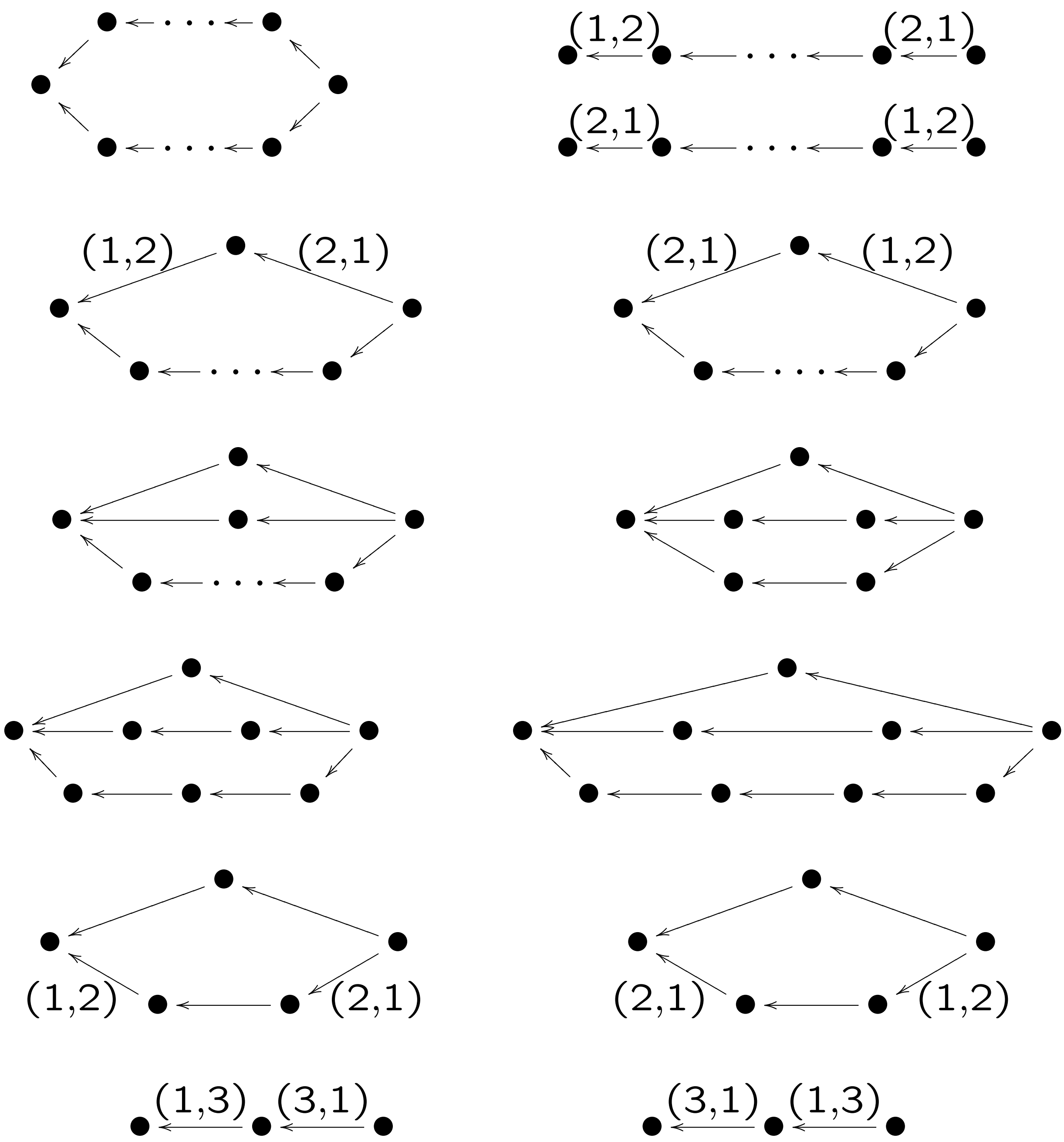
Λ canonical algebra $\Rightarrow$

- q_Λ positive semidefinite of corank one or two, or
- q_Λ is indefinite

Theorem. *Let Λ be a canonical algebra over a field K. The following conditions are equivalent:*

(1) q_Λ is positive semidefinite of corank one.

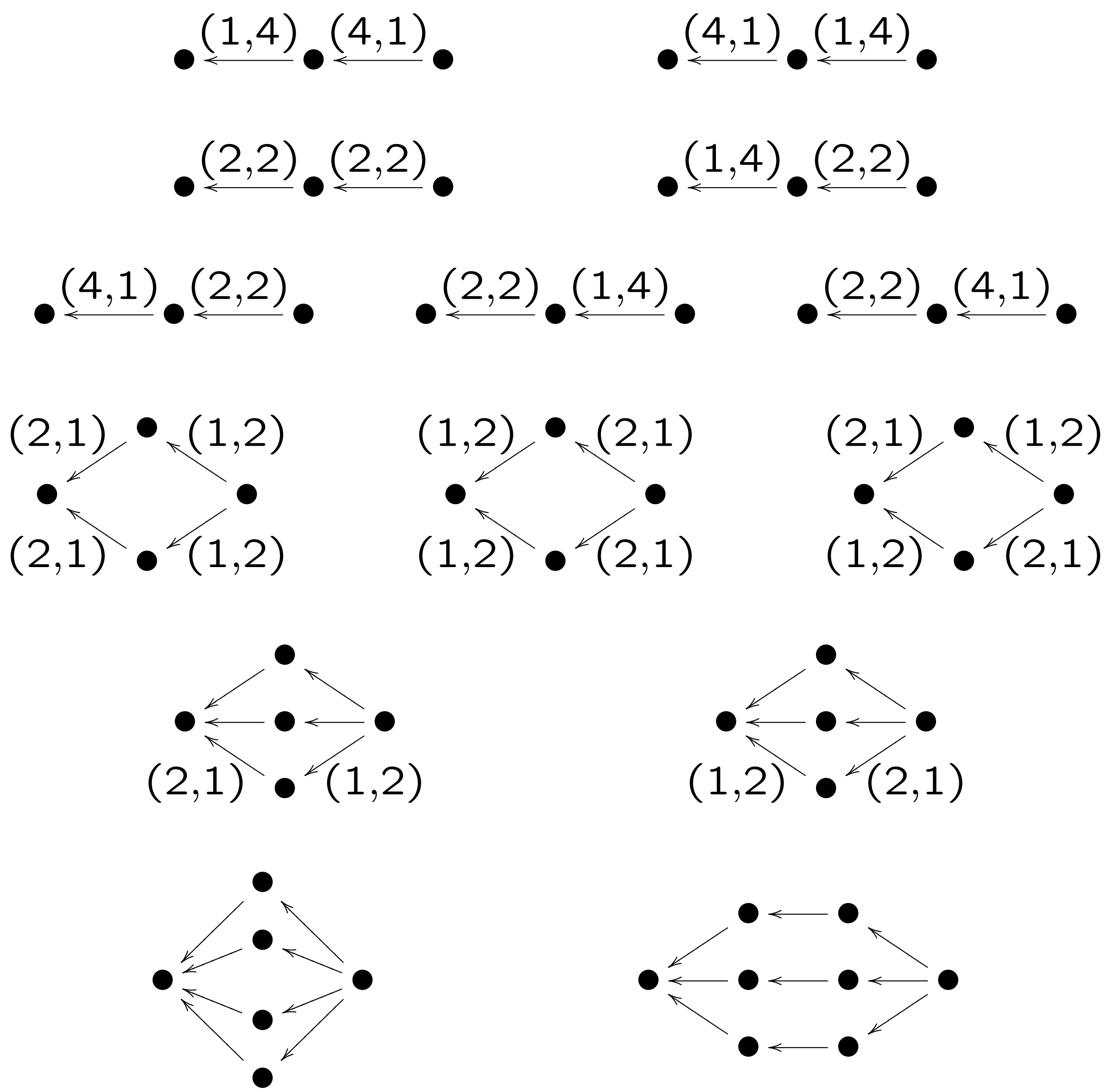
(2) Q_Λ is of one of the following forms

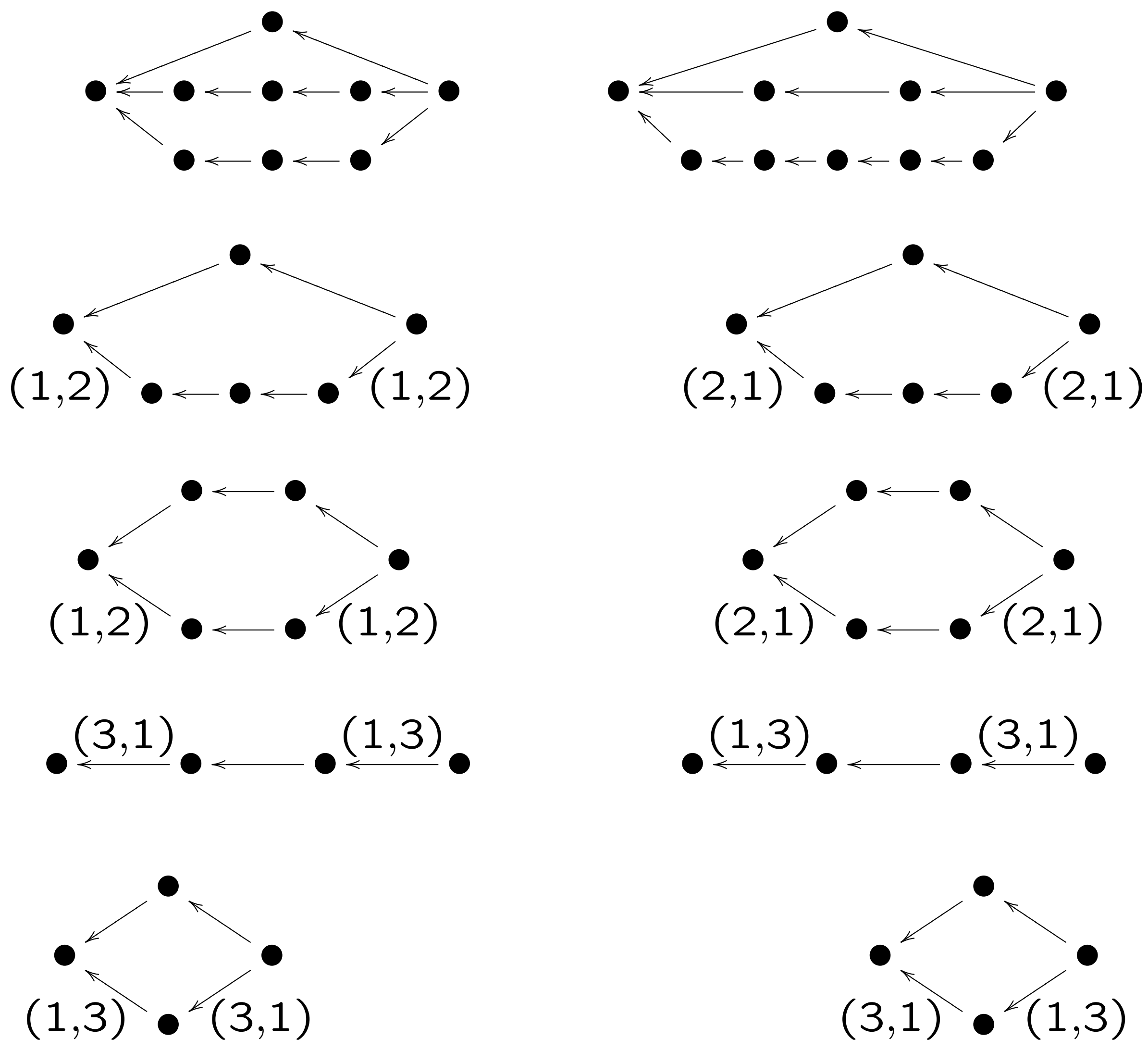


Theorem. *Let Λ be a canonical algebra over a field K. The following conditions are equivalent:*

(1) q_Λ is positive semidefinite of corank two.

