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Review of Matrix Algebra

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1 The Stokes Polarization Parameters

The description of light in terms of the polarization ellipse is very useful. It allows to describe any state of completely polarized light by means of a single equation. However, this representation is inadequate for several reasons

(1) As the beam of light propagates through space, the light vector traces out an ellipse or some special form of ellipse. e.g.,

The circle or a straight line in a time interval of the order of 10^{-15} sec. This period is clearly too short to allow us to follow the tracing of the ellipse as the beam propagates and prevents us from following the polarization ellipse in the optical time domain.

(2) The polarization ellipse is only applicable to describing light that is completely polarized.

(3) The polarization ellipse is an amplitude description of polarized light that can not be observed and measure. The measurable quantities are the time average of the square of the field amplitudes and the intensity.

The light can be

(i) Completely Polarized

(ii) Partially Polarized

(iii) Completely Un-polarized.

A mathematical description of all these states is required.

The fact that we can only measure the intensity of light and not the amplitudes requires that the polarization ellipse must be transformed, so that, only intensities are present, that is, measured or observable quantities.

The transverse components of the optical field are given by

$$E_x(z,t) = E_{0x}\cos(\omega t - kz + \delta x).$$

$$E_y(z,t) = E_{0y}\cos(\omega t - kz + \delta y).$$

The equation for polarization ellipse is

$$\frac{E_x^2(z,t)}{E_{0x}^2} + \frac{E_y^2(z,t)}{E_{0y}^2} - 2\frac{E_x(z,t)E_y(z,t)}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta, \tag{1}$$

where $\delta = \delta x - \delta y$. All information concerning the polarization behavior of the optical field is contained in Eq. (1).

In order to determine the observables of the polarization ellipse which are its intensity and polarization behavior, it is necessary to transform Eq. (1) to an intensity or observable representation.

In order to do this, we first take a time average of the time dependent quantities in Eq. (1)

$$\frac{\left\langle E_x^2(z,t)\right\rangle}{E_{0x}^2} + \frac{\left\langle E_y^2(z,t)\right\rangle}{E_{0y}^2} - 2\frac{\langle E_x(z,t)E_y(z,t)\rangle}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta. \tag{2}$$

The time average of the field components are defined as

$$\langle E_i(z,t)E_j(z,t)\rangle = Lim_{T\to\infty}\frac{1}{T}\int_0^T E_i(z,t)E_j(z,t)dt,$$

where i,j=x,y and T is the time of measurement.

In order to evaluate time averages, we must first remove the denominator of Eq. (2). By multiplying Eq. (2) to the factor $4E_{0x}^2E_{0y}^2$, we get

$$4E_{0y}^2\left\langle E_x^2(z,t)\right\rangle + 4E_{0x}^2\left\langle E_y^2(z,t)\right\rangle - 8E_{0x}E_{0y}\langle E_x(z,t)E_y(z,t)\rangle\cos\delta = (2E_{0x}E_{0y}\sin\delta)^2. \tag{3}$$

The time averages of the above terms can be written as

$$\begin{aligned}\left\langle E_x^2(z,t)\right\rangle &= \frac{1}{2}E_{0x}^2,\\ \left\langle E_x^2(z,t)\right\rangle &= \frac{1}{2}E_{0x}^2,\\ \langle E_x(z,t)E_y(z,t)\rangle &= \frac{1}{2}E_{0x}E_{0y}\cos\delta.\end{aligned}$$

Putting these values in Eq. (3), we get

$$2E_{0y}^2E_{0x}^2 + 2E_{0x}^2E_{0y}^2 - (2E_{0x}E_{0y}\cos\delta)^2 = (2E_{0x}E_{0y}\sin\delta)^2. \tag{4}$$

In order to describe both the intensity and the polarization, the form $E_{0x}^2 + E_{0y}^2$ must be included. Adding and Subtracting $E_{0x}^4 + E_{0y}^4$ on the L.H.S of Eq. (4) will lead to perfect square.

$$\begin{aligned}E_{0x}^4 + E_{0y}^4 + 2E_{0y}^2E_{0x}^2 + 2E_{0x}^2E_{0y}^2 - E_{0x}^4 - E_{0y}^4 - (2E_{0x}E_{0y}\cos\delta)^2 &= (2E_{0x}E_{0y}\sin\delta)^2,\\ \left(E_{0x}^2 + E_{0y}^2\right)^2 - \left(E_{0x}^2 - E_{0y}^2\right)^2 - (2E_{0x}E_{0y}\cos\delta)^2 &= (2E_{0x}E_{0y}\sin\delta)^2. \quad (5)\end{aligned}$$

The terms in the brackets can be written as

$$\begin{aligned}S_0 &= E_{0x}^2 + E_{0y}^2,\\ S_1 &= E_{0x}^2 - E_{0y}^2,\\ S_2 &= 2E_{0x}E_{0y}\cos\delta,\\ S_3 &= 2E_{0x}E_{0y}\sin\delta,\end{aligned}$$

where S_0, S_1, S_2, S_3 are the Stokes polarization parameters for a plane wave. So, Eq. (5) can be written as

$$S_0^2 = S_1^2 + S_2^2 + S_3^2.$$

The non-observable amplitude polarization ellipse given by

$$\frac{E_x^2(z,t)}{E_{0x}^2} + \frac{E_y^2(z,t)}{E_{0y}^2} - 2\frac{E_x(z,t)E_y(z,t)}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta,$$

is transformed to the stokes relation which is an intensity relation in the observable or measured domain.

Stokes parameters are intensities and they are real quantities.

(1) S_0 is the total intensity of the optical field.

(2) S_1 describes the prepouderance of the intensity of linearly horizontal polarized light over linearly vertical polarized light.

(3) S_2 describes the prepouderance of the intensity of linearly positive 45° polarized light over linearly negative 45° polarized light.

(4) S_3 describes the prepouderance of the intensity of right circularly polarized light over left circularly polarized light.

1.1 The Degenerate States of Polarized Light interms of Stokes Parameters

1.1.1 (i) Linearly Horizontal Polarized Light (LHP)

For this case, we have

$$E_{0y} = 0.$$

So, the Stokes polarization parameters are

$$\begin{aligned} S_0 &= S_1 = E_{0x}^2. \\ S_2 &= S_3 = 0. \end{aligned}$$

1.1.2 (ii) Linearly Vertical Polarized Light (LVP)

In this case, we have

$$E_{0x} = 0.$$

So, the Stokes polarization parameters are

$$\begin{aligned} S_0 &= E_{0y}^2. \\ S_1 &= -E_{0y}^2. \\ S_2 &= S_3 = 0. \end{aligned}$$

1.1.3 (iii) Linearly +45° Polarized Light (L+45°P)

The conditions to obtain L+45°P polarized light are

$$E_{0x} = E_{0y} = E_0, \delta = 0$$

So, the Stokes polarization parameters are

$$\begin{aligned} S_0 &= S_2 = 2E_0^2. \\ S_1 &= S_3 = 0. \end{aligned}$$

1.1.4 (iv) Linearly -45° Polarized Light (L-45°P)

The conditions on the amplitude are the same as for L+45°P light but the phase difference is $\delta = \pi$. In this case, the Stokes polarization parameters are

$$\begin{aligned} S_0 &= 2E_0^2. \\ S_2 &= -2E_0^2. \\ S_1 &= S_3 = 0. \end{aligned}$$

1.1.5 (v) Right Circularly Polarized Light (RCP)

The conditions for the right circularly polarized light (RCP) are

$$E_{0x} = E_{0y} = E_0, \delta = \frac{\pi}{2}$$

So, the Stokes polarization parameters are

$$\begin{aligned} S_0 &= S_3 = 2E_0^2. \\ S_1 &= S_2 = 0. \end{aligned}$$

1.1.6 (vi) Left Circularly Polarized Light (LCP)

The conditions on the amplitude are the same as that of right circularly polarized light. However, the phase shift between the orthogonal components is now $\delta = \frac{3\pi}{2}$. So, the Stokes polarization parameters are

$$\begin{aligned} S_0 &= 2E_0^2. \\ S_3 &= -2E_0^2. \\ S_1 &= S_2 = 0. \end{aligned}$$

The Stokes parameters for any state of elliptically polarized light is represented by

$$\begin{aligned} S_0 &= E_{0x}^2 + E_{0y}^2. \\ S_1 &= E_{0x}^2 - E_{0y}^2. \\ S_2 &= 2E_{0x}E_{0y}\cos\delta. \\ S_3 &= 2E_{0x}E_{0y}\sin\delta. \end{aligned}$$

1.2 The Stokes Vector

The inspection of the four Stokes parameters suggests that they can be arranged in the form of 4×1 column matrix. This column matrix is called " Stokes Vector ".

