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Preparatory School to the Winter College on Optics and Energy

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Image formation

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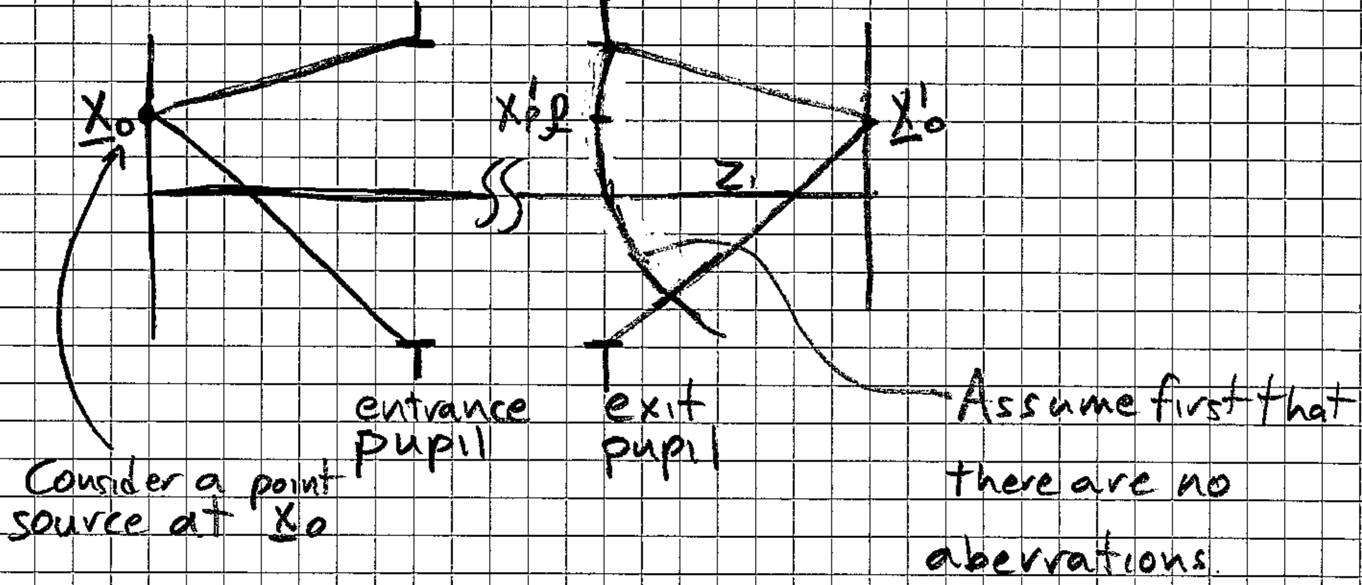
I Image formation

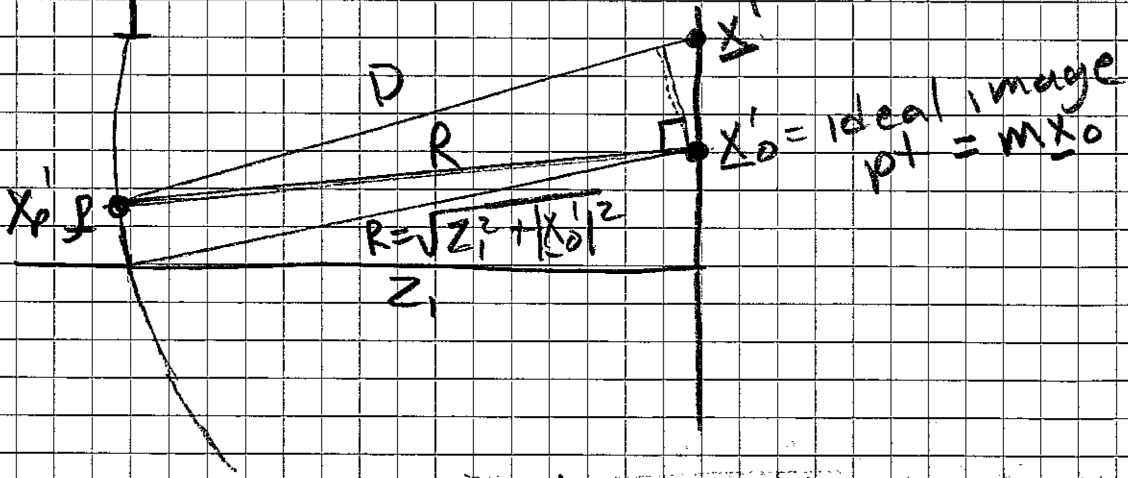
Assume incoherent objects.



The field in the image plane is roughly

$$U_{image}(\underline{X}') \propto \iint_{pupil} U_{pupil}(\underline{\rho})\, e^{ikD(\underline{\rho},\underline{X}')}\, d^2\rho$$

where $D(\underline{\rho}, \underline{X}')$ is the distance from a point ($\underline{\rho}$) in the ideal reference wavefront and the point $\underline{X}'$ at the image plane.



$$D \approx R - \frac{X'_\rho\,\underline{\rho}}{R}\cdot(\underline{X}' - \underline{X}'_0) \approx R - NA'\,\underline{\rho}\cdot(\underline{X}' - \underline{X}'_0)$$