(2) Q_Λ is of one of the following forms




(1,2)
(1,2)
(2,1)
(2,1)
(1,2)
(1,2)
(2,1)
(2,1)
(3,1)
(1,3)
(1,3)
(3,1)
(1,3)
(3,1)
(3,1)
(1,3)

Λ canonical algebra over a field K

Λ **canonical algebra of Euclidean type**:
q_Λ is positive semidefinite of corank one

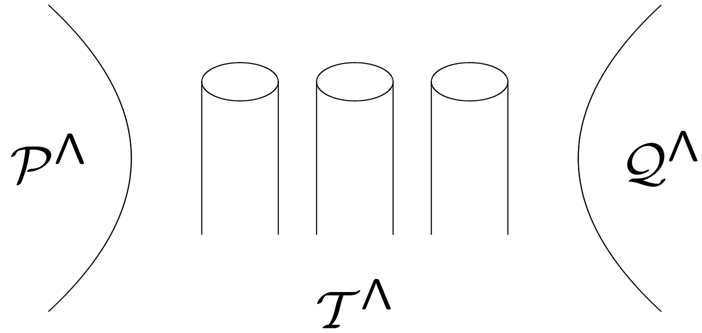
Λ **canonical algebra of tubular type**:
q_Λ is positive semidefinite of corank two

Λ **canonical algebra of wild type**:
q_Λ is indefinite

Q_Λ^* the valued quiver obtained from the valued quiver Q_Λ of Λ by removing the unique source and the arrows attached to it

- Λ canonical algebra of Euclidean type if and only if Q_Λ^* is a Dynkin valued quiver
- Λ canonical algebra of tubular type if and only if Q_Λ^* is a Euclidean valued quiver

Theorem (**Ringel**). *Let Λ be a canonical algebra of type $(n_1, \ldots, n_r)$ over a field K. Then the general shape of the Auslander-Reiten quiver Γ_Λ of Λ is as follows*



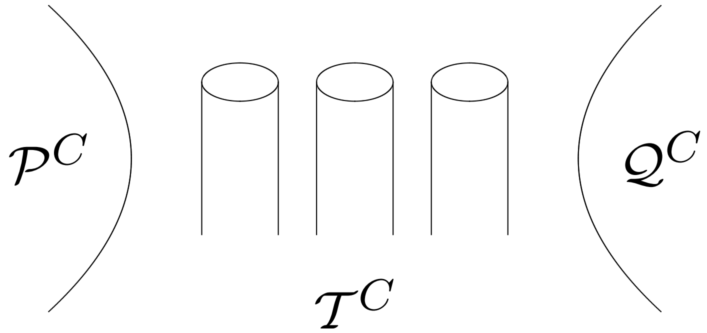
- $\mathcal{P}^\Lambda$ *is a family of components containing a unique postprojective component $\mathcal{P}(\Lambda)$ and all indecomposable projective Λ-modules.*
- $\mathcal{Q}^\Lambda$ *is a family of components containing a unique preinjective component $\mathcal{Q}(\Lambda)$ and all indecomposable injective Λ-modules.*
- $\mathcal{T}^\Lambda$ *is an infinite family of faithful pairwise orthogonal generalized standard stable tubes, having stable tubes of ranks $n_1, \ldots, n_r$ and the remaining tubes of rank one.*
- $\mathcal{T}^\Lambda$ *separates $\mathcal{P}^\Lambda$ from $\mathcal{Q}^\Lambda$.*
- $\mathrm{pd}_\Lambda X \le 1$ *for all modules X in $\mathcal{P}^\Lambda \cup \mathcal{T}^\Lambda$.*
- $\mathrm{id}_\Lambda Y \le 1$ *for all modules Y in $\mathcal{T}^\Lambda \cup \mathcal{Q}^\Lambda$.*
- $\mathrm{gl.\,dim}\,\Lambda \le 2$.

Let Λ be a canonical algebra of type $(n_1, \ldots, n_r)$

T tilting module in $\operatorname{add} \mathcal{P}^\Lambda$

$C = \operatorname{End}_\Lambda(T)$ **concealed canonical algebra of type Λ**

The general shape of Γ_C is a as follows



- $\mathcal{P}^C = \operatorname{Hom}_\Lambda(T, \mathcal{T}(T) \cap \mathcal{P}^\Lambda) \cup \operatorname{Ext}^1_\Lambda(T, \mathcal{F}(T))$ is a family of components containing a unique postprojective component $\mathcal{P}(C)$ and all indecomposable projective C-modules.
- $\mathcal{Q}^C = \operatorname{Hom}_\Lambda(T, \mathcal{Q}^\Lambda)$ is a family of components containing a unique preinjective component $\mathcal{Q}(C)$ and all indecomposable injective C-modules.
- $\mathcal{T}^C = \operatorname{Hom}_\Lambda(T, \mathcal{T}^\Lambda)$ is an infinite family of faithful pairwise orthogonal generalized standard stable tubes, having stable tubes of ranks $n_1, \ldots, n_r$ and the remaining tubes of rank one.
- $\mathcal{T}^C$ separates $\mathcal{P}^C$ from $\mathcal{Q}^C$.
- $\operatorname{pd}_C X \leq 1$ for all modules X in $\mathcal{P}^C \cup \mathcal{T}^C$.
- $\operatorname{id}_C Y \leq 1$ for all modules Y in $\mathcal{T}^C \cup \mathcal{Q}^C$.
- $\operatorname{gl.dim} C \leq 2$.

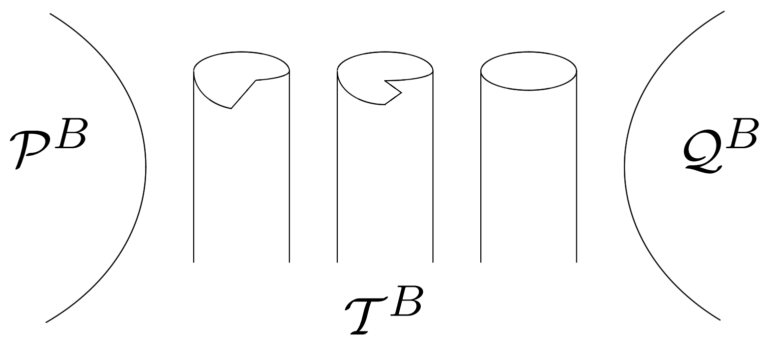
$C \cong \mathrm{End}_\Lambda(T)$, T tilting module in $\mathrm{add}\,\mathcal{P}^\Lambda$, if and only if $C \cong \mathrm{End}_\Lambda(T')$, T' tilting module in $\mathrm{add}\,\mathcal{Q}^\Lambda$.

Λ canonical algebra

T tilting module in $\mathrm{add}(\mathcal{P}^\Lambda \cup \mathcal{T}^\Lambda)$

$B = \mathrm{End}_\Lambda(T)$ **almost concealed canonical algebra** of type Λ

The general shape of Γ_B is as follows



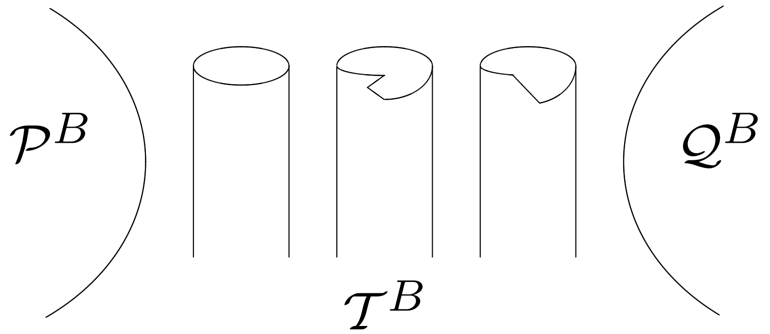
- $\mathcal{P}^B = \mathcal{P}^C$ for a concealed canonical factor algebra C of B.
- $\mathcal{Q}^B$ a family of components containing a unique preinjective component $\mathcal{Q}(B)$ and all indecomposable injective B-modules.
- $\mathcal{T}^B$ an infinite family of pairwise orthogonal generalized standard ray tubes, separating $\mathcal{P}^B$ from $\mathcal{Q}^B$.
- $\mathrm{pd}_B X \le 1$ for all modules X in $\mathcal{P}^B \cup \mathcal{T}^B$.
- $\mathrm{id}_B Y \le 1$ for all modules Y in $\mathcal{T}^B$.
- $\mathrm{gl.dim}\, B \le 2$.

Λ canonical algebra

T tilting module in add$(\mathcal{T}^\Lambda \cup \mathcal{Q}^\Lambda)$

$B = \mathrm{End}_\Lambda(T)$

The general shape of Γ_B is as follows



- $\mathcal{P}^B$ a family of components containing a unique postprojective component $\mathcal{P}(B)$ and all indecomposable projective B-modules.
- $\mathcal{Q}^B = \mathcal{Q}^C$ for a concealed canonical factor algebra C of B.
- $\mathcal{T}^B$ an infinite family of pairwise orthogonal generalized standard coray tubes, separating $\mathcal{P}^B$ from $\mathcal{Q}^B$.
- $\mathrm{pd}_B X \leq 1$ for all modules X in $\mathcal{P}^B$.
- $\mathrm{id}_B Y \leq 1$ for all modules Y in $\mathcal{T}^B \cup \mathcal{Q}^B$.
- $\mathrm{gl.dim}\, B \leq 2$.

$B \cong \mathrm{End}_\Lambda(T)$, T tilting module in add$(\mathcal{T}^\Lambda \cup \mathcal{Q}^\Lambda)$, if and only if $B^{\mathrm{op}} \cong \mathrm{End}_\Lambda(T')$, T' tilting module in add$(\mathcal{P}^\Lambda \cup \mathcal{T}^\Lambda)$ (B^{op} almost concealed canonical algebra)

Almost concealed canonical algebras of Euclidean type

Theorem. *(1) The class of concealed canonical algebras of Euclidean type coincides with the class of concealed algebras of Euclidean type.*

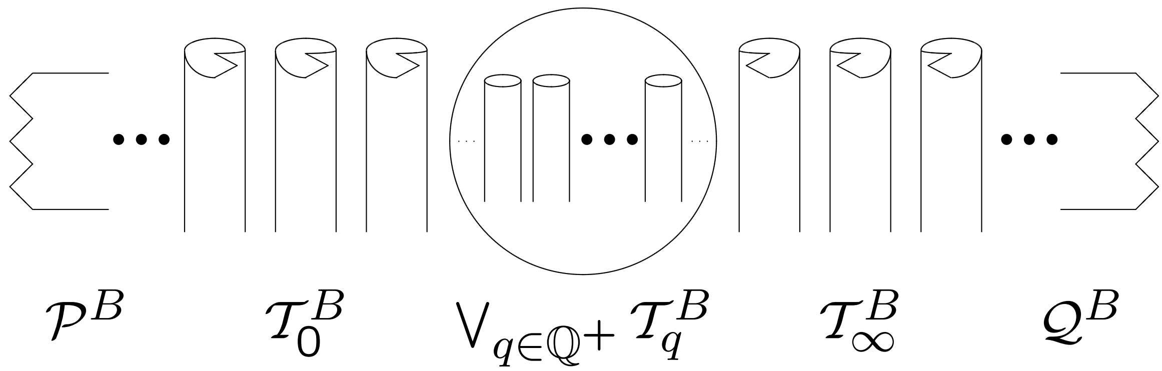
(2) The class of almost concealed canonical algebras of Euclidean types coincides with the class of tilted algebras of the form $\mathrm{End}_H(T)$*, where* H *is a hereditary algebra of a Euclidean type and* T *is a tilting* H*-module without preinjective direct summands.*

(3) The class of the opposite algebras of almost concealed canonical algebras of Euclidean types coincides with the class of tilted algebras of the form $\mathrm{End}_H(T)$*, where* H *is a hereditary algebra of a Euclidean type and* T *is a tilting* H*-module without postprojective direct summands.*

(4) An algebra A *is a representation-infinite tilted algebra of a Euclidean type if and only if* A *is isomorphic to* B *or* B^{op}*, for an almost concealed canonical algebra* B *of a Euclidean type.*

Tubular algebra = almost concealed canonical algebra of tubular type

Theorem. *Let B be a tubular algebra. Then the Auslander-Reiten quiver Γ_B of B is of the form*



where $\mathcal{P}^B$ is a postprojective component with a Euclidean section, $\mathcal{Q}^B$ is a preinjective component with a Euclidean section, $\mathcal{T}_0^B$ is an infinite family of pairwise orthogonal generalized standard ray tubes containing at least one indecomposable projective B-module, $\mathcal{T}_\infty^B$ is an infinite family of pairwise orthogonal generalized standard coray tubes containing at least one indecomposable injective B-module, and each $\mathcal{T}_q^B$, for $q \in \mathbb{Q}^+$ (the set of positive rational numbers) is an infinite family of pairwise orthogonal faithful generalized standard stable tubes.

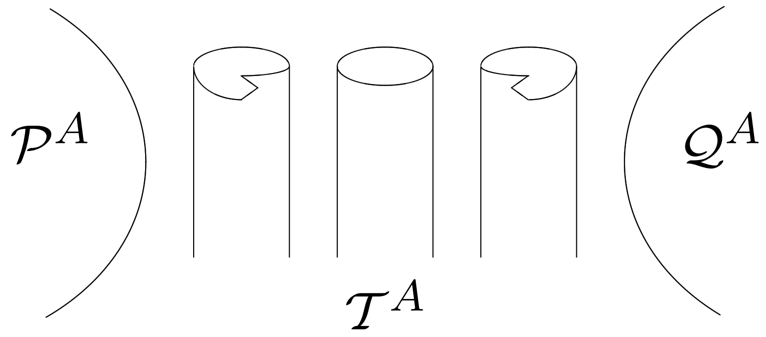
Quasitilted algebra of canonical type – an algebra A of the form $\mathrm{End}_{\mathscr{H}}(T)$, where T is a tilting object in an abelian hereditary K-category $\mathscr{H}$ whose derived category $D^b(\mathscr{H})$ of $\mathscr{H}$ is equivalent, as a triangulated category, to the derived category $D^b(\mathrm{mod}\,\Lambda)$ of the module category $\mathrm{mod}\,\Lambda$ of a canonical algebra Λ over K.

Theorem (**Happel-Reiten**). *Let A be a finite dimensional quasitilted K-algebra over a field K. Then A is either a tilted algebra or a quasitilted algebra of canonical type.*

Theorem (**Lenzing-Skowroński**). *Let A be a finite dimensional K-algebra over a field K. The following conditions are equivalent:*

(1) A is a representation-infinite quasitilted algebra of canonical type.