This step provides a formal method for treating numerous complicated problem using matrix algebra. The Stokes vector is

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}.$$

The Stokes vector for elliptically polarized light (EPL) is

$$S = \begin{pmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ 2E_{0x}E_{0y}\cos\delta \\ 2E_{0x}E_{0y}\sin\delta \end{pmatrix}.$$

The Stokes vector for the six degenerate polarization states are

1.2.1 (i) Linearly Horizontal Polarized Light (LHP)

The Stokes vector for linearly horizontal polarized light is

$$S = I_0 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

where $I_0 = E_{0x}^2$ is the total intensity.

1.2.2 (ii) Linearly Vertical Polarized Light (LVP)

The Stokes vector for linearly vertical polarized light is

$$S = I_0 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix},$$

where $I_0 = E_{0y}^2$ is the total intensity.

1.2.3 (iii) Linearly +45° Polarized Light (L+45°P)

The Stokes vector for linearly +45° polarized light is

$$S = I_0 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

where $I_0 = 2E_0^2$ is the total intensity.

1.2.4 (iv) Linearly -45° Polarized Light (L-45°P)

The Stokes vector for linearly -45° polarized light is

$$S = I_0 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

where $I_0 = 2E_0^2$ is the total intensity.

1.2.5 (v) Right Circularly Polarized Light (RCP)

The Stokes vector for right circularly polarized light is

$$S = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where $I_0 = 2E_0^2$ is the total intensity.

1.2.6 (vi) Left Circularly Polarized Light (LCP)

The Stokes vector for left circularly polarized light is

$$S = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix},$$

where $I_0 = 2E_0^2$ is the total intensity.

All this is based on the theoretical considerations.

1.3 Orientation and Ellipticity Angles

The Stokes polarization parameters can also be expressed in terms of the orientation and ellipticity angles of the polarization ellipse ψ and χ.

$$S = \begin{pmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ 2E_{0x}E_{0y}\cos\delta \\ 2E_{0x}E_{0y}\sin\delta \end{pmatrix}.$$

As orientation angle ψ of the polarization is given by

$$\tan 2\psi = \frac{2E_{0x}E_{0y}\cos\delta}{E_{0x}^2 - E_{0y}^2}.$$

This implies that

$$\tan 2\psi = \frac{S_2}{S_1}, 0 \leq \psi \leq \pi,$$

and ellipticity χ is given by

$$\sin 2\chi = \frac{2E_{0x}E_{0y}\sin\delta}{E_{0x}^2 + E_{0y}^2} = \frac{S_3}{S_0}, \frac{-\pi}{4} \leq \chi \leq \frac{\pi}{4}.$$

1.4 The Classical Measurement of the Stokes Polarization Parameters

The determination of the four Stokes parameters of an optical source requires an optical beam to pass squentially through two polarizing elements known as a " Wave plate " and a " Polarizer ". In the measurement, the waveplate is fixed with its fast axis along the transverse x-direction whereas polarizer can be rotated around its longitudinal z-axis.

The first three Stokes parameters S_0, S_1, S_2 are measured by removing the waveplate from the optical train and rotating the polarizer to three specific angles (θ).

The final Stokes parameter S_3 is obtained by inserting a quarter waveplate into the optical train.

In order to obtain the Stokes parameters of an optical beam, one must always take a time average of the polarization ellipse. However, the time averaging process can formally bypassed by representing the (real) optical amplitudes that is

$$E_x(z,t) = E_{0x}\cos(\omega t - kz + \delta x)$$

$$E_y(z,t) = E_{0y}\cos(\omega t - kz + \delta y)$$

in terms of complex amplitudes

$$E_x(t) = E_{0x}\exp(\imath\delta x)\exp(\imath\omega t).$$

$$E_y(t) = E_{0y}\exp(\imath\delta y)\exp(\imath\omega t).$$

The Stokes parameters for a plane wave are now defined by the equations

$$\begin{aligned} S_0 &= E_x E_x^* + E_y E_y^*. \\ S_1 &= E_x E_x^* - E_y E_y^*. \\ S_2 &= E_x E_y^* + E_y E_x^*. \\ S_3 &= \imath\left(E_x E_y^* - E_y E_x^*\right). \end{aligned}$$

Since the exponential time factor $\exp(\imath\omega t)$ disappears using the above definitions, it can be completely suppressed.

$$E_x = E_{0x}\exp(\imath\delta x).$$
$$E_y = E_{0y}\exp(\imath\delta y).$$

In order to measure Stokes parameters experimentally, we consider the complex components of the source. That is, the incident beam is represented in terms of its horizontal x and y-components E_x and E_y respectively.

The beam propagates through the waveplate. The waveplate has two orthogonal axes x and y. These axes are known as "fast" and "slow" axes respectively.

Along the x-axis (fast), the phase on the beam is advanced (increased) by $\frac{\phi}{2}$ and along the y-axis, the phase of the beam is retarted (decreased) by $\frac{\phi}{2}$ (written as -$\frac{\phi}{2}$).

The total phase shift between the orthogonal components is then $\frac{\phi}{2}-(-\frac{\phi}{2}) = \phi$. Then the complex components emerging from the wave plates are

$$E_x^{'} = E_x\exp(\imath\frac{\phi}{2}).$$
$$E_y^{'} = E_y\exp(-\imath\frac{\phi}{2}).$$

This beam is now incident on the polarizer. The polarizer has the property that it attenuates the optical components unequally along the x and y-directions. For an ideal polarizer along one axis, there is complete attenuation where as along the orthogonal axis there is complete (perfect) transmission.

The optical field only passes through the axis that allows complete transmission. This is called the transmission axis of the ideal polarizer.

If the transmission axis of the polarizer is rotated through an angle θ, so only the components of $E_x^{'}$ and $E_y^{'}$ along the rotated transmission axis can be perfectly transmitted. The component of $E_x^{'}$ along the transmission axis is $E_x^{'}\cos\theta$ and the component of $E_y^{'}$ along the transmission axis is $E_y^{'}\sin\theta$. The field transmitted along the transmission axis is the sum of these components.