Therefore:

$$U_{image}(\underline{X}') \,\tilde{\propto}\, e^{ikR} \iint_{pupil} U_{pupil}(\underline{\rho})\, e^{-ikNA'\underline{\rho}\cdot(\underline{X}'-\underline{X}_0')} d^2\rho$$

$$\propto e^{ikR}\, \hat{\mathcal{F}} U_{pupil} \Big|_{kNA'(\underline{X}'-\underline{X}_0')}$$

$$= e^{ikR}\, \tilde{U}_{pupil}\left(kNA'(\underline{X}'-\underline{X}_0')\right)$$

Irradiance E_ν (what we usually call intensity)

$$E_\nu \,\tilde{\propto}\, \left|\tilde{U}_{pupil}\left(kNA'(\underline{X}'-\underline{X}_0')\right)\right|^2$$

For a perfect imaging system:

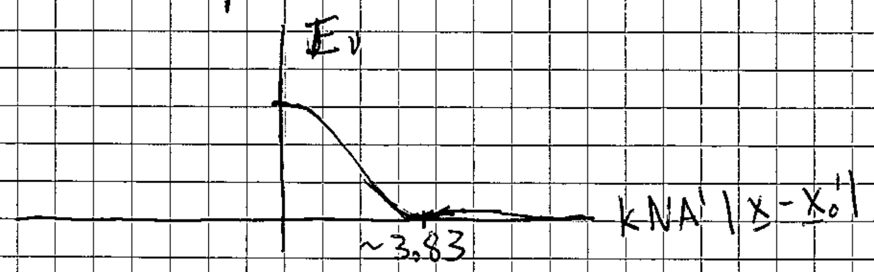
$$U_{pupil}(\underline{\rho}) = U_0 P(\underline{\rho}), \text{ where } P(\underline{\rho}) = \begin{cases} 1, & |\underline{\rho}| \le 1 \\ 0, & |\underline{\rho}| > 1. \end{cases}$$

$$\tilde{U}_{pupil}(\underline{q}) = \frac{1}{2\pi} \iint_{|\underline{\rho}|\le 1} U_0\, e^{-i\underline{q}\cdot\underline{\rho}}\, d^2\rho$$

$$= \frac{U_0}{2\pi} \int_0^{2\pi}\int_0^1 e^{-iq\rho\cos(\phi_\rho-\phi_q)}\, \rho\, d\rho\, d\phi_\rho = U_0 \int_0^1 \underbrace{J_0}_{\text{Bessel function}}(q\rho)\,\rho\, d\rho$$

$$= \frac{U_0}{q^2}\int_0^q J_0(u)\, u\, du = \frac{U_0}{q^2}\, q J_1(q) = U_0 \frac{J_1(q)}{q}$$

So $\quad E_\nu \,\tilde{\propto}\, \left|\dfrac{J_1\left(kNA'|\underline{X}-\underline{X}_0'|\right)}{kNA'|\underline{X}-\underline{X}_0'|}\right|^2$ Airy Pattern



If the system has aberrations:

$$U_{pupil}(\underline{\rho}) = U_0\, P(\underline{\rho})\, e^{-ikW(\underline{\rho}, \underline{h}_0)}$$

$$E_v(\underline{x}') \,\tilde{\propto}\, \left| \iint_{|\underline{\rho}|<1} e^{-ikW(\underline{\rho}, \underline{h}_0) - ik\underline{\rho}\cdot(\underline{x}'-\underline{x}_0')NA'}\, d^2\rho \right|^2$$

In general, no closed form solution, so must use numerical integration or fast Fourier transforms.

If instead of a point source object, we have an extended spatially incoherent source, then the irradiance images of all are superposed to give

$$E_{\nu,\text{image}}(\underline{X}') \propto \iint M_{\nu,\text{object}}(\underline{X}) \left| \tilde{U}_{\text{pupil}}\left(kNA'(\underline{X}' - m\underline{X})\right)\right|^2 d^2X$$

$$\propto \iint M_{\nu,\text{object}}\left(\frac{\underline{X}_0'}{m}\right) \left| \tilde{U}_{\text{pupil}}\left(kNA'(\underline{X}' - \underline{X}_0')\right)\right|^2 d^2X_0'$$

$$= M_{\nu,\text{object}}\left(\frac{\underline{X}'}{m}\right) * \left| \tilde{U}_{\text{pupil}}\left(kNA'\underline{X}'\right)\right|^2$$

Now, Fourier-transform both sides and use the convolution theorem:

$$\tilde{E}_{\nu,\text{image}}(\underline{K}') \propto \tilde{M}_{\nu,\text{object}}(m\underline{K}')\ \text{OTF}(m\underline{K}')$$

or

$$\tilde{E}_{\nu,\text{image}}(\underline{K}/m) \propto \tilde{M}_{\nu,\text{object}}(\underline{K})\ \text{OTF}(\underline{K})$$

where OTF = optical transfer function:

$$\text{OTF}(\underline{K}) = \frac{1}{\underset{\text{normalization}}{N}} \iint \left| \tilde{U}_{\text{pupil}}\left(k\underbrace{NA'm}_{NA}\underline{X}\right)\right|^2 e^{-i\underline{K}\cdot\underline{X}}\, d^2X$$

$$= \frac{\iint U^*_{\text{pupil}}\left(\frac{\underline{K}_0 - \underline{K}/2}{kNA}\right) U_{\text{pupil}}\left(\frac{\underline{K}_0 + \underline{K}/2}{kNA}\right) d^2K_0}{\iint \left|U_{\text{pupil}}\left(\frac{\underline{K}_0}{kNA}\right)\right|^2 d^2K_0}$$