(2) Γ_A admits a separating family $\mathcal{T}^A$ of pairwise orthogonal generalized standard semiregular (ray or coray) tubes.



- $\mathrm{Hom}_A(\mathcal{T}^A, \mathcal{P}^A) = 0$, $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{T}^A) = 0$, $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$
- every homomorphism $f : X \to Y$ with X in $\mathcal{P}^A$ and Y in $\mathcal{Q}^A$ factorizes through a module Z from $\mathrm{add}\,\mathcal{T}^A$

Moreover, A admits factor algebras A_l (left part of A) and A_r (right part of A) such that

- A_l is almost concealed of canonical type and $\mathcal{P}^A = \mathcal{P}^{A_l}$
- A_r^{op} is almost concealed of canonical type and $\mathcal{Q}^A = \mathcal{Q}^{A_r}$

Example. Let $A = KQ/I$ where Q is the quiver

$$\begin{tikzcd}
 & 4 & (1,1) \arrow[l, "\sigma"'] \arrow[dll, "\alpha_1"'] & & & \\
0 & (2,1) \arrow[l, "\beta_1"'] & & (2,2) \arrow[ll, "\beta_2"'] & & \omega \arrow[ulll, "\alpha_2"'] \arrow[ll, "\beta_3"'] \arrow[dll, "\gamma_3"] \\
 & (3,1) \arrow[ul, "\gamma_1"] & & (3,2) \arrow[ll, "\gamma_2"] & & \\
 & 5 \arrow[u, "\xi"] \arrow[r, "\eta"] & 6 & 7 \arrow[u, "\delta"] \arrow[r, "\varrho"] & 9 & \\
 & & & 8 \arrow[u, "\nu"'] & &
\end{tikzcd}$$

and I is the ideal of KQ generated by the elements

$$\alpha_2\alpha_1 + \beta_3\beta_2\beta_1 + \gamma_3\gamma_2\gamma_1,\ \alpha_2\sigma,\ \xi\gamma_1,\ \delta\gamma_2,\ \nu\varrho$$

Then A is a quasitilted algebra of canonical type

$A_l = KQ^{(l)}/I^{(l)}$ **tubular algebra** of type $(3,3,3)$

$Q^{(l)}$ obtained from Q by removing the vertices $5,6,7,8,9$ and the arrows $\xi,\eta,\delta,\varrho,\nu$

$I^{(l)}$ ideal of $KQ^{(l)}$ generated by $\alpha_2\alpha_1 + \beta_3\beta_2\beta_1 + \gamma_3\gamma_2\gamma_1, \quad \alpha_2\sigma$

$A_r = KQ^{(r)}/I^{(r)}$ almost concealed canonical algebra of wild type $(2,3,8)$

$Q^{(r)}$ obtained from Q by removing the vertex 4 and the arrow σ

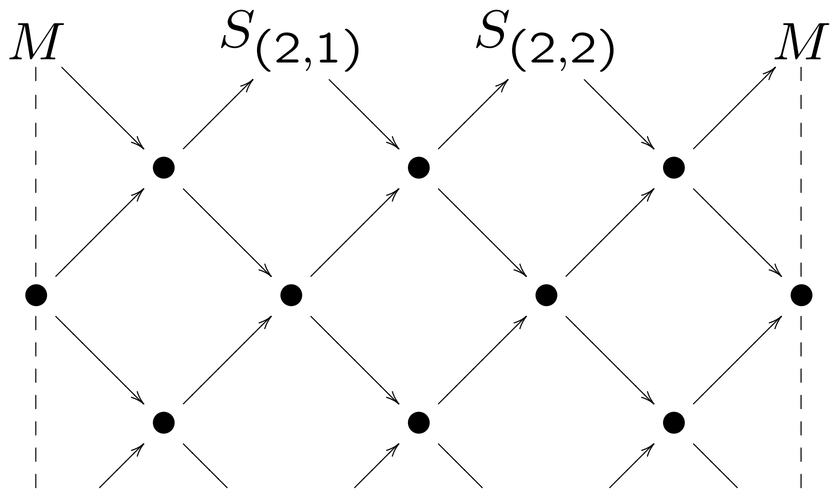
$I^{(r)}$ ideal of $KQ^{(r)}$ generated by $\alpha_2\alpha_1 + \beta_3\beta_2\beta_1 + \gamma_3\gamma_2\gamma_1, \quad \xi\gamma_1, \quad \delta\gamma_2, \quad \nu\varrho$

$$\Gamma_A = \mathcal{P}^A \vee \mathcal{T}^A \vee \mathcal{Q}^A$$

$$\mathcal{P}^A = \mathcal{P}^{A_l}, \qquad \mathcal{Q}^A = \mathcal{Q}^{A_r}$$

$\mathcal{T}^A$ semiregular family of tubes separating $\mathcal{P}^A$ from $\mathcal{Q}^A$

$\mathcal{T}^A$ consists of a stable tube $\mathcal{T}_1^A$ of rank 3

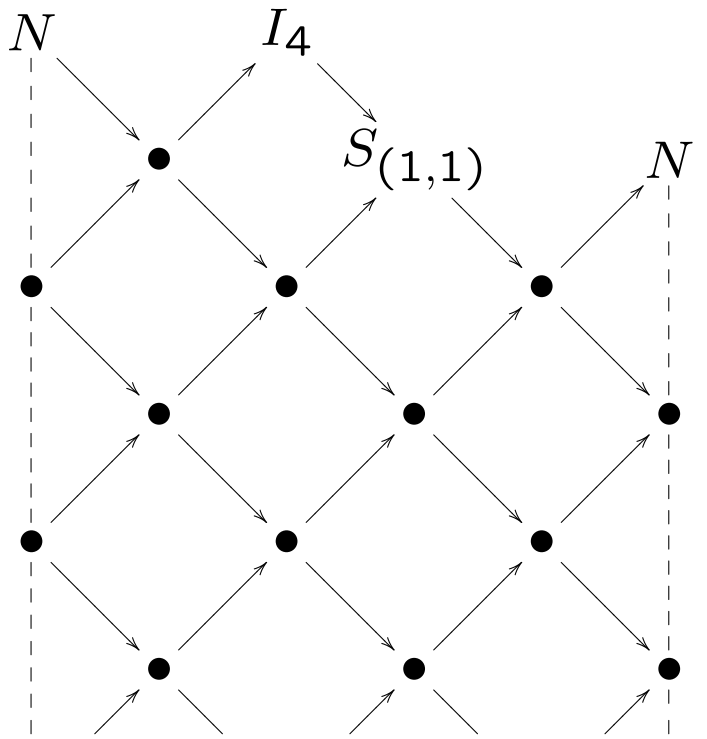


(identifying along the dashed lines)

consisting of indecomposable modules over the canonical algebra $C = K\Delta/J$, where Δ is the full subquiver of Q given by the vertices 0, ω, $(1,1)$, $(2,1)$, $(2,2)$, $(3,1)$, $(3,2)$ and J is the ideal of $K\Delta$ generated by $\alpha_2\alpha_1 + \beta_3\beta_2\beta_1 + \gamma_3\gamma_2\gamma_1$

$$\begin{tikzcd}
 & & & K \arrow[dll, "1"'] & & \\
M: & K & 0 \arrow[l] & & 0 \arrow[ll] & K \arrow[l] \arrow[ull, "1"'] \arrow[dl, "1"] \\
 & & K \arrow[ul, "-1"] & & K \arrow[ll, "1"] &
\end{tikzcd}$$

a coray tube $\mathcal{T}_0^A$ of the form

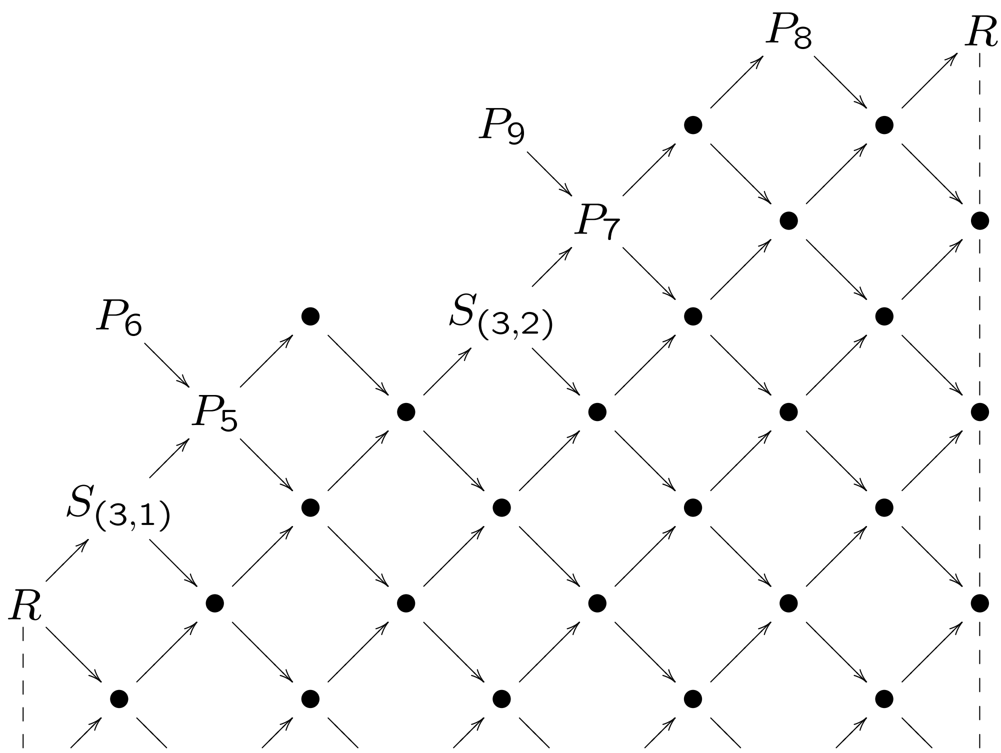


(identifying along the dashed lines)

obtained from the stable tube $\mathcal{T}_0^C$ of Γ_C of rank 2, with $S_{(1,1)}$ and N on the mouth, by one coray insertion

$$N : \quad \begin{tikzcd}
 & & 0 \arrow[dll] & & \\
K & K \arrow[l, "1"'] & & K \arrow[ll, "1"'] & K \arrow[l, "1"'] \arrow[ull] \arrow[dl, "1"] \\
 & K \arrow[ul, "-1"] & & K \arrow[ll, "1"] &
\end{tikzcd}$$

a ray tube $\mathcal{T}_2^A$ of the form



(identifying along the dashed lines)

obtained from the stable tube $\mathcal{T}_2^C$ of rank 3, with $S_{(3,1)}$, $S_{(3,2)}$ and R on the mouth, by 5 ray insertions

$$R: \quad \begin{tikzcd}
 & K \arrow[dl, "1"'] & & \\
K & K \arrow[l, "-1"'] & K \arrow[l, "1"'] & K \arrow[l, "1"'] \arrow[ull, "1"'] \arrow[dl] \\
 & 0 \arrow[ul] & 0 \arrow[l] &
\end{tikzcd}$$

and the infinite family of stable tubes of rank 1, consisting of indecomposable C-modules

7. Double tilted algebras

Theorem (Happel–Reiten–Smalø). *Let A be a finite dimensional K-algebra such that each indecomposable X in* $\operatorname{mod} A$ *satisfies* $\operatorname{pd}_A X \leq 1$ *or* $\operatorname{id}_A X \leq 1$. *Then* $\operatorname{gl.dim} A \leq 3$.

Following **Coelho and Lanzilotta** a finite dimenisional K-algebra A is said to be

- **shod** (small homological dimension) if every indecomposable module X in $\operatorname{mod} A$ satisfies $\operatorname{pd}_A X \leq 1$ or $\operatorname{id}_A X \leq 1$.
- **strict shod** if A is shod and $\operatorname{gl.dim} A = 3$.

Theorem (Coelho–Lanzilotta). *Let A be a finite dimensional K-algebra over a field K. The following conditions are equivalent:*

(1) A is a shod algebra.

(2) $\operatorname{ind} A = \mathcal{L}_A \cup \mathcal{R}_A$.

(3) There exists a splitting torsion pair $(\mathcal{Y}, \mathcal{X})$ in $\operatorname{mod} A$ *such that* $\operatorname{pd}_A Y \leq 1$, *for each module $Y \in \mathcal{Y}$ (torsion-free part), and* $\operatorname{id}_A X \leq 1$, *for each module $X \in \mathcal{X}$ (torsion part).*

Theorem. *Let A be a shod algebra. The following conditions are equivalent:*

(1) A is a strict shod algebra.