The field emerging from the rotated ideal polarizer is

$$E = E_x^{'}\cos\theta + E_y^{'}\sin\theta.$$

Putting values of $E_x^{'}$ and $E_y^{'}$, we get

$$E = E_x\exp(\imath\frac{\phi}{2})\cos\theta + E_y\exp(-\imath\frac{\phi}{2})\sin\theta.$$

The intensity of the beam is defined by

$$I = E.E^{*}.$$

The intensity of the beam emerging from the rotated ideal polarizer is

$$I(\theta,\phi) = E_xE_x^{*}\cos^2\theta + E_yE_y^{*}\sin^2\theta + E_xE_y^{*}e^{\imath\phi}\sin\theta\cos\theta + E_x^{*}E_ye^{-\imath\phi}\sin\theta\cos\theta.$$

By using trignometric half angle formulas such as

$$\begin{aligned}\cos^2\theta &= \frac{1+\cos 2\theta}{2}.\\ \sin^2\theta &= \frac{1-\cos 2\theta}{2}.\\ \sin\theta\cos\theta &= \frac{\sin 2\theta}{2},\end{aligned}$$

we have

$$\begin{aligned}I(\theta,\phi) &= \frac{1}{2}\left[(E_xE_x^* + E_yE_y^*) + (E_xE_x^* - E_yE_y^*)\cos 2\theta + (E_xE_y^* + E_x^*E_y)\sin 2\theta\cos\phi\right.\\ &\left. +\imath(E_xE_y^* - E_x^*E_y)\sin 2\theta\sin\phi\right].\end{aligned}$$

The terms in the brackets are exactly the Stokes parameters. So, the intensity formula for measuring the four Stokes parameters is

$$I(\theta,\phi) = \frac{1}{2}[S_0 + S_1\cos 2\theta + S_2\sin 2\theta\cos\phi + S_3\sin 2\theta\sin\phi]. \tag{6}$$

The intensity $I(\theta,\phi)$ is a linear superposition of the four parameters. Removing the wave plate is equivalent to setting $\phi = 0$ in Eq. (6), we get

$$I(\theta,0) = \frac{1}{2}[S_0 + S_1\cos 2\theta + S_2\sin 2\theta].$$

The polarizer is now related squentially to the angles $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$. The corresponding intensities are measured as

$$\begin{aligned}I(\theta,0) &= \frac{1}{2}[S_0 + S_1].\\ I(\frac{\pi}{4},0) &= \frac{1}{2}[S_0 + S_2].\\ I(\frac{\pi}{2},0) &= \frac{1}{2}[S_0 - S_1].\end{aligned}$$

The forth Stokes parameter is measured by inserting a quarter-wave plate ($\phi = \frac{\pi}{2}$) into the optical train and rotating the linear polarizer to $\theta = \frac{\pi}{4}$.

$$I(\frac{\pi}{4},\frac{\pi}{2}) = \frac{1}{2}[S_0 - S_3].$$

Exercise# 01

Solve the four equations for S_0, S_1, S_2, S_3. The result is

$$\begin{aligned} S_0 &= I(0,0) + I(\frac{\pi}{2},0). \\ S_1 &= I(0,0) - I(\frac{\pi}{2},0). \\ S_2 &= 2I(\frac{\pi}{4},0) - I(0,0) - I(\frac{\pi}{2},0), \\ &= 2I(\frac{\pi}{4},0) - S_0. \\ S_3 &= -2I(\frac{\pi}{4},\frac{\pi}{2}) + I(0,0) + I(\frac{\pi}{2},0), \\ &= S_0 - 2I(\frac{\pi}{4},\frac{\pi}{2}). \end{aligned}$$

Exercise#02

Consider that we make the following intensity measurements on an optical beam and find

$$I(0,0) = 0.50, I(\frac{\pi}{4},0) = 0.50, I(\frac{\pi}{2},0) = 0.50, I(\frac{\pi}{4},\frac{\pi}{2}) = 0.50.$$

Find the Stokes vector of the optical beam.

Answer

$$S = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

The Stokes vector for left circularly polarized light (LCP).

1.5 Unpolarized Light

Stokes parameters not only describes completely polarized light but also unpolarized light and partially polarized light as well. Stokes used experimental definition to explain unpolarized light.

Accoring to Stokes, unpolarized light is a light that is unaffected other than by a constant attenuation by its propagation through a waveplate and/or a linear polarizer even if either or both elements are rotated around their longituinal axis.

Using this definition, Stokes determined the parameters for unpolarized light from the expression of intensity of the beam incident on the optical detector. As

$$I(\theta,\phi) = \frac{1}{2}[S_0 + S_1 \cos 2\theta + S_2 \sin 2\theta \cos\phi + S_3 \sin 2\theta \sin\phi].$$

The intensity for unpolarized light can remain constant when the polarizer is rotated through the angle θ if and only if S_0= constant and $S_1 = S_2 = S_3 = 0$. So

$$I(\theta,\phi) = \frac{1}{2}S_0.$$

The stokes vector for an unpolarized light can be written as

$$S = S_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Normalized form is obtained by setting $S_0 = 1$.So

$$S = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This result can also be obtained from Stokes vector, derived from the theoretical grounds. i.e.,

$$S = \begin{pmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ 2E_{0x}E_{0y}\cos\delta \\ 2E_{0x}E_{0y}\sin\delta \end{pmatrix}.$$

Unpolarized light is defined if

$$\begin{aligned} \left\langle E_{0x}^2 \right\rangle + \left\langle E_{0y}^2 \right\rangle &= I_0, \left\langle E_{0x}^2 \right\rangle - \left\langle E_{0y}^2 \right\rangle = 0, \\ \langle \cos\delta \rangle &= \langle \sin\delta \rangle = 0, \end{aligned}$$

where I_0 is the intensity of the optical beam.

Thus, the Stokes parameters can be used to describe the extreme states of polarized light-completely polarized light and unpolarized light.

For completely polarized light

$$S_0^2 = S_1^2 + S_2^2 + S_3^2,$$

and for unpolarized light

$$S_0^2 > 0.$$

There must be an intermediate state between these two extremes called partially polarized light.

All polarization states are given by

$$S_0^2 \geq S_1^2 + S_2^2 + S_3^2,$$

where, for completely polarized light

$$S_0^2 = S_1^2 + S_2^2 + S_3^2,$$

and for partially polarized light

$$S_0^2 > S_1^2 + S_2^2 + S_3^2,$$

and for unpolarized light

$$S_0^2 > S_1^2 + S_2^2 + S_3^2,$$

provided that $S_1 = S_2 = S_3 = 0$.

1.6 Degree of Polarization (DOP)

Degree of polarization can be defined in terms of the Stokes parameters. A mathematical statement is obtained by decomposing the optical field into a polarize an an unpolarized portion. In all states, the intansity of the beam is S_0. Subtract the polarized intensity $\sqrt{S_1^2 + S_2^2 + S_3^2}$ from the total intensity to obtain the unpolarized intensity.

The components of the Stokes vector for unpolarized portion are

$$S^{(u)} = S_0 - \sqrt{S_1^2 + S_2^2 + S_3^2}, 0, 0, 0,$$

and for the polarized portion are

$$S^{(p)} = \sqrt{S_1^2 + S_2^2 + S_3^2}, S_1, S_2, S_3,$$

where $S^{(u)}$ is the unpolarized part and $S^{(p)}$ is the polarized part of the optical beam.

The degree of polarization (DOP) is then defined to be

$$DOP = \rho = \frac{I_{pol.}}{I_{tot.}} = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0},$$

where $0 \leq DOP \leq 1$.

If $DOP = 0$, then the light is unpolarized and if $DOP = 1$, then the light is completely polarized (elliptically polarized). The condition for partially polarized light is

$$0 < DOP < 1.$$

1.7 Partially Polarized Light

Partially polarized light lies between the extremes of unpolarize light and completely polarized light. This suggests that partially polarized light is a superposition of unpolarized light and completely polarized light.

The Stokes vector for partially polarized light has the form

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = a \begin{pmatrix} S_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}, \tag{7}$$

where

$$S_0 = aS_0 + bS_0,$$

which shows that $a = 1 - b$. So, Eq. (7) becomes

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = (1-b)\begin{pmatrix} S_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix},$$

where $0 \leq b \leq 1$. For unpolarized light $b = 0$ and for polarized light $b = 1$. This means that b is identical to $DOP = \rho$. So, Eq. (7) becomes

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = (1-\rho)\begin{pmatrix} S_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}, 0 \leq \rho \leq 1.$$

It gives the Stokes vector for partially polarized light as an incoherent superposition of unpolarized an completely polarized light.