Let $U_{pupil}(\underline{\rho}) = P(\underline{\rho})\, e^{ikW(\underline{\rho})}$ ← aberrations.

$$P(\underline{\rho}) = \begin{cases} 1, & |\underline{\rho}| \le 1 \\ 0, & |\underline{\rho}| > 1. \end{cases}$$

Then

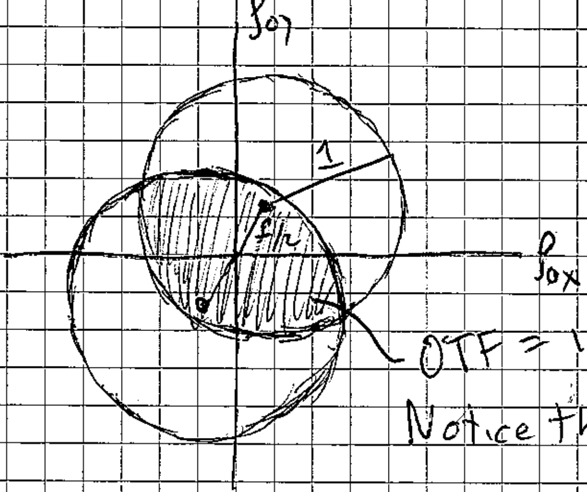
$$OTF(\underline{K}) = \frac{\iint P\left(\underline{\rho}_0 - \frac{\underline{\rho}}{2}\right) P\left(\underline{\rho}_0 + \frac{\underline{\rho}}{2}\right) e^{ik\left[W\left(\underline{\rho}_0 + \frac{\underline{\rho}}{2}\right) - W\left(\underline{\rho}_0 - \frac{\underline{\rho}}{2}\right)\right]} d^2\rho_0}{\iint |P(\underline{\rho}_0)|^2 d^2\rho_0}$$

$$= \frac{1}{\pi} \iint P\left(\underline{\rho}_0 - \frac{\underline{\rho}}{2}\right) P\left(\underline{\rho}_0 + \frac{\underline{\rho}}{2}\right) e^{ik\left[W\left(\underline{\rho}_0 + \frac{\underline{\rho}}{2}\right) - W\left(\underline{\rho}_0 - \frac{\underline{\rho}}{2}\right)\right]} d^2\rho_0 \Bigg|_{\underline{\rho} = \frac{\underline{K}}{kNA}}$$

π ← area of P.

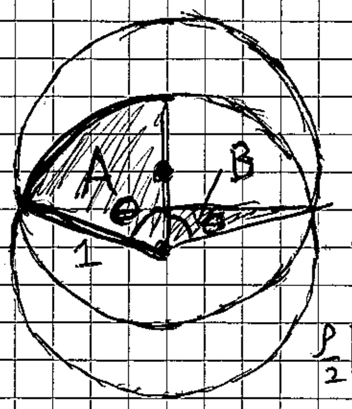
Example: if there are no aberrations ($W=0$)

$$OTF = \frac{1}{\pi} \iint P\left(\underline{\rho}_0 - \frac{\underline{\rho}}{2}\right) P\left(\underline{\rho}_0 + \frac{\underline{\rho}}{2}\right) d^2\rho_0 \Bigg|_{\underline{\rho} = \frac{\underline{K}}{kNA}}$$



OTF = intersection area

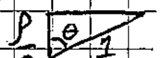
Notice that it only depends on $|\underline{\rho}|$



$A = \pi\left(\frac{\theta}{2\pi}\right) = \frac{\theta}{2}$

area of unit circle

fraction of unit circle



$B = \frac{\sin\theta\cos\theta}{2}$

But $\cos\theta = \frac{\rho}{2}$, $\sin\theta = \sqrt{1-\frac{\rho^2}{4}}$

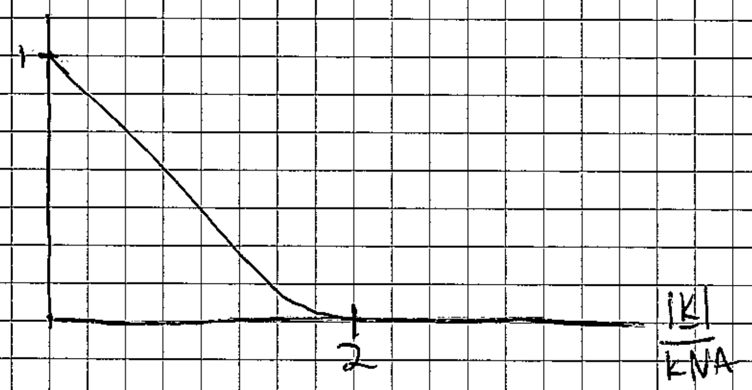
so $A = \frac{1}{2}\arccos\left(\frac{\rho}{2}\right)$

$B = \frac{\rho}{4}\sqrt{1-\rho^2/4}$

$$\text{area} = \frac{4(A-B)}{\pi} = \frac{2\arccos\left(\frac{\rho}{2}\right) - \rho\sqrt{1-\frac{\rho^2}{4}}}{\pi}$$

Test: area$(0) = 1$ ✓, area$(2) = 0$ ✓

$$\boxed{OTF(\underline{k}) = \frac{2\arccos\left(\frac{|\underline{k}|}{2kNA}\right) - \frac{|\underline{k}|}{kNA}\sqrt{1-\frac{|\underline{k}|^2}{4kNA^2}}}{\pi}}$$



If there are aberrations, the OTF cannot be calculated in closed form. Notice that, strictly speaking, we can only define the OTF if the aberrations are independent of $\underline{h}$ (for example defocus or spherical), since it is assumed in the use of the convolution theorem that the image of every point has the same shape. However, we can still compute the OTF for other aberations to get an idea of what the system does to the image at different regions.