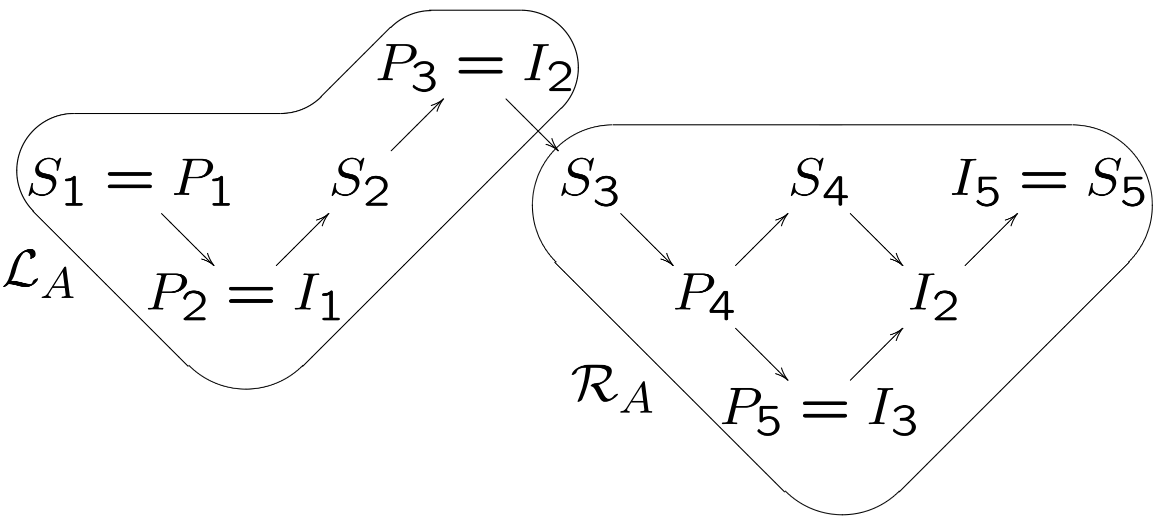
(2) $\mathcal{L}_A \setminus \mathcal{R}_A$ contains an indecomposable injective A-module.

(3) $\mathcal{R}_A \setminus \mathcal{L}_A$ contains an indecomposable projective A-module.

Example. $A = KQ/I$, Q the quiver

$$1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\gamma} 4 \xleftarrow{\sigma} 5$$

I ideal of KQ generated by $\beta\alpha$ and $\gamma\beta$. The Auslander-Reiten quiver Γ_A is of the form



$$0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow P_3 \longrightarrow P_4 \longrightarrow S_4 \longrightarrow 0$$

minimal projective resolution of S_4, so $\mathrm{pd}_A\, S_4 = 3$.

A strict shod algebra

A finite dimensional K-algebra over a field K

$\mathscr{C}$ a component of Γ_A.

A full translation subquiver Δ of $\mathscr{C}$ is said to be a **double section** of $\mathscr{C}$ if the following conditions are satisfied:

(a1) Δ is acyclic.

(a2) Δ is convex in $\mathscr{C}$.

(a3) For each τ_A-orbit $\mathcal{O}$ in $\mathscr{C}$, we have $1 \leq |\Delta \cap \mathcal{O}| \leq 2$.

(a4) If $\mathcal{O}$ is a τ_A-orbit $\mathcal{O}$ in $\mathscr{C}$ and $|\Delta \cap \mathcal{O}| = 2$ then $\Delta \cap \mathcal{O} = \{X, \tau_A X\}$, for some module $X \in \mathscr{C}$, and there exist sectional paths $I \to \cdots \to \tau_A X$ and $X \to \cdots \to P$ in $\mathscr{C}$ with I injective and P projective.

A double section Δ in $\mathscr{C}$ with $|\Delta \cap \mathcal{O}| = 2$, for some τ_A-orbit $\mathcal{O}$ in $\mathscr{C}$, is said to be a **strict double section** of $\mathscr{C}$.

A path $X_0 \to X_1 \to \cdots \to X_m$, with $m \geq 2$, in an Auslander-Reiten quiver Γ_A is said to be **almost sectional** if there exists exactly one index $i \in \{2, \ldots, m\}$ such that $X_{i-2} \cong \tau_A X_i$.

For a double section Δ of $\mathscr{C}$, we define the full subquivers of Δ:

$$\Delta'_l = \left\{ X \in \Delta; \begin{array}{l} \text{there is an almost sectional} \\ \text{path } X \to \cdots \to P \text{ with } P \\ \text{projective} \end{array} \right\},$$

$$\Delta'_r = \left\{ X \in \Delta; \begin{array}{l} \text{there is an almost sectional} \\ \text{path } I \to \cdots \to X \text{ with } I \text{ in-} \\ \text{jective} \end{array} \right\},$$

$\Delta_l = (\Delta \setminus \Delta'_r) \cup \tau_A \Delta'_r$, **left part** of Δ,

$\Delta_r = (\Delta \setminus \Delta'_l) \cup \tau_A^{-1} \Delta'_l$, **right part** of Δ.

Δ is a section if and only if $\Delta_l = \Delta = \Delta_r$

An indecomposable finite dimensional K-algebra B is said to be a **double tilted algebra** if the following conditions are satisfied:

(1) Γ_B admits a component $\mathscr{C}$ with a faithful double section Δ.

(2) There exists a tilted quotient algebra $B^{(l)}$ of B (not necessarily indecomposable) such that Δ_l is a disjoint union of sections of the connecting components of the indecomposable parts of $B^{(l)}$ and the category of all predecessors of Δ_l in $\operatorname{ind} B$ coincides with the category of all predecessors of Δ_l in $\operatorname{ind} B^{(l)}$.

(3) There exists a tilted quotient algebra $B^{(r)}$ of B (not necessarily indecomposable) such that Δ_r is a disjoint union of sections of the connecting components of the indecomposable parts of $B^{(r)}$, and the category of all successors of Δ_r in $\operatorname{ind} B$ coincides with the category of all successors of Δ_r in $\operatorname{ind} B^{(r)}$.

B is a **strict double tilted algebra** if the double section Δ is strict

$B^{(l)}$ **left tilted algebra** of B

$B^{(r)}$ **right tilted algebra** of B

B is a tilted algebra if and only if $B = B^{(l)} = B^{(r)}$

Theorem (**Reiten-Skowroński**). *An indecomposable finite dimensional K-algebra A is a double tilted algebra if and only if the quiver Γ_A contains a component $\mathscr{C}$ with a faithful double section Δ such that* $\mathrm{Hom}_A(U, \tau_A V) = 0$*, for all modules $U \in \Delta_r$ and $V \in \Delta_l$.*

Theorem (**Reiten-Skowroński**). *Let A be an indecomposable finite dimensional K-algebra. The following conditions are equivalent:*

(1) A is a strict shod algebra.

(2) A is a strict double tilted algebra.

(3) Γ_A admits a component $\mathscr{C}$ with a faithful strict double section Δ such that $\mathrm{Hom}_A(U, \tau_A V) = 0$*, for all modules $U \in \Delta_r$ and $V \in \Delta_l$.*

Corollary. *An indecomposable finite dimensional K-algebra A is a shod algebra if and only if A is one of the following*

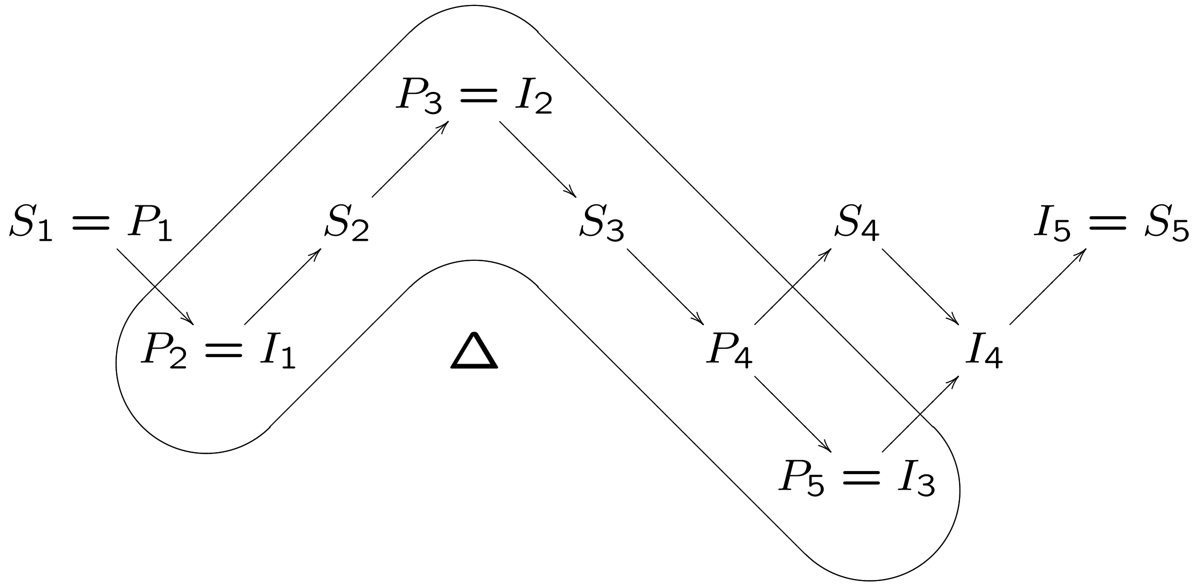
- *a tilted algebra,*
- *a strict double tilted algebra,*
- *a quasitilted algebra of canonical algebra.*

Example. $A = KQ/I$, Q the quiver

$$1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\gamma} 4 \xleftarrow{\sigma} 5$$

I ideal of KQ generated by $\beta\alpha$ and $\gamma\beta$

Γ_A is of the form



Δ faithful double section of $\mathscr{C} = \Gamma_A$

$\Delta_l' = \{P_2, S_2\}$

$\Delta_r' = \{S_3, P_4, P_5\}$

$\Delta_l = (\Delta \setminus \Delta_r') \cup \tau_A \Delta_r' = \{P_2, S_2, P_3\}$

$\Delta_r = (\Delta \setminus \Delta_l') \cup \tau_A^{-1} \Delta_l' = \{P_3, S_3, P_4, P_5\}$

$A^{(l)}$ left tilted algebra of A is hereditary of Dynkin type $\mathbb{A}_3$

$A^{(r)}$ right tilted algebra of A is hereditary of Dynkin type $\mathbb{A}_4$

B strict double tilted algebra

Γ_B admits a unique component $\mathscr{C} = \mathscr{C}_B$ with a faithful double section Δ

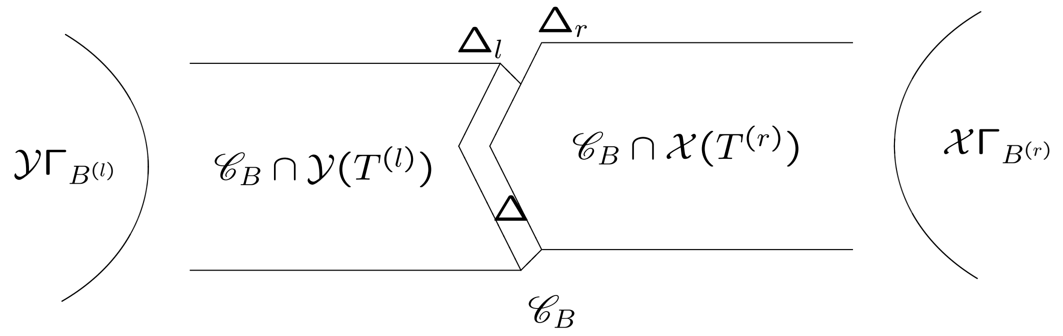
Moreover,

$$\Gamma_B = \mathcal{Y}\Gamma_{B^{(l)}} \cup \mathscr{C}_B \cup \mathcal{X}\Gamma_{B^{(r)}},$$

where

- $\mathcal{Y}\Gamma_{B^{(l)}}$ is the disjoint union of all components of $\Gamma_{B^{(l)}}$ contained entirely in the torsion-free part $\mathcal{Y}(T^{(l)})$ of $\operatorname{mod} B^{(l)}$, determined by a tilting module $T^{(l)}$ over a hereditary algebra $A^{(l)}$ of type Δ_l such that $B^{(l)} \cong \operatorname{End}_{A^{(l)}}(T^{(l)})$.
- $\mathcal{X}\Gamma_{B^{(r)}}$ is the disjoint union of all components of $\Gamma_{B^{(r)}}$ contained entirely in the torsion part $\mathcal{X}(T^{(r)})$ of $\operatorname{mod} B^{(r)}$, determined by a tilting module $T^{(r)}$ over a hereditary algebra $A^{(r)}$ of type Δ_r such that $B^{(r)} \cong \operatorname{End}_{A^{(r)}}(T^{(r)})$.

$\mathscr{C}_B$ **connecting component** of Γ_B



- $\mathrm{Hom}_B(\mathscr{C}_B, \mathcal{Y}\Gamma_{B(l)}) = 0$, $\mathrm{Hom}_B(\mathcal{X}\Gamma_{B(r)}, \mathscr{C}_B) = 0$, $\mathrm{Hom}_B(\mathcal{X}\Gamma_{B(r)}, \mathcal{Y}\Gamma_{B(l)}) = 0$.
- $\mathscr{C}_B$ is generalized standard, contains at least one projective module and at least one injective module.

Theorem (**Skowroński**). *Let A be an indecomposable finite dimensional K-algebra. The following conditions are equivalent:*

(1) A is a double tilted algebra.

(2) $\mathrm{ind}\, A = \mathcal{L}_A \cup \mathcal{R}_A$ *and* $\mathcal{L}_A \cap (\mathcal{R}_A \cup \tau_A \mathcal{R}_A)$ *contains a directing module.*

(3) $\mathrm{ind}\, A = \mathcal{L}_A \cup \mathcal{R}_A$ *and* $(\mathcal{L}_A \cup \tau_A^{-1} \mathcal{L}_A) \cap \mathcal{R}_A$ *contains a directing module.*

8. Generalized double tilted algebras

A finite dimensional K-algebra

Σ full translation subquiver of Γ_A is said to be **almost acyclic** if all but finitely many modules of Σ do not lie on oriented cycles in Γ_A

$\mathscr{C}$ component of Γ_A

A full translation subquiver Δ of $\mathscr{C}$ is said to be a **multisection** of $\mathscr{C}$ if the following conditions are satisfied:

(1) Δ is almost acyclic.