Exercise# 03

By using

$$I(\theta, \phi) = \frac{1}{2}[S_0 + S_1 \cos 2\theta + S_2 \sin 2\theta \cos\phi + S_3 \sin 2\theta \sin\phi].$$

Show that S_0 is the total intensity, S_1 describes the preponderance of the intensity of LHP over LVP light, S_2 describes the preponderance of the intensity of L+45P over L-45P light and S_3 describes the preponderance of the intensity of RCP over LCP light.

2 The Mueller Matrices for Polarizing Components

In classical optics, there are three types of polarizing elements which can be used to change the polarization state of an optical beam

(i) Polarizers

(ii) Waveplates

(iii) Rotators

The solution to the problem of the propagation of elliptically polarized light through several polarizing elements came from an entirely different area of mathematical analysis- namely- matrix algebra .

In late 1940 two new matrix algebras arose to solve these problems

(1) Jones Matrix Calculus

(2) Mueller-Stokes Matrix Calculus

Jones matrix calculus is suitable for describing the polarization behaviour in terms of amplitudes and phases.

Mueller-Stokes matrix calculus for describing the polarization behaviour in terms of intensities. Matrix approach to describe the polarization state of light is quite natural since the components of a optical field after a polarizing device are linearly related to its components before it entered the device.

2.1 The Mueller Matrix Calculus

Consider an incident beam with a given state of intensity and polarization interacting with a polarizing element.

Figure 1:

The beam propagates through the polarizing element and emerges with a new intensity and polarization state.

Both the induced beam and emerging beam are characterized by their four Stokes polarization parameters S_i and $S_i^{'}$ respectively, where $i = 0, 1, 2, 3$.

Assume that the Stokes parameters of the output beam $S_i^{'}$ can be linearly related to the Stokes parameters of the input beam S_i.

$$\begin{aligned} S_0^{'} &= m_{00}S_0 + m_{01}S_1 + m_{02}S_2 + m_{03}S_3, \\ S_1^{'} &= m_{10}S_0 + m_{11}S_1 + m_{12}S_2 + m_{13}S_3, \\ S_2^{'} &= m_{20}S_0 + m_{21}S_1 + m_{22}S_2 + m_{23}S_3, \\ S_3^{'} &= m_{30}S_0 + m_{31}S_1 + m_{32}S_2 + m_{33}S_3. \end{aligned}$$

Written as a matrix equation in terms of the Stokes vector

$$\begin{pmatrix} S_0^{'} \\ S_1^{'} \\ S_2^{'} \\ S_3^{'} \end{pmatrix} = \begin{pmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}.$$

In symbolic matrix form,

$$S^{'} = M.S,$$

where $S^{'}$ is the Stokes vector of output beam.

S is the Stokes vector of input beam.

M is 4×4 matrix called the Mueller matrix.

The elements of the Mueller matrix are real Quantities.

When an optical beam interacts with matter such as an optical polarization element-its polarization state is almost always changed.

The polarization state of an optical field can be changed by

(1) Changing the orthogonal amplitude(s)

(2) Changing the phase(s)

(3) Changing the direction of the components

A polarization element that changes the orthogonal amplitudes unequally is called a polarizer and is an anisotropic attenuator.

A polarizing element that introduces a phase shift between the orthogonal components is called a waveplate also called retarders or compensators.

A polarizing element that rotates the orthogonal components of the beam through an angle θ to a new direction is called a rotator.

3 The Mueller Matrix of the Polarizer

3.1 Polarizer

An optical elements that attenuates the orthogonal components of an optical beam unequally- that is- anisotropically- so a polarizer is an anisotropic attenuator.

It is described by two orthogonal transmission axes- characterized by transmission factors p_x and p_y respectively, the magnitudes are unequal to each other.

For equal transmission factors of each axis- the polarizer becomes a neutral density filter.

Consider a polarized beam incident on a polarizer.

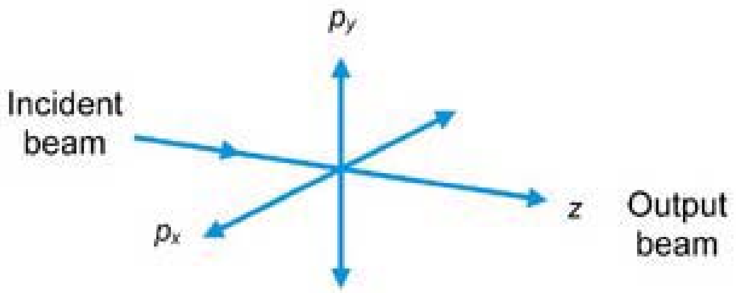

Figure 2:

The components of incident beam are represented by E_x and E_y and the components of the output beam are $E_x^{'}$ and $E_y^{'}$ respectively. The output and input field components are

$$
\begin{aligned}
E_x^{'} &= p_x E_x, 0 \leq p_x \leq 1, \\
E_y^{'} &= p_y E_y, 0 \leq p_y \leq 1,
\end{aligned}
$$

where p_x and p_y are the amplitude transmission co-efficients directed along the orthogonal x and y-axis.

(i) For perfect transmission $p_x(p_y) = 1$.

(ii) For perfect attenuation $p_x(p_y) = 0$.

If one of the transmission co-efficients is zero- polarizer has only single transmission axes- we have an ideal linear polarizer.

The Stokes parameters of incident beam are given by

$$\begin{aligned}
S_0 &= E_x E_x^* + E_y E_y^*. \\
S_1 &= E_x E_x^* - E_y E_y^*. \\
S_2 &= E_x E_y^* + E_y E_x^*. \\
S_3 &= \imath \left(E_x E_y^* - E_y E_x^* \right).
\end{aligned}$$

The Stokes parameters of the output beam are

$$\begin{aligned}
S_0^{'} &= E_x^{'} E_x^{*'} + E_y^{'} E_y^{*'}. \\
S_1^{'} &= E_x^{'} E_x^{*'} - E_y^{'} E_y^{*'}. \\
S_2^{'} &= E_x^{'} E_y^{*'} + E_y^{'} E_x^{*'}. \\
S_3^{'} &= \imath \left(E_x^{'} E_y^{*'} - E_y^{'} E_x^{*'} \right).
\end{aligned}$$

Putting values of $E_x^{'}$ and $E_y^{'}$ in terms of E_x and E_y, we find that

$$\begin{pmatrix} S_0^{'} \\ S_1^{'} \\ S_2^{'} \\ S_3^{'} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} p_x^2 + p_y^2 & p_x^2 - p_y^2 & 0 & 0 \\ p_x^2 - p_y^2 & p_x^2 + p_y^2 & 0 & 0 \\ 0 & 0 & 2p_x p_y & 0 \\ 0 & 0 & 0 & 2p_x p_y \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}.$$

The 4×4 matrix is written as

$$M_{pol}(p_x, p_y) = \frac{1}{2} \begin{pmatrix} p_x^2 + p_y^2 & p_x^2 - p_y^2 & 0 & 0 \\ p_x^2 - p_y^2 & p_x^2 + p_y^2 & 0 & 0 \\ 0 & 0 & 2p_x p_y & 0 \\ 0 & 0 & 0 & 2p_x p_y \end{pmatrix},$$

the Mueller matrix of a polarizer.