If there are aberrations, the OTF can become complex. Then we define

<u>Modulation transfer function</u>

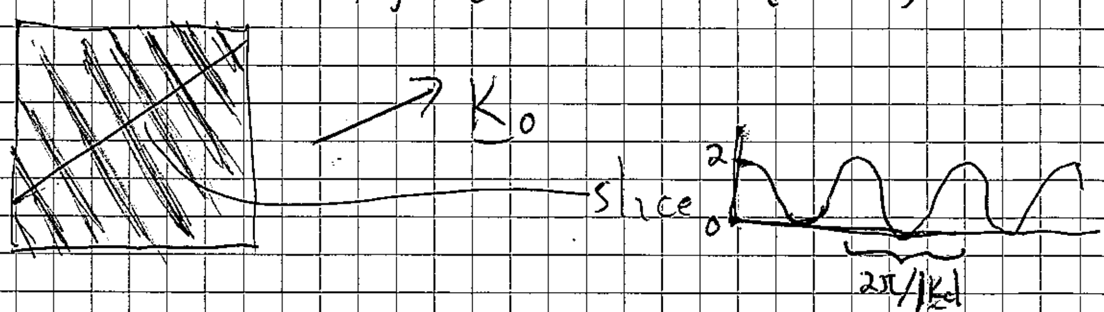
$$MTF(\underline{K}) = |OTF(\underline{K})|$$

<u>Phase transfer function</u>

$$PTF(\underline{K}) = Arg\{OTF(\underline{K})\}.$$

• <u>Interpretation of MTF, PTF</u>

Supose the object is a sinusoidal intensity distribution of the form: $M_{v,object}(\underline{X}) = 1 + \cos(\underline{K}_0 \cdot \underline{X})$



The Fourier transform of this object is

$$\tilde{M}_{\nu,\text{object}}(\underline{K}) \propto \delta(\underline{K}) + \frac{\delta(\underline{K}-\underline{K}_0)}{2} + \frac{\delta(\underline{K}+\underline{K}_0)}{2}$$

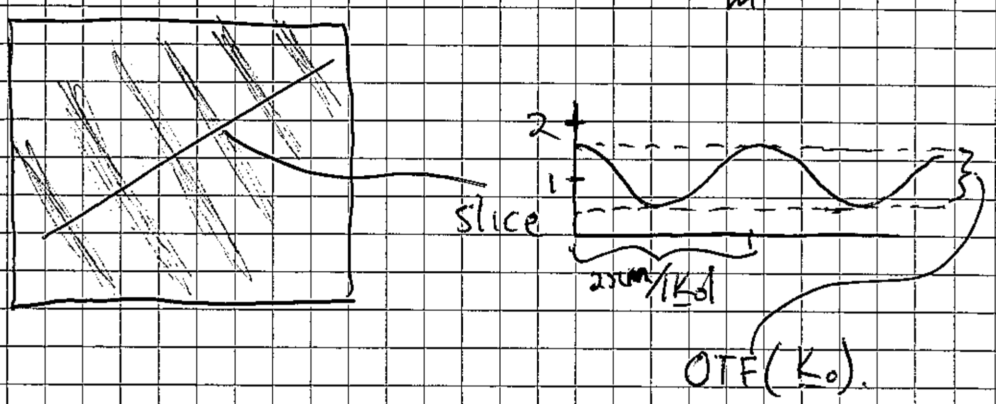
Therefore

$$\tilde{E}_{\nu,\text{image}}\left(\underline{K}/m\right) \propto OTF(\underline{K})\left[\delta(\underline{K}) + \frac{\delta(\underline{K}-\underline{K}_0)}{2} + \frac{\delta(\underline{K}+\underline{K}_0)}{2}\right]$$

$$= \cancel{OTF(\underline{0})}^{\,1}\,\delta(\underline{K}) + OTF(\underline{K}_0)\left[\frac{\delta(\underline{K}-\underline{K}_0)}{2} + \frac{\delta(\underline{K}+\underline{K}_0)}{2}\right]$$

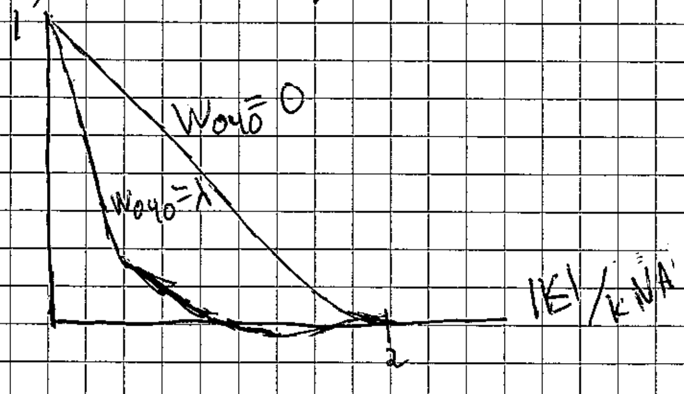
assume OTF real, & therefore even

So $E_{\nu,\text{image}}(\underline{X}') \propto 1 + OTF(\underline{K}_0)\cos\left(\frac{\underline{K}_0 \cdot \underline{X}'}{m}\right)$



So the visibility of the fringes is proportional to the MTF = |OTF| at the corresponding spatial frequency. If OTF<0, then the fringes are reversed (bright $\Leftrightarrow$ dark).

