STABLE CATEGORIES OF COHEN-MACAULAY MODULES
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In this series of lectures I will give an introduction to representation theory of Cohen-Macaulay modules over orders based on the fundamental articles [A, A2] due to Auslander. The definition is quite simple.

Definition 0.1. Let R be a complete regular local ring of Krull dimension d, for instance, the formal power series ring $k[[x_1, \cdots, x_d]]$ over a field k.

(a) We say that an R-algebra Λ is an *R-order* if Λ is a free R-module of finite rank.
(b) We say that a Λ-module X is *Cohen-Macaulay* (*CM*) if X is a free R-module of finite rank.

The following three examples are fundamental:

- ($d = 0$) finite dimensional modules over finite dimensional algebras over fields [ARS, ASS],
- ($d = 1$) lattices over orders over complete discrete valuation rings [CR1, CR2],
- maximal Cohen-Macaulay modules over commutative complete local Cohen-Macaulay rings [Y].

We denote by $\mathrm{CM}(\Lambda)$ the category of CM Λ-modules. This category behaves quite similarly to the categories of finite dimensional modules over finite dimensional algebras. Let us give some of basic properties of $\mathrm{CM}(\Lambda)$.

- $\mathrm{CM}(\Lambda)$ is Krull-Schmidt,
- we have a *canonical duality*

$$\mathrm{Hom}_R(-, R) : \mathrm{CM}(\Lambda) \overset{\sim}{\longleftrightarrow} \mathrm{CM}(\Lambda^{\mathrm{op}}), \tag{1}$$

- $\mathrm{CM}(\Lambda)$ is a resolving subcategory of $\mathrm{mod}\,\Lambda$,
- $\mathrm{CM}(\Lambda)$ is an exact category with enough projectives $\mathrm{add}\,\Lambda$ and enough injectives $\mathrm{add}\,\mathrm{Hom}_R(\Lambda, R)$.

The *stable category*

$$\underline{\mathrm{CM}}(\Lambda) \quad (\text{respectively, } \overline{\mathrm{CM}}(\Lambda))$$

is defined as the factor category of $\mathrm{CM}(\Lambda)$ by the ideal generated by Λ (respectively, $\mathrm{Hom}_R(\Lambda, R)$). The stable categories play more and more important role in representation theory and related subjects, e.g. [O, KST, LP, IY, KMV].

The Auslander-Reiten theory on $\mathrm{CM}(\Lambda)$ was developed under the following condition:

Definition 0.2. We say that an R-order Λ is an *isolated singularity* if

$$\mathrm{gl.dim}(\Lambda \otimes_R R_{\mathfrak{p}}) = \mathrm{Krull.dim} R_{\mathfrak{p}}$$

for any non-maximal prime ideal $\mathfrak{p}$ of R.

If $d = 0$, then the condition is always satisfied. If $d = 1$, then the condition means that $\Lambda \otimes_R K$ is a semisimple K-algebra for the quotient field K of R.

For an R-order Λ which is an isolated singularity, CM modules have an another meaning given in stable module theory [ABr]: We say that a finitely generated Λ-module X is *n-torsionfree* for a positive integer n if

$$\mathrm{Ext}^i_\Lambda(\mathrm{Tr}\, X, \Lambda) = 0$$

for any $0 < i \leq n$, where

$$\mathrm{Tr} : \underline{\mathrm{mod}}\Lambda \overset{\sim}{\longleftrightarrow} \underline{\mathrm{mod}}\Lambda^{\mathrm{op}}$$

is Auslander-Bridger transpose duality [ABr]. We denote by $\mathcal{F}_n(\Lambda)$ the category of n-torsionfree Λ-modules. Then we have a duality [ABr]

$$\Omega^n\, \mathrm{Tr} : \underline{\mathcal{F}}_n(\Lambda) \overset{\sim}{\longleftrightarrow} \underline{\mathcal{F}}_n(\Lambda^{\mathrm{op}}), \tag{2}$$

on the stable category $\underline{\mathcal{F}}_n(\Lambda)$ of $\mathcal{F}_n(\Lambda)$, where $\Omega : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda$ is the syzygy functor.

The following observation plays a crucial role.

Proposition 0.3. [A] *Let* Λ *be an* R*-order which is an isolated singularity.*

(a) $\mathrm{CM}(\Lambda) = \mathcal{F}_d(\Lambda)$.

(b) $\underline{\mathrm{CM}}(\Lambda)$ *is* Hom-*finite.*

Now we have the following fundamental equivalence.

Definition-Theorem 0.4. *Composing dualities* (1) *and* (2) *we have an equivalence*

$$\tau : \underline{\mathrm{CM}}(\Lambda) \xrightarrow{\Omega^d \operatorname{Tr}} \underline{\mathrm{CM}}(\Lambda^{\mathrm{op}}) \xrightarrow{\operatorname{Hom}_R(-,R)} \overline{\mathrm{CM}}(\Lambda)$$

called the Auslander-Reiten translation.

Using τ we have Auslander-Reiten duality and existence theorem of almost split sequences as in the case of finite dimensional algebras.

In the lecture I will explain the following subjects:

- orders in dimension 1: hereditary orders and Bass orders [CR1, CR2],
- Auslander-Reiten theory using $\tau : \underline{\mathrm{CM}}(\Lambda) \to \overline{\mathrm{CM}}(\Lambda)$ [A, A2],
- tilting theory aspects of CM modules: Auslander-Buchweitz approximation [ABu],
- orders in dimension 2: fundamental sequences and algebraic McKay correspondence [A3],
- triangulated categories associated with orders,
- toward higher dimensional Auslander-Reiten theory via cluster tilting [I].

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