For a neutral density filter $p_x = p_y = p$. The Mueller matrix reduces to

$$M_{ND} = p^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, 0 \leq p^2 \leq 1.$$

This is the unit diagonal matrix and p^2 is the transmission factor. Now

$$\begin{pmatrix} S_0^{'} \\ S_1^{'} \\ S_2^{'} \\ S_3^{'} \end{pmatrix} = p^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

$$= p^2 \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}.$$

The polarization state of output beam and input beam are same, apart from a factor p^2. The total intensity of incident beam is reduced by a factor p^2.

$$I^{'} = p^2 I, 0 \leq p^2 \leq 1.$$

(1) An Ideal Linear Polarizer

Transmission of the polarized beam takes place along only one axis. For x-direction $p_x \neq 0, p_y = 0$, we have

$$M_{pol}(\|) = \frac{p_x^2}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The output beam that propagates through a linear polarizer of this kind is described by

$$\begin{pmatrix} S_0^{'} \\ S_1^{'} \\ S_2^{'} \\ S_3^{'} \end{pmatrix} = \frac{p_x^2}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

$$= \frac{p_x^2}{2} \begin{pmatrix} S_0 + S_1 \\ S_0 + S_1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{p_x^2}{2}(S_0 + S_1) \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

This shows regardless of the state of polarization of the input beam- the state of the output beam is always linearly horizontal polarized (LHP).

A linear polarizer is simply a polarizing element in which the polarization state of the output beam is always linearly polarized (LP) regardless of the polarization state of the input beam.

If transmission takes place only along the y-axis, then $p_y \neq 0, p_x = 0$, so

$$M_{pol}(\perp) = \frac{p_y^2}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\perp$ indicates that the transmission axis is perpendicular to the x-axis, i.e., along the y-axis.

The Stokes vector of output beam are now given by

$$\begin{pmatrix} S_0^{'} \\ S_1^{'} \\ S_2^{'} \\ S_3^{'} \end{pmatrix} = \frac{p_y^2}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

$$= \frac{p_y^2}{2}\begin{pmatrix} S_0 - S_1 \\ -S_0 + S_1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{p_y^2}{2}(S_0 - S_1)\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

This is the Stokes vector for LVP light.

Regardless of the polarization of the input state, the polarization state of the output beam is always linearly vertically polarized (LVP).

For an ideal linear polarizer, LHP or LVP light, are the only two states of polarization that can emerge from either a linear horizontal or linear vertical polarizer, respectively.

Exercise

(i) A linear horizontally polarized (LHP) beam is incident on a linear polarizer with its transmission axis in the x-direction. Find the Stokes vector of output beam.

Soln.

$$\begin{pmatrix} S_0^{'} \\ S_1^{'} \\ S_2^{'} \\ S_3^{'} \end{pmatrix} = \frac{p_x^2}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= p_x^2\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Output beam is still LHP and the intensity of the incident beam is reduced by the transmission factor p_x^2.

(ii) If the incident beam is LVP, then

$$\begin{aligned}\begin{pmatrix} S_0^{'} \\ S_1^{'} \\ S_2^{'} \\ S_3^{'} \end{pmatrix} &= \frac{p_x^2}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \\ &= p_x^2\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.\end{aligned}$$

So, the incident beam is completely blocked and no light is transmitted through the polarizer.

(iii) A linear polarizer with its transmission axis parallel to the x-axis followed by another linear polarizer with its transmission axis perpendicular to x-axis, this arrangement is called " Crossed polarizers ".

The Stokes vector of the incident beam is S and the Stokes vector of the output beam emerging from the first linear polarizer is given by

$$S^{'} = M_{pol.}(\|)S.$$

The beam $S^{'}$ then propagates through the second polarizer and we have

$$\begin{aligned}S^{''} &= M_{pol.}(\perp)S^{'} \\ &= M_{pol.}(\perp).M_{pol.}(\|)S \\ &= M.S.\end{aligned}$$

where $M = M_{pol.}(\perp).M_{pol.}(\|)$. Putting values of $M_{pol.}(\perp)$ and $M_{pol.}(\|)$, we get

$$\begin{aligned}M &= \frac{p_x^2 p_y^2}{4}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \frac{p_x^2 p_y^2}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

This is a null Mueller matrix.

Regardless of the input state of polarization, no light emerges from the crossed polarizer pair. It suggests a means of controlling the passage or blocking of light.

The Mueller matrix of the polarizer can be expressed in trigonometric rather than algebric terms.

$$M_{pol.} = \frac{1}{2}\begin{pmatrix} p_x^2+p_y^2 & p_x^2-p_y^2 & 0 & 0 \\ p_x^2-p_y^2 & p_x^2+p_y^2 & 0 & 0 \\ 0 & 0 & 2p_xp_y & 0 \\ 0 & 0 & 0 & 2p_xp_y \end{pmatrix}, 0 \leq p_x, p_y \leq 1.$$

By writing $p_x = p\cos\beta$ and $p_y = p\sin\beta$, we get

$$M_{pol.} = \frac{p}{2}\begin{pmatrix} 1 & \cos 2\beta & 0 & 0 \\ \cos 2\beta & 1 & 0 & 0 \\ 0 & 0 & \sin 2\beta & 0 \\ 0 & 0 & 0 & \sin 2\beta \end{pmatrix}, 0 \leq 2\beta \leq \pi.$$

(i) For $2\beta = 0$, the Mueller matrix is for linear horizontal polarizer.
(ii) For $2\beta = \frac{\pi}{2}$, the Mueller matrix is for neutral density filter.
(iii) For $2\beta = \pi$, the Mueller matrix is for linear vertical polarizer.

4 The Mueller Matrix of a Waveplate

Waveplate is a polarizing element that introduces a phase shift ϕ between the orthogonal components of an optical beam. A waveplate is a phase shifter. It is also called retarder or a compensator.

The phase of a propagating optical beam is given by

$$\phi = kz = \frac{2\pi}{\lambda}z,$$

where k is the wave number, z is the distance and λ is the wavelength.

If a wave travels a distance, for example $z = \frac{\lambda}{4}$ (a quarter wavelength) then the phase shift is

$$\phi = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}.$$

i.e., The distance of $\frac{\lambda}{4}$ corresponds to a phase of $\frac{\pi}{2}$. Similarly, a propagation distance of $z = \frac{\lambda}{2}$ (a half wavelength) corresponds to a phase shift of π.

A wavelength is characterized by two orthogonal axes called the fast axis and the slow axis- taken to be along the x-axis and y-axis respectively. The phase shift along the fast axis is $+\frac{\phi}{2}$ and along the slow axis is -$\frac{\phi}{2}$. The total phase shift between the two axes is

$$\frac{\phi}{2} - (-\frac{\phi}{2}) = \phi.$$

The field components of the emerging beam are related to the incident field components by

$$E_x^{'} = E_x \exp(\imath\frac{\phi}{2}).$$
$$E_y^{'} = E_y \exp(-\imath\frac{\phi}{2}).$$

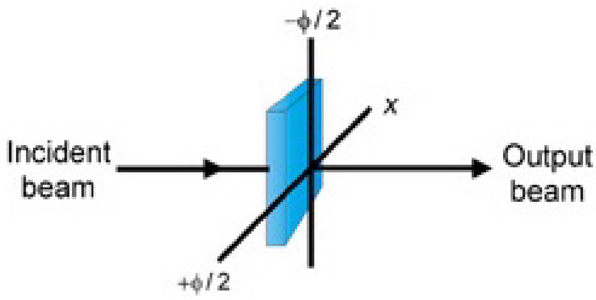

Figure 3:

Putting these values in the definition of the Stokes parameters for the input and output beams.

$$\begin{aligned} S_0' &= S_0, S_1' = S_1, \\ S_2' &= S_2\cos\phi - S_3\sin\phi, \\ S_3' &= S_2\sin\phi + S_3\cos\phi. \end{aligned}$$

In matrix form

$$\begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}.$$

The Mueller matrix for a wave plate with a phase shift of ϕ is given by

$$M_{WP}(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

There are two special cases

(i) The quarter-waveplate ($\phi = \frac{\pi}{2}$)

(ii) The half-waveplate ($\phi = \pi$)

(i) For a quarter-waveplate

$$M_{WP}(\frac{\pi}{2}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The quarter-waveplate transforms linearly polarized light to right or left circularly polarized light (RCP or LCP).