When there are aberrations, the MTF drops drastically. For example, for $W_{040} = \lambda$:



Group theory

Studies problems in terms of their "symmetries".

Group: a set of quantities, together with an operation ($\bullet$) between them, such that the following axioms are satisfied:

Axioms

1) **Closure:** if a & b are in the group, then $a \bullet b$ is also in the group
2) **Associativity:** $(a \bullet b) \bullet c = a \bullet (b \bullet c)$
3) **Identity:** there is an element e so that $e \bullet a = a \bullet e = a$ for any a in the group
4) **Inverse:** for any a in the group, one can define a^{-1}, so that $a \bullet a^{-1} = a^{-1} \bullet a = e$.

There is another "axiom" that only some groups satisfy (it is not required):

5) **Commutativity:** $a \bullet b = b \bullet a$ for any a, b.
Groups that satisfy this are called "Abelian", and the ones that do not "non-Abelian".

Examples

I) A very small group containing only the elements $\{-1, 1\}$, with $\bullet$ being multiplication.

Cayley table

$\bullet$	-1	1
-1	1	-1
1	-1	1

1 is the identity

Each element is its own inverse.

This group is Abelian.

II) $\{1, i, -1, -i\}$, with $\bullet$ = multiplication

Cayley table

$\cdot$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

1 = identity

1, −1 their own inverses

i, −i each other's inverses

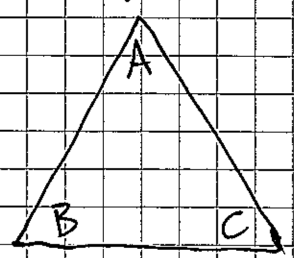
Abelian (table symmetric about diagonal).

Exercise: which of the following are groups? If they are groups, indicate the identity, inverse, and whether they are Abelian.

1. Naturals with a) $\bullet = +$, b) $\bullet = -$, c) $\bullet = \times$, d) $\bullet = \div$
2. Reals " " " "
3. Complex " " " "
4. 2×2 nonsingular matrices with $\bullet$ = product
5. 3D vectors with a) $\bullet$ = dot product, b) $\bullet$ = cross product.

Notice that these groups have infinite numbers of elements.

The elements of a group need not be numbers or matrices. They can be, for example, operations or actions. To illustrate the relation to symmetry, consider an equilateral triangle:



What can we do to it so that it looks the same?

rotate 120°, rotate −120°

flip wrt. vertical axis

flip wrt −30° axis

flip wrt 30° axis

These 5 operations together with "leave it as is", which is the identity, form a group. "$\bullet$" is successive action.

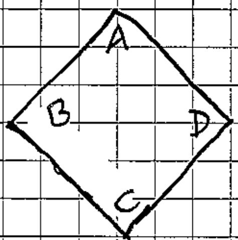
Call these operations: RL, RR, FV, F-, F+, I.
Generate the Cayley table. Identify there the inverse operations. Is this group Abelian?

- Note that this group is equivalent to the group of all possible permutations of three objects $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \square \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ where $\square$ can be:

These two groups form an "isomorphism".

Consider the generalization to a square:



How many elements does the group have? Is it Abelian? Is it isomorphic to the group of all permutations of four objects, or is it a "subgroup"?

Continuous groups.

- **Euclidean group**, called $E(n)$ or $ISO(n)$, is the group of transformations in n-dimensional Euclidean space that preserve distances (isometries).

Consider $E(2)$; that is, the 2D euclidean plane.

The distance between two points (x_1, y_1) & (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

The transformations that preserve distance are:

Translations: $T_{(a,b)}\ (x,y) = (x+a, y+b)$

Rotations: $R_\theta\ (x,y) = (x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta)$.

(we could also include inversions)

"$\bullet$" is the successive action of these operations.

Note that T and R form each a subgroup:

$$T_{(a_1,b_1)}\, T_{(a_2,b_2)} = T_{(a_1+a_2,\, b_1+b_2)}.$$

$$R_{\theta_1} R_{\theta_2} = R_{\theta_1+\theta_2}.$$

These two subgroups are Abelian.

However, the group $E(2)$ composed of all T & R is not Abelian.

This group has an infinite (continuous) # of elements.

How many parameters do we need to specify to label an element?

- <u>Special unitary group</u>, called $SU(n)$, is the group of $n \times n$ unitary matrices with determinant $= 1$. [It is a subgroup of $U(n)$, the unitary group, composed of all unitary matrices.] Very useful/common in physics. In particular, $SU(2)$ turns out to correspond to transformations on the unit sphere.

- <u>Symplectic group</u>, called $Sp(2n, F)$, where F can be $\mathbb{R}$ (reals), $\mathbb{C}$ (complex), is the group of $2n \times 2n$ matrices whose elements are of the class F, and that are symplectic. A symplectic matrix $\mathbb{M}$ satisfies:

$$\mathbb{M}^T \mathbb{O}_{2n} \mathbb{M} = \mathbb{O}_{2n}, \text{ where } \mathbb{O}_{2n} = \begin{pmatrix} \mathbb{O}_n & \mathbb{I}_n \\ -\mathbb{I}_n & \mathbb{O}_n \end{pmatrix}.$$

Symplectic matrices have determinant $= 1$.
This group is clearly connected to paraxial optics! It preserves phase space area, and maps lines to lines in phase space. It includes phase space shearings, squeezings and rotations.