(2) Δ is convex.

(3) For each τ_A-orbit $\mathcal{O}$ in $\mathscr{C}$, we have $1 \leq |\Delta \cap \mathcal{O}| < \infty$.

(4) $|\Delta \cap \mathcal{O}| = 1$, for all but finitely many τ_A-orbits $\mathcal{O}$ in $\mathscr{C}$.

(5) No proper full convex subquiver of Δ satisfies the conditions (1)–(4).

For a multisection Δ of a component $\mathscr{C}$ of Γ_A we define the following full subquivers of $\mathscr{C}$:

$$\Delta'_l = \left\{ X \in \Delta;\, \begin{array}{l}\text{there is a nonsectional path} \\ X \to \cdots \to P \text{ with } P \text{ projec-} \\ \text{tive}\end{array} \right\},$$

$$\Delta'_r = \left\{ X \in \Delta;\, \begin{array}{l}\text{there is a nonsectional path} \\ I \to \cdots \to X \text{ with } I \text{ injective}\end{array} \right\},$$

$$\Delta''_l = \left\{ X \in \Delta'_l;\, \tau_A^{-1} X \notin \Delta'_l \right\},$$

$$\Delta''_r = \left\{ X \in \Delta'_r;\, \tau_A X \notin \Delta'_r \right\},$$

$\Delta_l = (\Delta \setminus \Delta'_r) \cup \tau_A \Delta''_r$ **left part** of Δ,

$\Delta_r = (\Delta \setminus \Delta'_l) \cup \tau_A^{-1} \Delta''_l$ **right part** of Δ,

$\Delta_c = \Delta'_l \cap \Delta'_r$, **core** of Δ.

Theorem (Reiten-Skowroński). *Let A be a finite dimensional K-algebra. A component $\mathscr{C}$ of Γ_A is almost acyclic if and only if $\mathscr{C}$ admits a multisection.*

Theorem (Reiten-Skowroński). *Let A be a finite dimensional K-algebra, $\mathscr{C}$ a component of Γ_A and Δ a multisection of $\mathscr{C}$. Then*

(1) Every cycle of $\mathscr{C}$ lies in Δ_c.

(2) Δ_c is finite.

(3) Every indecomposable module X in $\mathscr{C}$ is in Δ_c, or a predecessor of Δ_l or a successor of Δ_r in $\mathscr{C}$.

(4) Δ is faithful if and only if $\mathscr{C}$ is faithful.

Δ multisection of a component of Γ_A

$w(\Delta) \in \mathbb{N} \cup \{\infty\}$ **width** of Δ (numerical invariant of Δ)

Take a path p in Δ. Then a subpath q of p

$$M \to Z^{(1)} \to \tau_A^{-1} M \to Z^{(2)} \to \tau_A^{-2} M \to \ldots \to Z^{(n)} \to \tau_A^{-n} M$$

is called a **hook path** of length n (if $n \geq 1$), and q is a **maximal hook subpath** of p if q is not contained in any hook subpath of p of larger length.

We associate to the path p a sequence of maximal hook subpaths of p as follows (if there are hook subpaths of p):

- Start with a maximal hook subpath

$$M \to Z^{(1)} \to \tau_A^{-1} M \to Z^{(2)} \to \tau_A^{-2} M \to \ldots \to Z^{(n)} \to \tau_A^{-n} M$$

  of p, where M is the first module on p which is a source of hook subpath of p.
- Then take a maximal hook subpath of p with the source at the first possible successor of $\tau_A^{-n} M$ on p.
- Continue the process.

$i(p) =$ the sum of lengths of these hook subpaths of p

Then $i(p) = 0$ if and only if the path p is sectional

$w(\Delta) =$ maximum of $i(p) + 1$ for all paths p in Δ

$w(\Delta) \in (\mathbb{N} \setminus \{0\}) \cup \{\infty\}$

A multisection Δ of $\mathscr{C}$ with $w(\Delta) = n$ is called n-**section**.

Observe that

- $w(\Delta) < \infty$ if and only if Δ is acyclic.
- Δ is a 1-section if and only if Δ is a section.
- Δ is a 2-section if and only if Δ is a strict double section.

Proposition. *Let A be an algebra, $\mathscr{C}$ a component of Γ_A and Δ, Σ are multisections of $\mathscr{C}$. Then*

$$\Delta_c = \Sigma_c \quad \textit{and} \quad w(\Delta) = w(\Sigma).$$

Hence the core and the width of a multisection of an almost acyclic component $\mathscr{C}$ of Γ_A are invariants of $\mathscr{C}$.

Every finite component of Γ_A is trivially almost acyclic, and hence admits a multisection.

An indecomposable finite dimensional K-algebra B is said to be a **generalised double tilted algebra** if the following conditions are satisfied:

(1) Γ_B admits a component $\mathscr{C}$ with a faithful multisection Δ.

(2) There exists a tilted quotient algebra $B^{(l)}$ of B (not necessarily indecomposable) such that Δ_l is a disjoint union of sections of the connecting components of the indecomposable parts of $B^{(l)}$ and the category of all predecessors of Δ_l in $\operatorname{ind} B$ coincides with the category of all predecessors of Δ_l in $\operatorname{ind} B^{(l)}$.

(3) There exists a tilted quotient algebra $B^{(r)}$ of B (not necessarily indecomposable) such that Δ_r is a disjoint union of sections of the connecting components of the indecomposable parts of $B^{(r)}$, and the category of all successors of Δ_r in $\operatorname{ind} B$ coincides with the category of all successors of Δ_r in $\operatorname{ind} B^{(r)}$.

B is said to be an n-**double tilted algebra** if Γ_B admits a component $\mathscr{C}$ with a faithful n-section Δ and the conditions (2) and (3) hold.

Observe that every indecomposable algebra of finite representation type is a generalized double tilted algebra.

Theorem (**Reiten-Skowroński**). *Let B be an n-double tilted algebra. Then*

$$\operatorname{gl.dim} B \leq n + 1.$$

Theorem (**Reiten-Skowroński**). *Let A be an indecomposable finite dimensional K-algebra. The following conditions are equivalent:*

(1) A is a generalized double tilted algebra.

(2) Γ_A admits a component $\mathscr{C}$ with a faithful multisection Δ such that $\mathrm{Hom}_A(U, \tau_A V) = 0$, for all modules $U \in \Delta_r$ and $V \in \Delta_l$.

(3) Γ_A admits a faithful generalized standard almost cyclic component.

Corollary. *Let A be an indecomposable finite dimensional K-algebra. The following equivalences hold:*

(1) A is an n-double tilted algebra, for some $n \geq 2$, if and only if Γ_A contains a faithful generalized standard almost cyclic component $\mathscr{C}$ with a nonsectional path from an injective module to a projective module.

(2) A is an n-double tilted algebra, for some $n \geq 3$, if and only if Γ_A contains a faithful generalized standard component $\mathscr{C}$ with a multisection Δ such that $\Delta_c \neq \emptyset$.

A an algebra

$\mathscr{C}$ component of Γ_A

$\mathcal{L}_{\mathscr{C}}$ the set of all modules X in $\mathscr{C}$ such that $\mathrm{pd}_A Y \leq 1$ for any predecessor Y of X in $\mathscr{C}$.

$\mathcal{R}_{\mathscr{C}}$ the set of all modules X in $\mathscr{C}$ such that $\mathrm{id}_A Y \leq 1$ for any successor Y of X in $\mathscr{C}$.

Observe that, if Δ is a multisection of $\mathscr{C}$, then

$$\Delta_c \subseteq \mathscr{C} \setminus (\mathcal{L}_{\mathscr{C}} \cup \mathcal{R}_{\mathscr{C}}).$$

Theorem (**Reiten-Skowroński**). *Let A be an indecomposable finite dimensional K-algebra, $\mathscr{C}$ a faithful component of Γ_A with a multisection Δ, and $\mathscr{C}$ is not semiregular (contains both a projective module and an injective module). Then the following conditions are equivalent:*

(1) $\mathscr{C}$ is generalized standard.

(2) $\mathscr{C} = \mathcal{L}_{\mathscr{C}} \cup \Delta_c \cup \mathcal{R}_{\mathscr{C}}$.

Example. $A = KQ/I$, Q the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xleftarrow{\gamma} 4 \xleftarrow{\sigma} 5 \xleftarrow{\delta} 6 \xleftarrow{\varepsilon} 7 \xleftarrow{\eta} 8$$

I ideal of KQ generated by $\sigma\gamma$, $\delta\sigma$ and $\varepsilon\delta$. Then Γ_A is of the form

$$\begin{tikzcd}[column sep=tiny, row sep=small]
 & P_4 \arrow[dr] & & S_2 \arrow[dr] & & S_1 = I_1 & & & & & & \\
P_3 = S_3 \arrow[ur] \arrow[dr] & & M \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] & & & & P_8 = S_8 \arrow[dr] & & I_6 \arrow[dr] & \\
 & P_2 \arrow[ur] \arrow[dr] & & I_3 \arrow[ur] \arrow[dr] & & & & & & P_7 \arrow[ur] \arrow[dr] & & S_7 = I_7 \\
 & & P_1 \arrow[ur] & & S_4 \arrow[dr] & & S_5 \arrow[dr] & & S_6 \arrow[ur] & & I_8 \arrow[ur] & \\
 & & & & & P_5 = I_4 \arrow[ur] & & P_6 = I_5 \arrow[ur] & & & &
\end{tikzcd}$$

$\Delta^{(1)} = \mathcal{X} \cup \{P_4, M, I_6\}$ $\quad$ $\Delta^{(2)} = \mathcal{X} \cup \{P_4, M, S_8\}$

$\Delta^{(3)} = \mathcal{X} \cup \{S_2, M, I_6\}$ $\quad$ $\Delta^{(4)} = \mathcal{X} \cup \{S_2, M, S_8\}$

$\Delta^{(5)} = \mathcal{X} \cup \{S_2, I_2, I_6\}$ $\quad$ $\Delta^{(6)} = \mathcal{X} \cup \{S_2, I_2, S_8\}$

$\Delta^{(7)} = \mathcal{X} \cup \{S_1, I_2, P_7\}$ $\quad$ $\Delta^{(8)} = \mathcal{X} \cup \{S_1, I_2, S_8\}$

where $\mathcal{X} = \{I_3, S_4, P_5, S_5, P_6, S_6, P_7\}$, are all multisections of $\mathscr{C} = \Gamma_A$. Moreover,

$w(\Delta^{(i)}) = 3$ and $\Delta_c^{(i)} = \{S_5\}$ for $i \in \{1, \ldots, 8\}$

$\operatorname{gl.dim} A = 4 = w(\Delta^{(i)}) + 1$

$$0 \to P_3 \to P_4 \to P_5 \to P_6 \oplus P_8 \to P_7 \to S_7 \to 0$$

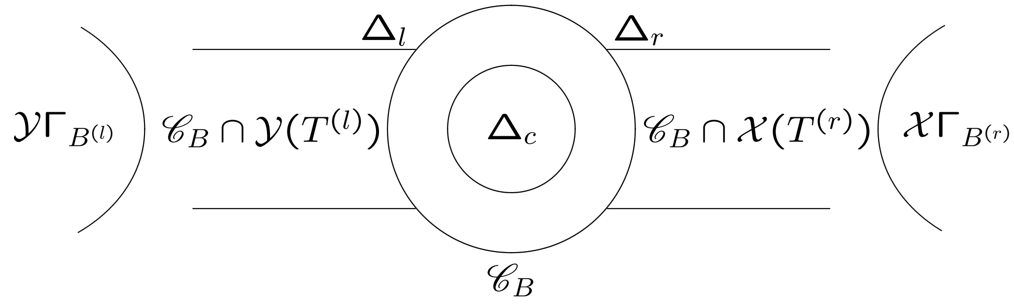
minimal projective resolution of S_7 in $\operatorname{mod} A$, so $\operatorname{pd}_A S_7 = 4$

B n-tilted algebra, $n \geq 2$

Γ_B admits a unique component $\mathscr{C} = \mathscr{C}_B$ with a faithful n-section Δ

$\mathscr{C}_B$ **connecting component** of Γ_B

Γ_B is of the form



- $\mathcal{Y}\Gamma_{B^{(l)}}$ is the disjoint union of all components of $\Gamma_{B^{(l)}}$ contained entirely in the torsion-free part $\mathcal{Y}(T^{(l)})$ of $\operatorname{mod} B^{(l)}$, determined by a tilting module $T^{(l)}$ over a hereditary algebra $A^{(l)}$ of type Δ_l with $B^{(l)} \cong \operatorname{End}_{A^{(l)}} T^{(l)}$.
- $\mathcal{X}\Gamma_{B^{(r)}}$ is the disjoint union of all components of $\Gamma_{B^{(r)}}$ contained entirely in the torsion part $\mathcal{X}(T^{(r)})$ of $\operatorname{mod} B^{(r)}$, determined by a tilting module $T^{(r)}$ over a hereditary algebra $A^{(r)}$ of type Δ_r with $B^{(r)} \cong \operatorname{End}_{A^{(r)}} T^{(r)}$.
- $\operatorname{Hom}_B(\mathscr{C}_B, \mathcal{Y}\Gamma_{B^{(l)}}) = 0$, $\operatorname{Hom}_B(\mathcal{X}\Gamma_{B^{(r)}}, \mathscr{C}_B) = 0$, $\operatorname{Hom}_B(\mathcal{X}\Gamma_{B^{(r)}}, \mathcal{Y}\Gamma_{B^{(l)}}) = 0$.
- $\mathscr{C}_B$ is generalized standard, contains at least one projective module and at least one injective module.