Consider the Stokes vector for incident $L \pm 45P$ light.

$$S = \begin{pmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \end{pmatrix}.$$

The Stokes vector of the output beam is

$$\begin{aligned} S' &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ \pm 1 \end{pmatrix}. \end{aligned}$$

This is the Stokes vector for right or left ($\pm$) circularly polarized light (RCP, LCP).

The transformation of LPL to circularly polarized light is an important application of the quarter-waveplate. Circularly polarized light is obtained only if the incident linearly polarized light is oriented exactly at $\pm 45^\circ$ with respect to the fast axes of the quarter-waveplate.

Exercise

Show that the quarter-waveplate can transform RCP or LCP light to $L+45P$ or $L-45P$ respectively.

$$\begin{aligned} S &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \pm 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ \mp 1 \\ 0 \end{pmatrix}. \end{aligned}$$

The QWP can be used to transform linearly polarized light to circularly polarized light or circularly polarized light to linearly polarized light.

(ii) For half-waveplate ($\phi = \pi$)

$$M_{WP}(\pi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

A half-waveplate is characterized by a diagonal matrix wiath $m_{22} = m_{33} = -1$. So,

$$
\begin{aligned}
S^{'} &= \begin{pmatrix} S_0^{'} \\ S_1^{'} \\ S_2^{'} \\ S_3^{'} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \\
&= \begin{pmatrix} S_0 \\ S_1 \\ -S_2 \\ -S_3 \end{pmatrix}.
\end{aligned}
$$

As the orientation and the ellipticity angles ψ and χ of an optical beam are given by

$$\tan 2\psi = \frac{S_2}{S_1},$$

and

$$\sin 2\chi = \frac{S_3}{S_0}.$$

Comparing the elements of Stokes parameters, we see that

$$
\begin{aligned}
\psi^{'} &= \frac{\pi}{2} - \psi. \\
\chi^{'} &= \chi - \frac{\pi}{2}.
\end{aligned}
$$

Thus, the effect of $m_{22} = m_{33} = -1$ is that it reverses the orientation and ellipticity of the polarization state of an optical beam.

Another useful property of the waveplates is that their phases add if one waveplate has a phase shift ϕ_1 and another has phase shift ϕ_2.

The product of the Mueller matrices of the two waveplates leads to a Mueller metrix whose phase is the sum of the phases $\phi = \phi_1 + \phi_2$.

$$
\begin{aligned}
M_{WP} &= M_{WP}(\phi_1).M_{WP}(\phi_2). \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi_2 & -\sin\phi_2 \\ 0 & 0 & \sin\phi_2 & \cos\phi_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi_1 & -\sin\phi_1 \\ 0 & 0 & \sin\phi_1 & \cos\phi_1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\phi_2+\phi_1) & -\sin(\phi_2+\phi_1) \\ 0 & 0 & \sin(\phi_2+\phi_1) & \cos(\phi_2+\phi_1) \end{pmatrix}.
\end{aligned}
$$

5 The Mueller Matrix of a Rotator

The final way to change the polarization state of an optical field is to allow a beam to propagates through the polarizing element that rotates its orthogonal field components $E_x(z.t)$ through an angle θ.

The angle θ gives the rotation of $E_x(z.t)$ to $E_x^{'}(z.t)$ and $E_y(z.t)$ to $E_y^{'}(z.t)$. The angle γ is the angle between E located at the ouput point P and $E_x(z.t)$. Point P is described in the primed co-ordinate system.

$$\begin{aligned} E_x^{'} &= E\cos(\gamma-\theta). \\ E_y^{'} &= E\sin(\gamma-\theta). \end{aligned}$$

In unprimed co-ordinate system, we have

$$E_x = E\cos\gamma, E_y = E\sin\gamma. \tag{8}$$

Expaniding the trignometric function, we get

$$\begin{aligned} E_x^{'} &= E\left(\cos\gamma\cos\theta+\sin\gamma\sin\theta\right). \\ E_y^{'} &= E\left(\sin\gamma\cos\theta-\sin\theta\cos\gamma\right). \end{aligned} \tag{9}$$

Substituting Eq. (8) into Eq. (9), we get the amplitude equations of rotation.

$$\begin{aligned} E_x^{'} &= E_x\cos\theta+E_y\sin\theta. \\ E_y^{'} &= -E_x\sin\theta+E_y\cos\theta. \end{aligned}$$

In order to find the Mueller matrix for the amplitude equations of rotation, we form the Stokes parameters and find the Mueller mtarix for rotation is

$$M_{ROT}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is found that the physical rotation of an angle θ leads to the appearence 2θ in M_{ROT} because we are working in the intensity domain. In amplitude domain we would expect just θ.

Rotators are used primarily to change the orientation angle of the polarization ellipse. To see this behaviour, The Stokes vector of the input and output beams are

$$\begin{pmatrix} S_0^{'} \\ S_1^{'} \\ S_2^{'} \\ S_3^{'} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}.$$

Consider that the angle of orientation of an incident beam is ψ. The orientation angle is defined as

$$\tan 2\psi = \frac{S_2}{S_1}. \tag{10}$$

For the emerging beam, the orientation angle $\psi^{'}$ is

$$\tan 2\psi^{'} = \frac{-S_1\sin 2\theta + S_2\cos 2\theta}{S_1\cos 2\theta + S_2\sin 2\theta}. \tag{11}$$

Substituting Eq. (10) inti Eq. (11), we find that

$$\tan 2\psi' = \tan(2\psi - 2\theta).$$

This implies that

$$\psi' = \psi - \theta. \tag{12}$$

Eq. (12) shows that a rotator merely rotates the polarization ellipse of the incident beam through an angle θ. The sign of θ is negative because the rotation is defined to be clockwise. If the rotation is counterclockwise, then θ is positive. i.e.,

$$\psi' = \psi + \theta.$$

For ellipticity

$$\sin 2\chi = \frac{S_3}{S_0}.$$

For output beam

$$\sin 2\chi = \frac{S_3'}{S_0'}.$$

This implies that the ellpiticity remains unchanged.

6 The Mueller matrices for Rotated Polarizing Components

In the derivation of the Mueller matrices for a polarizer,a waveplate, and a rotator, we have assumed that the axes of these devices are aligned along the x- and y-axes, respectively. Consequently, it is necessary to know the form of the Mueller matrices for the rotated polarizing elements.

In practice, the polarization elements are often rotated which greatly enhances their use and application.

Nearly all polarizing elements are rotated in an optical system. Furthermore, when the polarizing component is rotated, its usefulness is extended. For example, rotating a linear horizontal polarizer through an angle 90°, the polarizer becomes a linear vertical polarizer.

The axes of the polarizing component are rotated through an angle θ and are along the x' and y' axes. The axes of the incident beam, however, are along the x- and y-axes. In order for the incident beam to interact with the rotating polarizing element, we must determine the components of the incident beam along the axes of the rotated polarizing axis. Then after the beam has passed through the polarizing element, we must then determine the components of the emerging beam that are along the original x- and y-axes.

In order to derive the Mueller matrix for the rotated polarizing element, we must determine the components of the Stokes vector S of the incident beam that can propagate along the x' and y' axes of the rotated polarizing element.

The Stokes vector $S^{'}$ of the beam whose components are along the x$^{'}$ and y$^{'}$ axes are

$$S^{'} = M_{ROT}(\theta).S.$$

The $S^{'}$ beam now interacts with the rotated polarizing element characterized by a Mueller matrix M. The Stokes vector $S^{''}$ of the beam emerging from the rotated polaring component is

$$S^{''} = M.S^{'} = M.M_{ROT}(\theta).S.$$

We now must determine the components of the emerging beam $S^{''}$ along the original x- and y-axes. This can be done by a counterclockwise rotation of $S^{''}$ through an angle -θ.

$$\begin{aligned} S^{'''} &= M_{ROT}(-\theta).S^{''} = M_{ROT}(-\theta).M.M_{ROT}(\theta).S \\ &= M(\theta).S, \end{aligned}$$

where

$$M(\theta) = M_{ROT}(-\theta).M.M_{ROT}(\theta). \tag{13}$$

Eq. (13) is the Mueller matrix of a rotated polarizing component.