Lie Algebras and Groups

- A <u>Lie group</u> is a group that is also associated to a "differentiable manifold". The three groups mentioned earlier are Lie groups.

- A <u>Lie algebra</u> is a mathematic structure used to study a Lie group. It is composed of: a set of "vectors" g, and a binary operation $[\cdot,\cdot]$ so that: Lie bracket

1) $[g_1, g_2] \propto g_3$
2) Bilinearity: $[\underbrace{a}_{\text{constants}} X + \underbrace{b}_{} Y, Z] = a[X,Z] + b[Y,Z]$
3) $[X,Y] = -[Y,X]$ and therefore $[X,X]=0$
4) Jacobi identity: $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.

Examples: 1) vectors, where $[\vec{a},\vec{b}] = \vec{a}\times\vec{b}$.
2) some matrices and operators, where $[A,B] = AB - BA$.

Connection to groups.

<u>Generators.</u>

Consider SU(2), namely the transformations over the unit sphere. One of these transformations is a rotation by ϕ around the z direction. This is described by the matrix:

$$R^{(z)}_{(\phi)} = \begin{pmatrix} \cos\phi, & -\sin\phi, & 0 \\ \sin\phi, & \cos\phi, & 0 \\ 0, & 0, & 1 \end{pmatrix}.$$

For small ϕ, $$R^{(z)}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\phi & 0 \\ \phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(\phi^2) = \mathbb{I} + \phi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(\phi^2)$$

Let $\mathbb{T}_z = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (where the i's were included to make the matrix Hermitian).

Then $\mathbb{R}^{(z)}(\phi) = \mathbb{I} - i\phi\,\mathbb{T}_z + \mathcal{O}(\phi^2)$, for small ϕ.

For larger ϕ, we can write

$$\underline{\mathbb{R}^{(z)}(\phi) = e^{-i\phi\mathbb{T}_z}},$$

where the exponential is understood in terms of its Taylor series.

$\mathbb{T}_z$ is the "generator" of rotations around the z axis.

We can also find:

$$\mathbb{T}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \mathbb{T}_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

If we choose $[\cdot,\cdot]$ as the commutator, then it is easy to show that:

$$[\mathbb{T}_x, \mathbb{T}_y] = i\,\mathbb{T}_z, \quad [\mathbb{T}_y, \mathbb{T}_z] = i\,\mathbb{T}_x, \quad [\mathbb{T}_z, \mathbb{T}_x] = i\,\mathbb{T}_y.$$

These three relations define the algebra, and contain the deep symmetry of the group.

Note that, besides these specific matrices, others exist with the same algebra, for example, the Pauli matrices, defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with $[\,]$ being again the commutator.

This algebra describes many physical systems, like quantum angular momentum, or polarization.

In general, for $SU(n)$, the size of the Lie group (number of generators) is n^2-1, which can be represented by these many $n \times n$ matrices. For $SU(2)$ it is the Pauli matrices. For $SU(3)$ it can be the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \Big/ \sqrt{3}.$$

Relation between Maxwell's Equations, Coherence and Radiometry

1. From Maxwell's Equations to the Helmholtz Equation

Maxwell's equations in free space:

(i) $\nabla \cdot \vec{E} = 0$ (because $\rho = 0$)

(ii) $\nabla \cdot \vec{B} = 0$

(iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(iv) $\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ (because $\vec{J} = \vec{0}$)

To get to the wave equation, take curl of (iii)

$$\nabla \times (\nabla \times \vec{E}) = -\nabla \times \frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} \nabla \times \vec{B}$$

Now use $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$ on LHS, and (iv) on RHS

$$\nabla(\underbrace{\cancel{\nabla \cdot \vec{E}}}_{0 \text{ because (i)}}) - \nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E}$$

So

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} = \vec{0}$$

This is a wave equation for $\vec{E}(\vec{r}, t)$, where the characteristic speed is $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ = speed of light.

If we now assume that the field is monochromatic, that is: $\vec{E}(\vec{r}, t) = \vec{U}(\vec{r})\, e^{-i\omega t}$, then

$$\left[\nabla^2 \vec{U} + \mu_0 \varepsilon_0 \omega^2 \vec{U}\right] \cancel{e^{-i\omega t}} = \vec{0} \quad \text{or}$$

$$\nabla^2 \vec{U} + k^2 \vec{U} = 0 \quad \text{where } k = \sqrt{\mu_0 \varepsilon_0}\, \omega = \frac{\omega}{c}$$

is the wavenumber.

2. From the Helmholtz equation to paraxial optics

A solution to the Helmholtz equation is a "plane wave":

$$\vec{U} = \vec{A}\, e^{ik\hat{u}\cdot\vec{r}}.$$

Substituting this in the Helmholtz equation gives us

$$-k^2\,\hat{u}\cdot\hat{u} + k^2 = 0 \Rightarrow \underline{\hat{u}\cdot\hat{u} = 1}$$

Solutions for which all components of $\hat{u}$ are real are called "homogeneous" or "traveling" plane waves. There are also solutions where the components of $\hat{u}$ can be complex. These are called "evanescent waves". Let us ignore these for now.