Theorem (Skowroński). *Let B be an indecomposable basic finite dimensional K-algebra over a field K. The following conditions are equivalent:*

(1) *B is either a generalized double tilted algebra or a quasitilted algebra.*

(2) $\operatorname{ind} B \setminus (\mathcal{L}_B \cup \mathcal{R}_B)$ *is finite.*

(3) *There is a finite set $\mathcal{X}$ of modules in* $\operatorname{ind} B$ *such that every path in* $\operatorname{ind} B$ *from an injective module to a projective module consists entirely of modules from $\mathcal{X}$.*

Open problem. *Let B be an indecomposable basic finite dimensional K-algebra over a field K such that, for all but finitely many modules X in* $\operatorname{ind} B$, *we have* $\operatorname{pd}_B X \leq 1$ *or* $\operatorname{id}_B X \leq 1$. *Is then B a generalized double tilted algebra or a quasitilted algebra?*

Confirmed only in special cases

Theorem (**Skowroński**). *Let A be a finite dimensional K-algebra over a field K. The following conditions are equivalent:*

(1) A is a generalized double tilted algebra and Γ_A admits a connecting component $\mathscr{C}_A$ containing all indecomposable projective modules.

(2) $\mathrm{rad}_A^\infty(-, A_A) = 0$.

(3) $\mathrm{id}_A X \leq 1$ *for all but finitely many (up to isomorphism) modules X in* $\mathrm{ind}\, A$.

$\mathscr{C}_A \cap \mathcal{Y}(T^{(l)}) = \mathcal{Y}(T^{(l)})$ *finite* ($\mathcal{Y}\Gamma_{A^{(l)}}$ *empty*)

Theorem (**Skowroński**). *Let A be a finite dimensional K-algebra over a field K. The following conditions are equivalent:*

(1) A is a generalized double tilted algebra and Γ_A admits a connecting component $\mathscr{C}_A$ containing all indecomposable injective modules.

(2) $\mathrm{rad}_A^\infty(D({}_A A), -) = 0$.

(3) $\mathrm{pd}_A X \leq 1$ *for all but finitely many (up to isomorphism) modules X in* $\mathrm{ind}\, A$.

$\mathscr{C}_A \cap \mathcal{X}(T^{(r)}) = \mathcal{X}(T^{(r)})$ *finite* ($\mathcal{X}\Gamma_{A^{(r)}}$ *empty*)

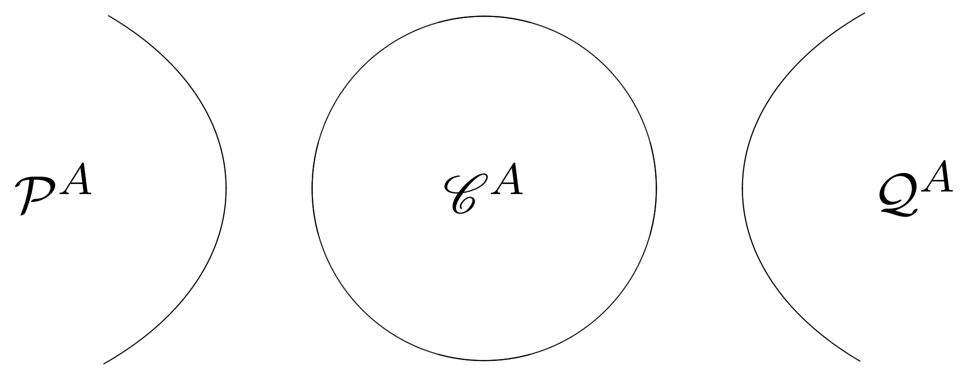
9. Generalized multicoil enlargements of concealed canonical algebras

A finite dimensional K-algebra over a field K

A family $\mathscr{C} = (\mathscr{C}_i)_{i \in I}$ of components of Γ_A is called **separating** in $\operatorname{mod} A$ if the modules in $\operatorname{ind} A$ split into three disjoint classes $\mathcal{P}^A$, $\mathscr{C}^A = \mathscr{C}$ and $\mathcal{Q}^A$ such that

- $\mathscr{C}^A$ is a sincere family of pairwise orthogonal generalized standard components
- $\operatorname{Hom}_A(\mathscr{C}^A, \mathcal{P}^A) = 0$, $\operatorname{Hom}_A(\mathcal{Q}^A, \mathscr{C}^A) = 0$, $\operatorname{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$.
- any homomorphism from $\mathcal{P}^A$ to $\mathcal{Q}^A$ factors through $\operatorname{add} \mathscr{C}^A$.

Then we say that $\mathscr{C}^A$ **separates** $\mathcal{P}^A$ from $\mathcal{Q}^A$. Moreover, then $\mathcal{P}^A$ and $\mathcal{Q}^A$ are uniquely determined in $\operatorname{ind} A$ by $\mathscr{C}^A$.



We write $\Gamma_A = \mathcal{P}^A \vee \mathscr{C}^A \vee \mathcal{Q}^A$

Theorem (Lenzing-Peña). *An indecomposable finite dimensional K-algebra over a field K is a concealed canonical algebra if and only if Γ_A admits a separating family $\mathcal{T}^A$ of stable tubes.*

Theorem (Lenzing-Skowroński). *An indecomposable finite dimensional K-algebra over a field K is a quasitilted algebra of canonical type if and only if Γ_A admits a separating family $\mathcal{T}^A$ of semiregular tubes (ray or coray tubes).*

Theorem (Reiten-Skowroński). *An indecomposable finite dimensional K-algebra over a field K is a generalized double tilted algebra if and only if Γ_A admits a separating almost acyclic component $\mathscr{C}$.*

A finite dimensional K-algebra

$\mathscr{C}$ component of Γ_A

$\mathscr{C}$ is said to be **almost cyclic** if all but finitely many modules of $\mathscr{C}$ lie on oriented cycles of $\mathscr{C}$.

$\mathscr{C}$ is said to be **coherent** if the following two conditions are satisfied:

- For each projective module P in $\mathscr{C}$ there is an infinite sectional path
$$P = X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to \dots$$
in $\mathscr{C}$
- For each injective module I in $\mathscr{C}$ there is an infinite sectional path
$$\cdots \to Y_{i+1} \to Y_i \to \cdots \to Y_2 \to Y_1 = I$$
in $\mathscr{C}$.

Every stable tube (more generally, every semi-regular tube) of Γ_A is an almost cyclic and coherent component

Theorem (**Malicki-Skowroński**). *Let A be a finite dimensional K-algebra and $\mathscr{C}$ be a component of Γ_A. Then $\mathscr{C}$ is almost cyclic and coherent if and only if $\mathscr{C}$ is a generalized multicoil (obtained from a finite family of stable tubes by a sequence of admissible operations).*

For a finite family of $C_1, \ldots, C_m$ of concealed canonical algebras and $C = C_1 \times \cdots \times C_m$ one defines a generalized multicoil enlargement B of C by iterated application of admissible operations (ad 1)–(ad 5) and their dual operations (ad 1*)–(ad 5*).

Theorem (Malicki-Skowroński). *Let A be a finite dimensional K-algebra over a field K. The following statements are equivalent:*

(1) Γ_A admits a separating family of almost cyclic coherent components.

(2) A is a generalized multicoil enlargement of a product C of concealed canonical K-algebras.

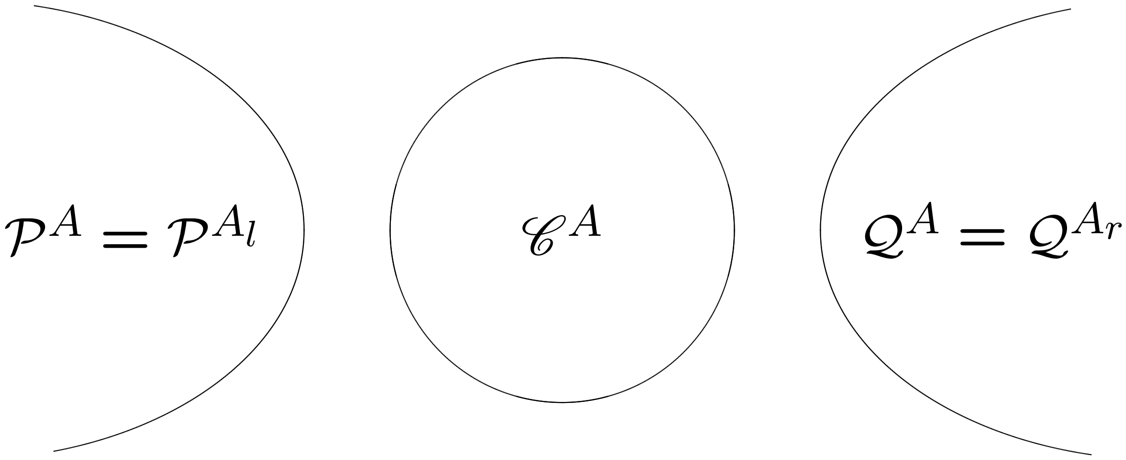
Theorem (Malicki-Skowroński). *Let A be a finite dimensional K-algebra over a field K with a separating family $\mathscr{C}^A$ of almost cyclic coherent components in Γ_A, and $\Gamma_A = \mathcal{P}^A \vee \mathscr{C}^A \vee \mathcal{Q}^A$. Then*

(1) There is a unique factor algebra A_l of A which is a (not necesarily indecomposable) quasitilted algebra of canonical type with a separating family $\mathcal{T}^{A_l}$ of coray tubes such that $\Gamma_{A_l} = \mathcal{P}^{A_l} \vee \mathcal{T}^{A_l} \vee \mathcal{Q}^{A_l}$ and $\mathcal{P}^A = \mathcal{P}^{A_l}$.

(2) There is a unique factor algebra A_r of A which is a (not necesarily indecomposable) quasitilted algebra of canonical type with a separating family $\mathcal{T}^{A_r}$ of ray tubes such that $\Gamma_{A_r} = \mathcal{P}^{A_r} \vee \mathcal{T}^{A_r} \vee \mathcal{Q}^{A_r}$ and $\mathcal{Q}^A = \mathcal{Q}^{A_r}$.

A_l **left quasitilted algebra** of A

A_r **right quasitilted algebra** of A



- Every component of Γ_A not in $\mathscr{C}^A$ lies entirely in $\mathcal{P}^A$ or lies entirely in $\mathcal{Q}^A$
- Every component of Γ_A contained in $\mathcal{P}^A$ is either postprojective, a stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for some $r \geq 1$, of the form $\mathbb{Z}\mathbb{A}_\infty$, or can be obtained from a stable tube or a component of type $\mathbb{Z}\mathbb{A}_\infty$ by a finite number of ray insertions.
- Every component of Γ_A contained in $\mathcal{Q}^A$ is either preinjective, a stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for some $r \geq 1$, of the form $\mathbb{Z}\mathbb{A}_\infty$, or can be obtained from a stable tube or a component of type $\mathbb{Z}\mathbb{A}_\infty$ by a finite number of coray insertions.

Theorem (**Malicki-Skowroński**). *Let A be a finite dimensional K-algebra over a field K with a separating family $\mathscr{C}^A$ of almost cyclic coherent components in Γ_A, and $\Gamma_A = \mathcal{P}^A \vee \mathscr{C}^A \vee \mathcal{Q}^A$. Then the following statements hold:*

(1) $\mathrm{pd}_A X \leq 1$ *for any module X in $\mathcal{P}^A$.*

(2) $\mathrm{id}_A Y \leq 1$ *for any module Y in $\mathcal{Q}^A$.*

(3) $\mathrm{pd}_A Z \leq 2$ *and* $\mathrm{id}_A Z \leq 2$ *for any module Z in $\mathscr{C}^A$.*

(4) $\mathrm{gl.dim}\, A \leq 3$.

One-point extensions and coextensions of algebras

A finite dimensional K-algebra over a field K
F finite dimensional division K-algebra
$M = {}_FM_A$ F-A-bimodule

M_A module in $\operatorname{mod} A$
K acts centrally on ${}_FM_G$
(hence $\dim_K {}_FM = \dim_K M_A$)

One-point extension of A by M is the matrix K-algebra of the form

$$A[M] = \begin{bmatrix} A & 0 \\ {}_FM_A & F \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ m & f \end{bmatrix} ; \begin{matrix} f \in F,\ a \in A, \\ m \in M \end{matrix} \right\}$$

with the usual addition and multiplication. Then the valued quiver $Q_{A[M]}$ of $A[M]$ contains the valued quiver Q_A of A as a convex subquiver, and there is an additional (extension) vertex which is a source. We may identify the category $\operatorname{mod} A[M]$ with the category whose objects are triples (V, X, φ), where $X \in \operatorname{mod} A$, $V \in \operatorname{mod} F$, and $\varphi : V_F \to \operatorname{Hom}_A(M, X)_F$ is an F-linear map. A morphism $h : (V, X, \varphi) \to (W, Y, \psi)$ is given by a pair (f, g), where $f : V \to W$ is F-linear, $g : X \to Y$ is a morphism in $\operatorname{mod} A$ and $\psi f = \operatorname{Hom}_A(M, g)\varphi$. Then the new indecomposable projective $A[M]$-module P is given by the triple $(F, M, \bullet)$, where $\bullet : F_F \to \operatorname{Hom}_A(M, M)_F$ assigns to the identity element of F the identity morphism of M.