We recall that the Mueller matrix for rotation $M_{ROT}(\theta)$ is

$$M_{ROT}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Mueller matrix for a rotated polarizer is most conveniently found by expressing the Mueller matrix of a polarizer in trigonometric form, namely,

$$M_{POL}(\theta) = \frac{p^2}{2} \begin{pmatrix} 1 & \cos 2\beta & 0 & 0 \\ \cos 2\beta & 1 & 0 & 0 \\ 0 & 0 & \sin 2\beta & 0 \\ 0 & 0 & 0 & \sin 2\beta \end{pmatrix}, 0 \leq 2\beta \leq \pi.$$

The Mueller matrix for a rotated polarizer is

$$M_{POL}(\theta) = \frac{p^2}{2} \begin{pmatrix} 1 & \cos 2\beta \cos 2\theta & \cos 2\beta \sin 2\theta & 0 \\ \cos 2\beta \cos 2\theta & \cos^2 2\theta + \sin 2\beta \sin^2 2\theta & (1-\sin 2\beta)\sin 2\theta \cos 2\theta & 0 \\ \cos 2\beta \sin 2\theta & (1-\sin 2\beta)\sin 2\theta \cos 2\theta & \sin^2 2\theta + \sin 2\beta \cos^2 2\theta & 0 \\ 0 & 0 & 0 & \sin 2\beta \end{pmatrix}$$

The Mueller matrix for an ideal linear polarizer ($\beta = 0$) is

$$M_{POL}(\theta) = \frac{1}{2} \begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta \cos 2\theta & 0 \\ \sin 2\theta & \sin 2\theta \cos 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $p^2 = 1$ for an ideal linear polarizer.

(1) Linear Horizontal Polarizer

For linear horizontal polarizer $\theta = 0$. So, the Mueller matrix is

$$M_{LHP}(0) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(2) Linear Vertical Polarizer

For linear vertical polarizer $\theta = \frac{\pi}{2}$. So, the Mueller matrix is

$$M_{LVP}(\frac{\pi}{2}) = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(3) Linear +45° Polarizer

For linear +45° polarizer $\theta = \frac{\pi}{4}$. So, the Mueller matrix is

$$M_{L+45P}(\frac{\pi}{4}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(4) Linear -45° Polarizer

For linear -45° polarizer $\theta = \pi$. So, the Mueller matrix is

$$M_{L-45P}(\pi) = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

These forms of the Mueller matrix for a polarizer appear often in problems involving the generation and analysis of polarized light.

6.1 The Mueller matrix for a waveplate

The Mueller matrix for a waveplate is given by

$$M_{WP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

The Mueller matrix for a waveplate rotated through an angle θ is

$$M_{WP}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\theta + \cos\phi \sin^2 2\theta & (1-\cos\phi)\sin 2\theta \cos 2\theta & \sin\varphi \sin 2\theta \\ 0 & (1-\cos\phi)\sin 2\theta \cos 2\theta & \sin^2 2\theta + \cos\phi \cos^2 2\theta & -\sin\varphi \cos 2\theta \\ 0 & -\sin\varphi \sin 2\theta & \sin\varphi \cos 2\theta & \cos\phi \end{pmatrix}.$$

Now, the Mueller matrix for half-waveplate is at $\phi = \pi$.

$$M_{HWP}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 4\theta & \sin 4\theta & 0 \\ 0 & \sin 4\theta & -\cos 4\theta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

It looks similar to the Mueller matrix for rotation

$$M_{ROT}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Both differs in some essential ways.

(1) The presence of -ive sign along the diagonal term $m_{22} = m_{33} = -1$. This behaviour showed that the ellipticity and orientation of the emerging beam is reversed from that of the incident beam. For above case, it shows that a clockwise rotation of the half-waveplate causes the polarization ellipse of the output beam to rotate counterclockwise.

As the orientation and the ellipticity of an incident beam using a half-waveplate is reversed (in comparison to a true rotator), a half-waveplate is called a pseudo-rotator.

(2) For a mechanical rotation of θ with the half-waveplate, the polariztion ellipse is rotated by 2θ and in a direction opposite to the direction of the mechanical rotation.

For a true mechanical rotation using a rotator, the polarization ellipse is rotated by an amount θ and in the same direction as rotation.

Half-waveplates can be use as a polarization rotators but accuracy reduces to half as compared to true rotators.

If the objective is to rotate the polarization ellipse by a fixed amount it is better to use a true rotator rather than a half-waveplate. Half-waveplates are most suitable when there is need to reverse the ellipticity or the orientation of the polarization ellipse.

Consider an either right (+) or left (-) circularly polarized incident beam. The stokes vector for these two states is written as

$$S = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \pm 1 \end{pmatrix}.$$

Multiplying with $M_{HWP}(\theta)$ for $\theta = 0$, we get

$$S = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \mp 1 \end{pmatrix}.$$

It is again a circularly polarized light but its ellipticity is reversed. i.e. right circularly polarized light (RCP) is transformed to left circularly polarized light (LCP) and vice versa.

If we have incident beams that are $L+45P$ and $L-45P$, their Stokes vectors are, respectively.

$$S = \begin{pmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \end{pmatrix}.$$

After the beam pass through a half-waveplate, the Stokes vector of the emerging beas is

$$S = \begin{pmatrix} 1 \\ 0 \\ \mp 1 \\ 0 \end{pmatrix}.$$

This shows that the orientations of linearly polarized beams are reversed. i.e.,

$L + 45P$ light is transformed to $L - 45P$ and vice versa.

These are the properties of reversing the elipticity and orientation manifested by the -ive sign in m_{22} and m_{33} in the mueller matrix for the half-waveplate and its rotational behaviour that makes half-waveplates so useful.

For the Mueller matrix of a rotated quarter-waveplate, set $\phi = \frac{\pi}{2}$ in the Mueller matrix $M_{WP}(\theta)$, we get

$$M_{QWP}(\theta) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\theta & \sin 2\theta \cos 2\theta & \sin 2\theta \\ 0 & \sin 2\theta \cos 2\theta & \sin^2 2\theta & -\cos 2\theta \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \end{pmatrix}.$$

Consider an incident linearly horizontally polarized beam (LHP) so its normalized Stokes vector is

$$S = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Multiplying the above two matrices , we get the output beam.

$$S = \begin{pmatrix} 1 \\ \cos^2 2\theta \\ \sin 2\theta \cos 2\theta \\ -\sin 2\theta \end{pmatrix}.$$

The orientation angle $\psi^{'}$ and the ellipticity angle $\chi^{'}$ of emerging beam (its polarization ellipse) are given by

$$\begin{aligned} \tan 2\psi^{'} &= \frac{S_2^{'}}{S_1^{'}} = \frac{\sin 2\theta \cos 2\theta}{\cos^2 2\theta} \\ &= \tan 2\theta. \end{aligned}$$

and

$$\sin 2\chi^{'} = \frac{S_3^{'}}{S_0^{'}} = -\sin 2\theta.$$

This implies that

$$\begin{aligned} \psi^{'} &= \psi. \\ \chi^{'} &= \theta + \frac{\pi}{2}. \end{aligned}$$

It shows that the rotated quarter-waveplate can be used to generate a polarized optical beam with a specific desired orientation and ellipticity.