For traveling plane waves, $\hat{u}\cdot\hat{u}=1$ means that $\hat{u}$ is a unit vector. Also, because $\nabla\cdot\vec{E}=0$,

$$\nabla\cdot\vec{U} = 0 \quad \therefore \quad \hat{u}\cdot\vec{A} = 0, \text{ so } \vec{A}\perp\hat{u}.$$

A general field can be built as a combination of plane waves:

$$\vec{U}(\vec{r}) = \int \vec{A}(\hat{u})\, e^{ik\hat{u}\cdot\vec{r}}\, d\Omega$$

where the integration is over $\hat{u}$. Let us assume that this superposition involves only traveling waves. $\vec{A}(\hat{u})$ is called the "angular spectrum" of $\vec{U}(\vec{r})$.

The paraxial approximation results from assuming that $\vec{U}$ is composed only of plane waves traveling at very small angles from the z-axis, that is:

$$\vec{A}(\hat{u}) \approx 0 \text{ except for } \hat{u}\cdot\hat{z} \approx 1 \text{ i.e. } \theta \ll \frac{\pi}{2}.$$

Then we can approximate:

$$u_z = \sqrt{1-u_x^2-u_y^2} \approx 1 - \frac{u_x^2+u_y^2}{2}$$

The superposition of plane waves can then be written as:

$$\vec{U}(\vec{r}) = \int \vec{A}\, e^{ik\hat{u}\cdot\vec{r}}\, d\Omega = \iint \underbrace{\frac{\vec{A}}{u_z}}_{\tilde{\vec{U}}_0(u_x,u_y)} e^{ik\hat{u}\cdot\vec{r}}\, du_x du_y$$

$$\approx e^{ikz} \iint \tilde{\vec{U}}_0(u_x,u_y)\, e^{ik\left(\underline{u}\cdot\underline{x} - \frac{|\underline{u}|^2}{2} z\right)}\, du_x du_y,$$

where $\underline{u} = (u_x, u_y)$, $\underline{x} = (x, y)$.

Note that, because $\hat{u}$ points almost in the z direction, $\vec{A}$ and $\tilde{\vec{U}}_0$ have very small z components, so their x & y components dominate. Let us then write a scalar equation

$$\underline{U(\vec{r}) \approx e^{ikz} \iint \tilde{U}_0(\underline{u})\, e^{ik\underline{u}\cdot\underline{x}}\, e^{-ik\frac{|\underline{u}|^2 z}{2}}\, d^2u} \quad (1)$$

where this scalar represent either component of $\vec{U}$.

Note also that, at $z=0$, $\hat{U}_0(\underline{u})$ is proportional to the Fourier transform of $U(x,y,0)$:

$$U(x,y,0) \approx \iint \hat{U}_0(\underline{u})\, e^{ik\underline{u}\cdot\underline{x}}\, d^2u,$$

which can be inverted to

$$\hat{U}_0(\underline{u}) = \left(\frac{k}{2\pi}\right)^2 \iint U(x',y',0)\, e^{-ik\underline{u}\cdot\underline{x}'}\, d^2x'. \quad (2)$$

The substitution of (2) into (1) can be shown to lead to the well-known Fresnel diffraction formula of paraxial optics:

$$U(\vec{r}) = e^{ikz}\left(\frac{k}{2\pi}\right)^2 \iiiint U(\underline{x}',0)\, e^{ik\underline{u}\cdot(\underline{x}-\underline{x}')}\, e^{-ik\frac{|\underline{u}|^2 z}{2}}\, d^2u\, d^2x'$$

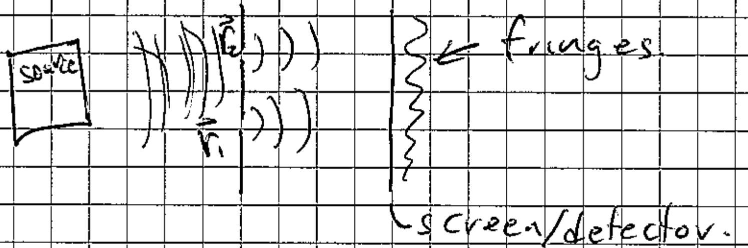
$$= \left(\frac{k}{2\pi}\right)^2 e^{ikz} \iint U(\underline{x}',0) \underbrace{\iint e^{-ik\left(\frac{z}{2}|\underline{u}|^2 - \underline{u}\cdot(\underline{x}-\underline{x}')\right)} d^2u}_{\frac{2\pi}{ikz}\, e^{\frac{ik|\underline{x}-\underline{x}'|^2}{2z}}}\, d^2x'$$

$$= \frac{k}{2\pi i z}\, e^{ikz} \iint U(\underline{x}',0)\, e^{ik\frac{|\underline{x}-\underline{x}'|^2}{2z}}\, d^2x' \quad (3)$$

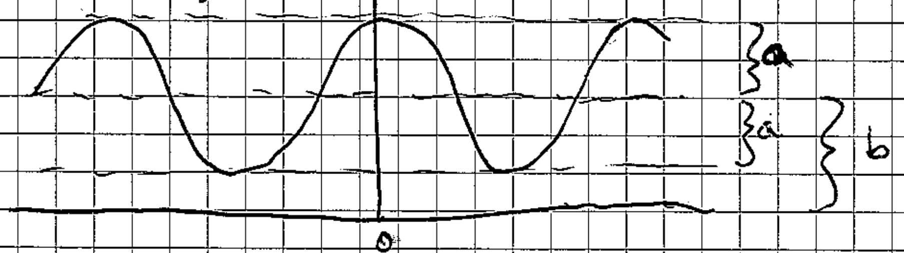
Partially coherent fields

If a field is not purely monochromatic and is generated by uncorrelated (or partially correlated) sources, then it can be described by the theory of partial coherence. Let us continue to work in the frequency domain, i.e. let us assume that we are considering quasimonochromatic fields with frequencies around ω.