An important class of such one-point extensions occurs in the following situation. Let Λ be a finite dimensional K-algebra, P an indecomposable projective Λ-module, ${}_\Lambda\Lambda = P \oplus Q$, and assume that $\mathrm{Hom}_\Lambda(P, Q \oplus \mathrm{rad}\, P) = 0$. Since P is indecomposable projective, $S = P/\mathrm{rad}\, P$ is a simple Λ-module and hence $\mathrm{End}_\Lambda(S)$ is a division K-algebra. Moreover, the canonical homomorphism of algebras $\mathrm{End}_\Lambda(P) \to \mathrm{End}_\Lambda(S)$ is an isomorphism. Then we obtain isomorphisms of algebras

$$\Lambda \cong \mathrm{End}_\Lambda(\Lambda_\Lambda) \cong \begin{bmatrix} A & 0 \\ {}_F M_A & F \end{bmatrix} = A[M],$$

where $F = \mathrm{End}_\Lambda(P)$, $A = \mathrm{End}_\Lambda(Q)$, and $M = {}_F M_A = \mathrm{Hom}_\Lambda(Q, P) \cong \mathrm{rad}\, P$. Clearly K acts centrally on ${}_F M_A$.

Dually, **one-point coextension** of A by M is the matrix K-algebra of the form

$$[M]A = \begin{bmatrix} F & 0 \\ D({}_F M_A) & A \end{bmatrix} = \left\{ \begin{bmatrix} f & 0 \\ x & a \end{bmatrix} ; \begin{matrix} f \in F,\ a \in A, \\ x \in D(M) \end{matrix} \right\}$$

where $D(M) = \mathrm{Hom}_K({}_F M_A, K)$ is an A-F-bimodule.

For a finite dimensional division K-algebra F and $r \geq 1$ natural number, $T_r(F)$ the $r \times r$-lower triangular matrix algebra

$$\begin{bmatrix} F & 0 & 0 & \dots & 0 & 0 \\ F & F & 0 & \dots & 0 & 0 \\ F & F & F & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F & F & F & \dots & F & 0 \\ F & F & F & \dots & F & F \end{bmatrix}$$

A finite dimensional K-algebra

Γ a component of Γ_A

X a module in Γ

$\mathcal{S}(X)$ the **support** of the functor $\mathrm{Hom}_A(X, -)|_\Gamma$ is the K-linear category defined as follows

$\mathcal{H}_X$ the full subcategory of $\operatorname{ind} A$ consisting of the indecomposable modules M in Γ such that $\mathrm{Hom}_A(X, M) \neq 0$,

$\mathcal{I}_X$ the ideal of $\mathcal{H}_X$ consisting of homomorphisms $f : M \to N$ (with M, N in $\mathcal{H}_X$) such that $\mathrm{Hom}_A(X, f) = 0$.

$\mathcal{S}(X) = \mathcal{H}_X / \mathcal{I}_X$ the quotient category

Admissible operations

A finite dimensional K-algebra over a field K

Γ a family of pairwise orthogonal generalized standard infinite components of Γ_A

X indecomposable module in Γ

Assume X is a **brick**: $F = F_X = \mathrm{End}_A(X)$ is a division K-algebra

$X = {}_F X_A$ is an F-A-bimodule, K acts centrally on X

For X with $\mathcal{S}(X)$ of certain shape, called the **pivot**, five admissible operations (ad 1)–(ad 5) and their duals (ad 1*)–(ad 5*) are defined, modifying

A to a new algebra A'

$\Gamma = (\Gamma, \tau)$ to a new translation quiver (Γ', τ')

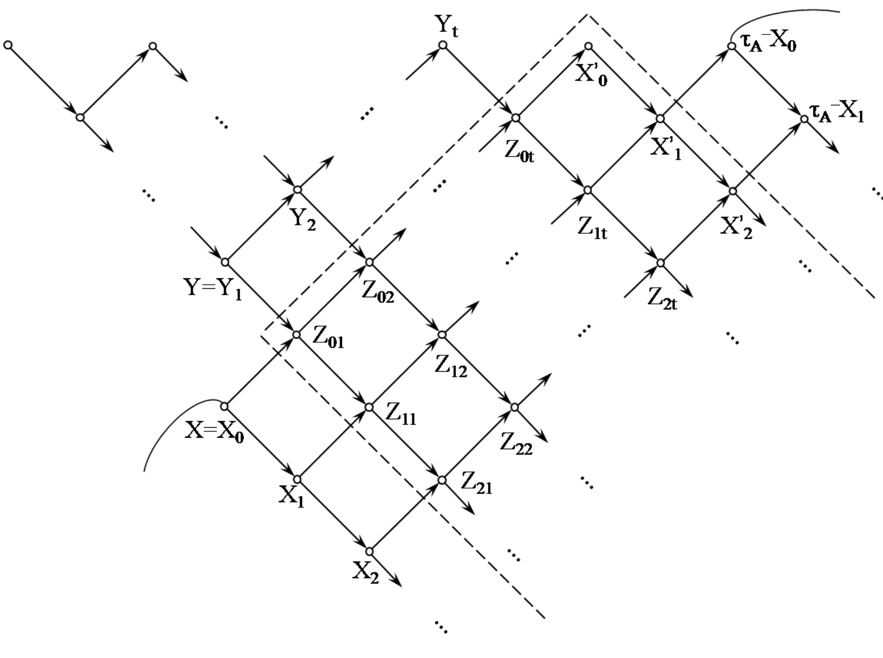
(ad 1) Assume $\mathcal{S}(X)$ consists of an infinite sectional path starting at X:

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

In this case, we let $t \geq 1$ be a positive integer, $D = T_t(F)$ and Y_1, Y_2, ..., Y_t denote the indecomposable injective D-modules with $Y = Y_1$ the unique indecomposable projective-injective D-module. We define the *modified algebra* A' of A to be the one-point extension

$$A' = (A \times D)[X \oplus Y]$$

and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \left(F, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ for $i \geq 0$, $1 \leq j \leq t$, and $X'_i = (F, X_i, 1)$ for $i \geq 0$ as follows:



The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{0j} = Y_{j-1}$ if $j \geq 2, Z_{01}$ is projective, $\tau' X'_0 = Y_t, \tau' X'_i = Z_{i-1,t}$ if $i \geq 1, \tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ, or Γ_D, respectively.

If $t = 0$ we define the modified algebra A' to be the one-point extension $A' = A[X]$ and the modified translation quiver Γ' to be the translation quiver obtained from Γ by inserting only the sectional path consisting of the vertices X'_i, $i \geq 0$.

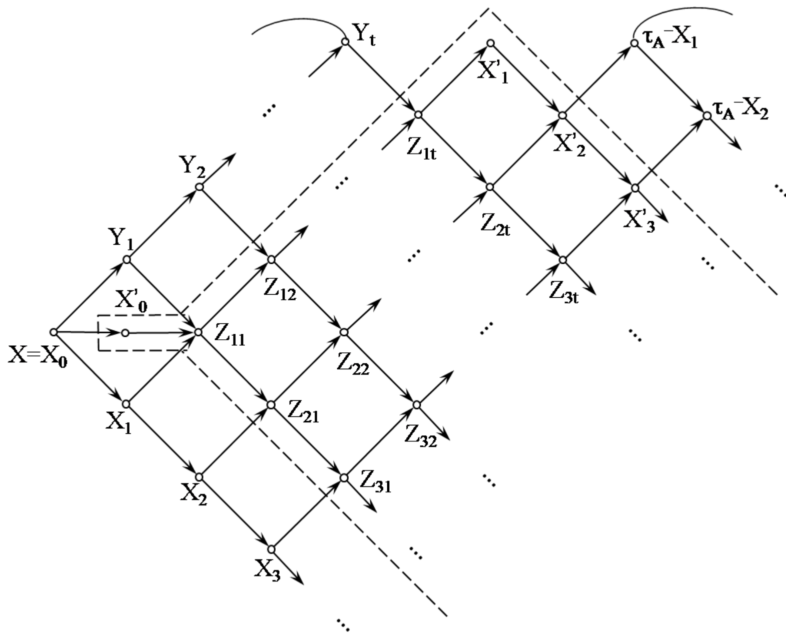
The nonnegative integer t is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t + 1$. We call t the *parameter* of the operation.

In case Γ is a stable tube, it is clear that any module on the mouth of Γ satisfies the condition for being a pivot for the above operation.

(ad 2) Suppose that $\mathcal{S}(X)$ admits two sectional paths starting at X, one infinite and the other finite with at least one arrow:

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$$

where $t \geq 1$. In particular, X is necessarily injective. We define the *modified algebra* A' of A to be the one-point extension $A' = A[X]$ and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \left(F, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ for $i \geq 1$, $1 \leq j \leq t$, and $X'_i = (F, X_i, 1)$ for $i \geq 1$ as follows:



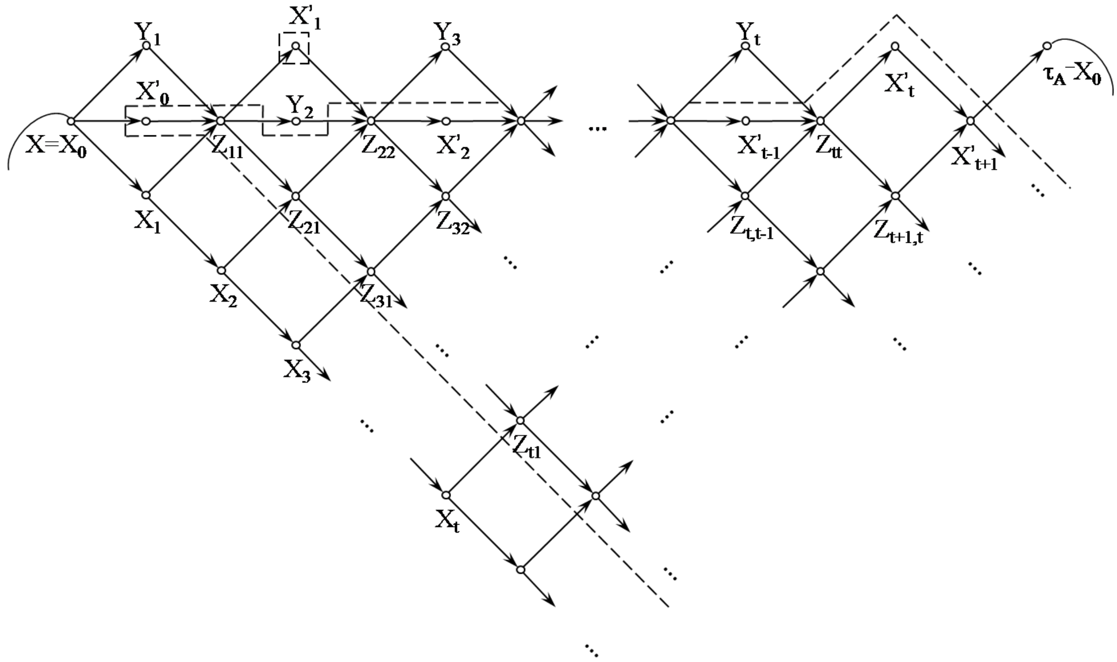
The translation τ' of Γ' is defined as follows: X'_0 is projective-injective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{1j} = Y_{j-1}$ if $j \geq 2, \tau' X'_i = Z_{i-1,t}$ if $i \geq 2, \tau' X'_1 = Y_t, \tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ.

The integer $t \geq 1$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t + 1$. We call t the *parameter* of the operation.

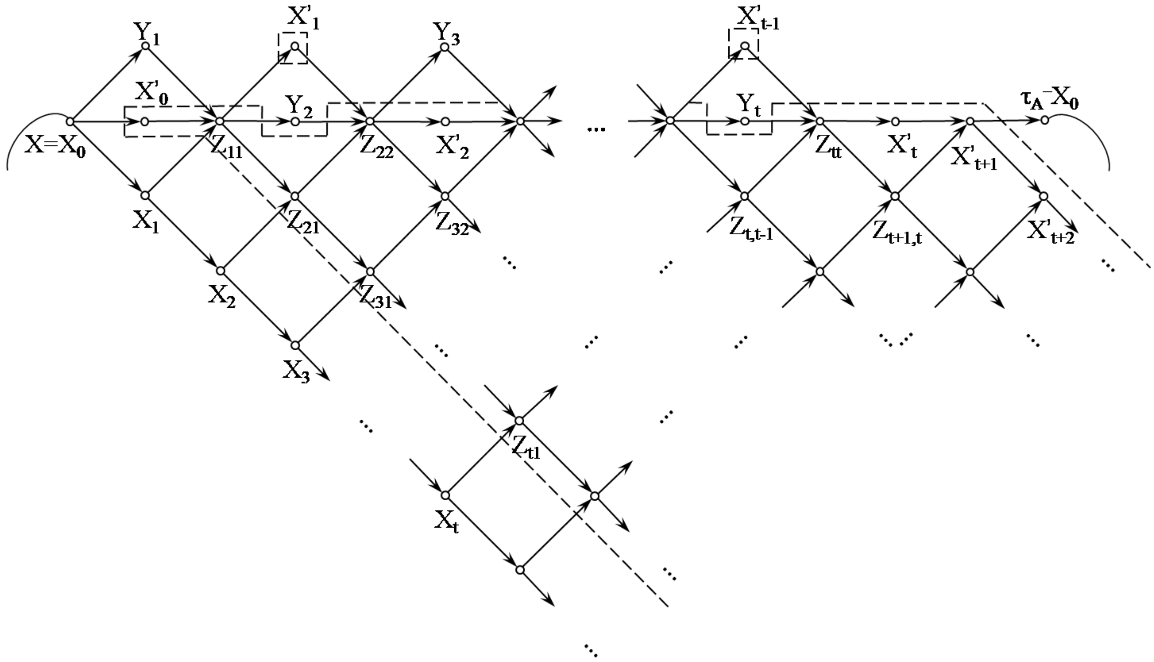
(ad 3) Assume $\mathcal{S}(X)$ is the mesh-category of two parallel sectional paths:

$$\begin{array}{ccccccccccc} & Y_1 & \to & Y_2 & \to & \cdots & \to & Y_t & & & \\ & \uparrow & & \uparrow & & & & \uparrow & & & \\ X = & X_0 & \to & X_1 & \to & \cdots & \to & X_{t-1} & \to & X_t & \to \cdots \end{array}$$

where $t \geq 2$. In particular, X_{t-1} is necessarily injective. Moreover, we consider the translation quiver $\overline{\Gamma}$ of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1}$. We assume that the union $\widehat{\Gamma}$ of connected components of $\overline{\Gamma}$ containing the vertices $\tau_A^{-1} Y_{i-1}$, $2 \leq i \leq t$, is a finite translation quiver. Then $\overline{\Gamma}$ is a disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver Γ^*, containing the pivot X. We define the *modified algebra* A' of A to be the one-point extension $A' = A[X]$ and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangle consisting of the modules $Z_{ij} = \left(F, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ for $i \geq 1$, $1 \leq j \leq t$, and $X_i' = (F, X_i, 1)$ for $i \geq 1$ as follows:



if t is odd, while



if t is even.

The translation τ' of Γ' is defined as follows: X'_0 is projective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2$, $2 \leq j \leq t$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' X'_i = Y_i$ if $1 \leq i \leq t$, $\tau' X'_i = Z_{i-1,t}$ if $i \geq t+1$, $\tau' Y_j = X'_{j-2}$ if $2 \leq j \leq t$, $\tau'(\tau^{-1} X_i) = X'_i$, if $i \geq t$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ^*. We note that X'_{t-1} is injective.

The integer $t \geq 2$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t+1$. We call t the *parameter* of the operation.

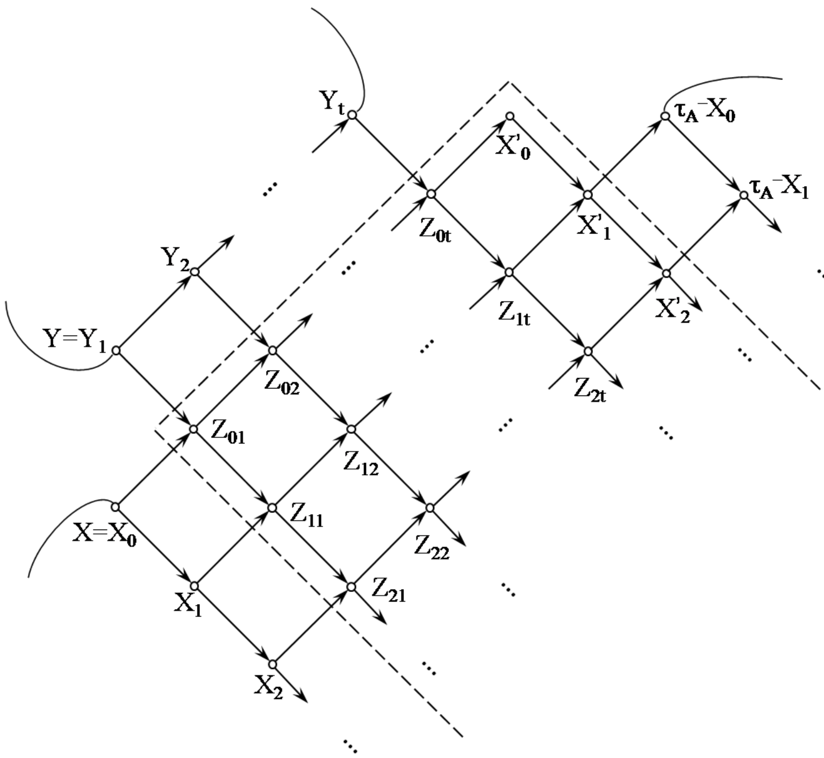
(ad 4) Suppose that $\mathcal{S}(X)$ consists an infinite sectional path, starting at X

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

and

$$Y = Y_1 \to Y_2 \to \cdots \to Y_t$$

with $t \geq 1$, be a finite sectional path in Γ_A such that $F_Y = F = F_X$. Let r be a positive integer. Moreover, we consider the translation quiver $\overline{\Gamma}$ of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1}$. We assume that the union $\widehat{\Gamma}$ of connected components of $\overline{\Gamma}$ containing the vertices $\tau_A^{-1} Y_{i-1}$, $2 \leq i \leq t$, is a finite translation quiver. Then $\overline{\Gamma}$ is a disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver Γ^*, containing the pivot X. For $r = 0$ we define the *modified algebra* A' of A to be the one-point extension $A' = A[X \oplus Y]$ and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangle consisting of the modules $Z_{ij} = \left(F, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ for $i \geq 0$, $1 \leq j \leq t$, and $X_i' = (F, X_i, 1)$ for $i \geq 1$ as follows:

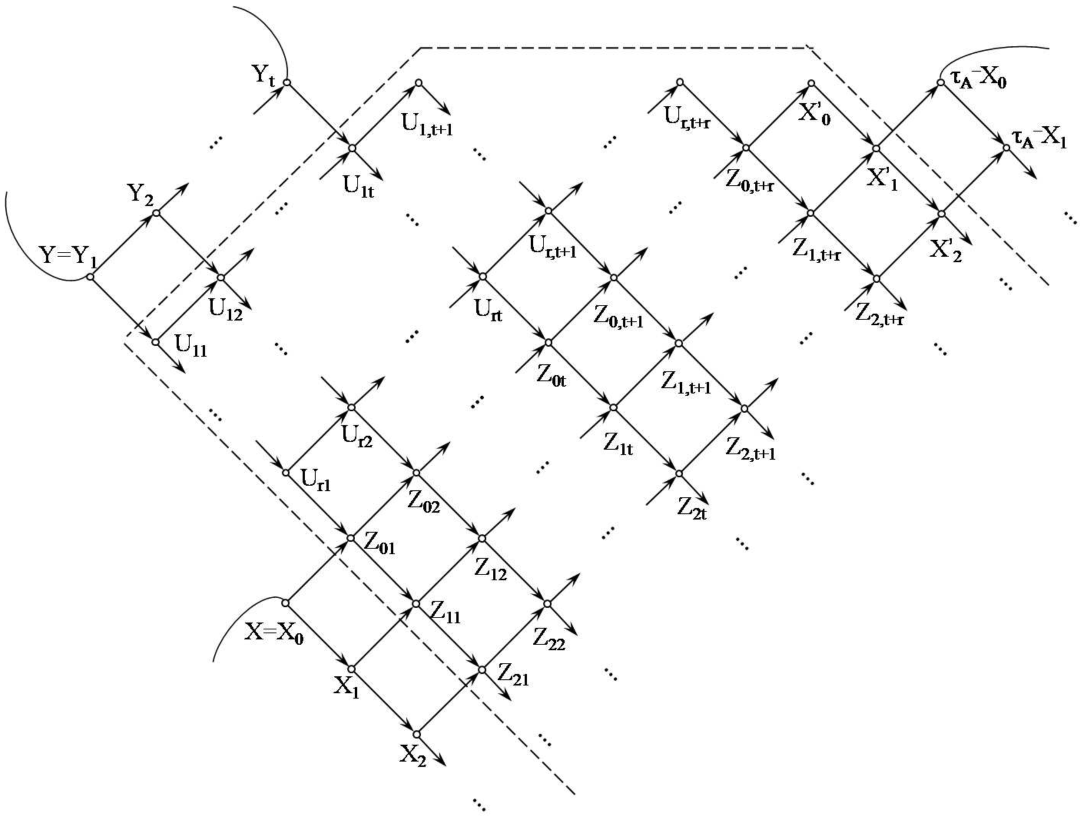


The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{0j} = Y_{j-1}$ if $j \geq 2, Z_{01}$ is projective, $\tau' X'_0 = Y_t, \tau' X'_i = Z_{i-1,t}$ if $i \geq 1, \tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ^*.

For $r \geq 1$, let $G = T_r(F)$, $U_{1,t+1}$, $U_{2,t+1}$, $\ldots$, $U_{r,t+1}$ denote the indecomposable projective G-modules, $U_{r,t+1}$, $U_{r,t+2}$, $\ldots$, $U_{r,t+r}$ denote the indecomposable injective G-modules, with $U_{r,t+1}$ the unique indecomposable projective-injective G-module. We define the *modified algebra* A' of A to be the triangular matrix algebra of the form:

$$A' = \begin{bmatrix} A & 0 & 0 & \ldots & 0 & 0 \\ Y & F & 0 & \ldots & 0 & 0 \\ Y & F & F & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y & F & F & \ldots & F & 0 \\ X \oplus Y & F & F & \ldots & F & F \end{bmatrix}$$

with $r+2$ columns and rows and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangles consisting of the modules $U_{kl} = Y_l \oplus U_{k,t+k}$ for $1 \leq k \leq r$, $1 \leq l \leq t$, and $Z_{ij} = \left(F, X_i \oplus U_{rj}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ for $i \geq 0$, $1 \leq j \leq t+r$, and $X'_i = (F, X_i, 1)$ for $i \geq 0$ as follows:



The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{0j} = U_{r,j-1}$ if $2 \leq j \leq t+r$, $Z_{01}, U_{k1}, 1 \leq k \leq r$ are projective, $\tau' U_{kl} = U_{k-1,l-1}$ if $2 \leq k \leq r$, $2 \leq l \leq t+r$, $\tau' U_{1l} = Y_{l-1}$ if $2 \leq l \leq t+1$, $\tau' X'_0 = U_{r,t+r}, \tau' X'_i = Z_{i-1,t+r}$ if $i \geq 1, \tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ^*, or Γ_G, respectively.

We note that the quiver $Q_{A'}$ of A' is obtained from the quiver of the double one-point extension $A[X][Y]$ by adding a path of length $r + 1$ with source at the extension vertex of $A[X]$ and sink at the extension vertex of $A[Y]$.

The integers $t \geq 1$ and $r \geq 0$ are such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangles equals $t + r + 1$. We call $t + r$ the *parameter* of the operation.

To the definition of the next admissible operation we need also the finite versions of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4), which we denote by (fad 1), (fad 2), (fad 3) and (fad 4), respectively. In order to obtain these operations we replace all infinite sectional paths of the form $X_0 \to X_1 \to X_2 \to \cdots$ (in the definitions of (ad 1), (ad 2), (ad 3), (ad 4)) by the finite sectional paths of the form $X_0 \to X_1 \to X_2 \to \cdots \to X_s$. For the operation (fad 1) $s \geq 0$, for (fad 2) and (fad 4) $s \geq 1$, and for (fad 3) $s \geq t - 1$. In all above operations X_s is injective.

(ad 5) We define the *modified algebra* A' of A to be the iteration of the extensions described in the definitions of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4), and their finite versions corresponding to the operations (fad 1), (fad 2), (fad 3) and (fad 4). The *modified translation quiver* Γ' of Γ is obtained in the following three steps: first we are doing on Γ one of the operations (fad 1), (fad 2) or (fad 3), next a finite number (possibly empty) of the operation (fad 4) and finally the operation (ad 4), and in such a way that the sectional paths starting from all the new projective vertices have a common cofinite (infinite) sectional subpath.

C finite dimensional K-algebra

$\mathcal{T}^C$ a family of pairwise orthogonal generalized standard stable tubes of Γ_C.

A finite dimensional K-algebra algebra A is a **generalized multicoil enlargement** of C, with respect to $\mathcal{T}^C$, if A is obtained from C by an iteration of admissible operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*) performed either on stable tubes of $\mathcal{T}^C$, or on generalized multicoils obtained from stable tubes of $\mathcal{T}^C$ by means of operations done so far.

A **generalized multicoil** is a translation quiver obtained from a finite family $\mathcal{T}_1, \ldots, \mathcal{T}_s$ of stable tubes by an iteration of admissible (translation quiver) operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*).