To measure the coherence of a field at two points $\vec{r}_1$ and $\vec{r}_2$, one can use Young's pinhole experiment:



If the intensities at $\vec{r}_1$ and $\vec{r}_2$ are the same, then the visibility of the fringes is related to the "spectral degree of coherence:



$$\text{Visibility} = \frac{a}{b} = |\mu(\vec{r}_1, \vec{r}_2)| \quad \text{degree of coherence}$$

The spectral degree of coherence can be written as

$$\mu(\vec{r}_1,\vec{r}_2) = \frac{W(\vec{r}_1,\vec{r}_2)}{\sqrt{W(\vec{r}_1,\vec{r}_1)\,W(\vec{r}_2,\vec{r}_2)}}$$

where $W(\vec{r}_1,\vec{r}_2)$ is the "cross-spectral density", which gives the statistical correlation of the fluctuating field at two points:

$$W(\vec{r}_1,\vec{r}_2) = \langle U^*(\vec{r}_1)\, U(\vec{r}_2)\rangle$$

where $\langle\cdot\rangle$ denotes a statistical correlation. In particular, $W(\vec{r},\vec{r})$ gives the intensity (or, in radiometric terms, the spectral irradiance) at $\vec{r}$.

The propagation of W is ruled by the same laws as that for $U(\vec{r})$, but applied to both $\vec{r}_1$ and $\vec{r}_2$. So, for example, paraxial propagation is given by

$$W(\underline{x}_1, z_1, \underline{x}_2, z_2) = e^{ik(z_2-z_1)} \iiiint \langle \tilde{U}_0^*(\underline{u}_1)\,\tilde{U}_0(\underline{u}_2)\rangle \, e^{ik(\underline{u}_2\cdot\underline{x}_2-\underline{u}_1\cdot\underline{x}_1)}\, e^{-ik\left(\frac{|\underline{u}_2|^2}{2}z_2-\frac{|\underline{u}_1|^2}{2}z_1\right)} d^2u_1\, d^2u_2, \quad \text{(1bis)}$$

where $\langle \tilde{U}_0^*(\underline{u}_1)\,\tilde{U}_0(\underline{u}_2)\rangle$ can be found from the initial conditions

$$\langle \tilde{U}_0^*(\underline{u}_1)\,\tilde{U}_0(\underline{u}_2)\rangle = \left(\frac{k}{2\pi}\right)^4 \iiiint W(\underline{x}_1', 0, \underline{x}_2', 0)\, e^{-ik(\underline{u}_2\cdot\underline{x}_2'-\underline{u}_1\cdot\underline{x}_1')}\, d^2x_1'\, d^2x_2' \quad \text{(2bis)}$$

Again, the substitution of Eq. (2bis) into Eq. (1bis) gives the Fresnel propagation formula, this time for partially coherent fields

$$W(\underline{x}_1, z_1, \underline{x}_2, z_2) = \left(\frac{k}{2\pi z}\right)^2 e^{ik(z_2 - z_1)} \iiiint W(\underline{x}_1', 0, \underline{x}_2', 0) \quad (3bis)$$
$$e^{ik\left[\frac{|\underline{x}_2 - \underline{x}_2'|^2}{2z_2} - \frac{|\underline{x}_1 - \underline{x}_1'|^2}{2z_1}\right]} d^2x_1' \, d^2x_2' .$$

Van Cittert – Zernike Theorem

Suppose that there is a planar, spatially incoherent source at $z=0$, so that

$$W(\underline{x}_1', 0, \underline{x}_2', 0) = \left(\frac{2\pi}{k}\right)^2 I_0(\underline{x}_1') \; \delta(\underline{x}_2' - \underline{x}_1')$$

The substitution into Eq. (3bis) then becomes, at $z_1 = z_2 = z$,

$$W(\underline{x}_1, z, \underline{x}_2, z) = \frac{1}{z^2} \iint I_0(\underline{x}') \, e^{\frac{ik}{2z}\left[|\underline{x}_2 - \underline{x}'|^2 - |\underline{x}_1 - \underline{x}'|^2\right]} d^2x'$$

$$= \frac{1}{z^2} \iint I_0(\underline{x}') \, e^{\frac{ik}{2z}\left(|\underline{x}_2|^2 - |\underline{x}_1|^2 - 2\underline{x}' \cdot (\underline{x}_2 - \underline{x}_1)\right)} d^2x'$$

$$= \frac{e^{ik\frac{|\underline{x}_2|^2 - |\underline{x}_1|^2}{2z}}}{z^2} \underbrace{\iint I_0(\underline{x}') \, e^{-ik\underline{x}' \cdot \frac{(\underline{x}_2 - \underline{x}_1)}{z}} d^2x'}_{\propto \tilde{I}_0\left(k\frac{\underline{x}_2 - \underline{x}_1}{z}\right)}$$

The spectral degree of coherence is

$$|\mu(\underline{x}_1, z, \underline{x}_2, z)| = \left| \frac{\tilde{I}_0\left(\frac{k}{z}(\underline{x}_2 - \underline{x}_1)\right)}{\tilde{I}_0(0)} \right|$$

So, as z increases, the field becomes more spatially coherent.