7 The Jones Matrix Calculus

In early 1940's Jones expressed the classical electric field components (amplitudes and absolute phases) in terms of a 2×1 column matrix and the polarizing elements (polarizer, waveplate and rotator) as 2×2 matrix.

The Jones matrix calculus describes the polarization behaviour in terms of amplitudes and phases. Jones matrices helps to solve very complex problems.

The Jones matrix for two of the polarizing elements, namely, the polarizer. The Jones matrix describes only completely polarized light. Jones matrix are simpler than the Mueller matrix.

The Jones matrix calculus has played a significant role in the development of fibre optics. We begin the development of the Jones matrix calculus by considering the following figure. The figure shows the components of the incident optical field that propagates through a polarizing element from which the output optical field emerges.

The optical field components for the emerging beam are assumed to be linearly related to the incident field components. Furthermore, the polarizing element is also assumed to respond differentialy to the input components $E_x(z.t)$ and $E_y(z.t)$ and the components can be coupled.

The plane-wave components of the optical field in terms of the complex quantities can be written as

$$E_x(z,t) = E_{0x} \exp \imath(\omega t - kz + \delta_x).$$

$$E_y(z,t) = E_{0y} \exp \imath(\omega t - kz + \delta_y).$$

The propagator $\omega t - kz$ is now suppressed , so

$$E_x(z,t) = E_{0x} \exp \imath\delta_x. \tag{14}$$

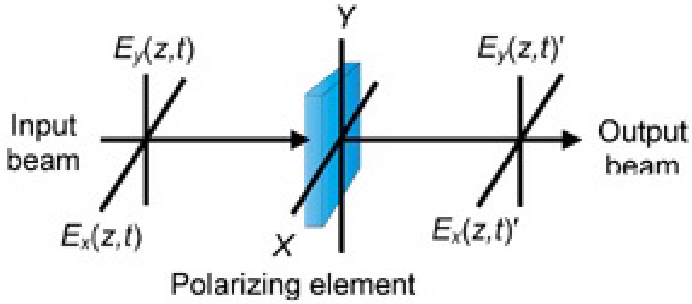

Figure 4:

$$E_y(z,t) = E_{0y} \exp \imath\delta_y. \tag{15}$$

In matrix form, above equation becomes

$$E = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_{0x} \exp \imath\delta_x \\ E_{0y} \exp \imath\delta_y \end{pmatrix}. \tag{16}$$

This 2×1 matrix is called the Jones column matrix or, simply, the Jones vector.

In the Jones vector the amplitudes E_{0x} and E_{0y} are real quantities. Before we peoceed to find the Jones vectors for various states of polarized light, we discuss the normalization of the Jones vector.

The total intensity of the optical field is given by

$$I = E_x E_x^* + E_y E_y^*.$$

In matrix form,

$$I = \begin{pmatrix} E_x^* & E_y^* \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}.$$

The row matrix $\begin{pmatrix} E_x^* & E_y^* \end{pmatrix}$ is the complex transpose of the Jones vector matrix E. i.e.,

$$E^\dagger = \begin{pmatrix} E_x^* & E_y^* \end{pmatrix}.$$

So, the intensity of the optical field is

$$I = E^\dagger E. \tag{17}$$

Substituiting Eq. (14) and (15) into Eq. (17), we get

$$E_{0x}^2 + E_{0y}^2 = I = E_0^2.$$

It is customary to set $E_0^2 = 1$, whereupon the Jones vector is said to be normalized. The normalized condition is $E^\dagger E = 1$.

We now find the Jones vector for the following degenerate states of completely polarized light.

(1) LHP Light

For this state $E_y = 0$, so Eq.(16) becomes

$$E_{LHP} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_{0x} \exp \imath\delta_x \\ 0 \end{pmatrix}.$$

From the normalization condition, we see that $E_{0x}^2 = 1$. Thus, the normalized Jones vector for LHP light is written as

$$E_{LHP} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{18}$$

(2) LVP Light

For this state $E_x = 0$, so Eq.(16) becomes

$$E_{LVP} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{19}$$

(3) L +45P Light

For this state $E_x = E_y$, so $2E_{0x}^2 = 1$ and Eq.(16) becomes

$$E_{L+45P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(4) L -45P Light

For this state $E_x = -E_y$, so $2E_{0x}^2 = 1$ and Eq.(16) becomes

$$E_{L-45P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(5) RCP Light

For this case $E_{0x} = E_{0y}$ and $\delta_y - \delta_x = \frac{\pi}{2}$. Then $2E_{0x}^2 = 1$ and Eq.(16) becomes

$$E_{RCP} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \imath \end{pmatrix}. \tag{20}$$

(6) LCP Light

For this case $E_{0x} = E_{0y}$ and $\delta_y - \delta_x = -\frac{\pi}{2}$. Then $2E_{0x}^2 = 1$ and Eq.(16) becomes

$$E_{LCP} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\imath \end{pmatrix}. \tag{21}$$

7.1 Orthogonality of the Jones Vectors

An additional property of the Jones vector is that they are orthogonal vectors.

Two vectors A and B are said to be orthogonal if $A^{\dagger}.B = 0$. For example, for LHP and LVP light we have

$$A^{\dagger}.B = \begin{pmatrix} 1 & 0 \end{pmatrix}^{*} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0,$$

so the two states are orthogonal.

Similarly, for RCP and LCP light we have

$$A^{\dagger}.B = \frac{1}{2} \begin{pmatrix} 1 & \imath \end{pmatrix}^{*} \begin{pmatrix} 0 \\ \imath \end{pmatrix} = 0,$$

so oppositely circularly polarized states are orthogonal.

Thus, the normalized condition for two Jones vectors E_1 and E_2 is

$$E_1^{\dagger} E_2 = 0.$$

The orthonormal and normalizing conditions allows us to write

$$E_i^{\dagger} E_j = \delta_{ij}, i, j = 1, 2,$$

where δ_{ij} is the kronecker delta and is defined to be

$$\begin{aligned} \delta_{ij} &= 1, i = j \\ &= 0, i \neq j. \end{aligned}$$

Any pair of orthonormal polariztaion states forms a complete set. Therefore, any arbitrary state of polarization can be expressed as a linear combination of the polarization states belonging to any pair of orhthogonal polarization states.

In a manner analogous to the superposition of incoherent intensities or Stokes vectors, we can superpose coherent amplitudes, that is, Jones vectors. To show this the Jones vectors for LHP and LVP light are E_{LHP} and E_{LVP}, we have

$$E_{LHP} = \begin{pmatrix} E_{0x} \exp \imath\delta_x \\ 0 \end{pmatrix}. \tag{22}$$

$$E_{LVP} = \begin{pmatrix} 0 \\ E_{0y} \exp \imath\delta_y \end{pmatrix}. \tag{23}$$

Adding Eq. (22) and Eq. (23), we get

$$\begin{aligned} E &= E_{LHP} + E_{LVP} \qquad (24) \\ &= \begin{pmatrix} E_{0x} \exp \imath\delta_x \\ E_{0y} \exp \imath\delta_y \end{pmatrix} \\ &= E_{ELP}, \end{aligned}$$

Which is the Jones vector for elliptically polarized light (ELP). Thus, superposing the two orthogonal linear polarizations gives rise to elliptically polarized light. If $E_{0x} = E_{0y}$ and $\delta_x = \delta_y$, then Eq. (24) becomes

$$E = E_{0x} \exp \imath\delta_x \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This implies that

$$E = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{25}$$

which aside from the normalizing factor is $L + 45P$ light. Eq. (25) can also be obtained from Eq. (18) and Eq. (19)

$$\begin{aligned} E &= E_{LHP} + E_{LVP} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Example

Consider the case of superposing left and right circularly polarized light of equal amplitudes. Then from Eq. (20) and Eq. (21), we see that

$$\begin{aligned} E &= E_{RCP} + E_{LCP} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \imath \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\imath \end{pmatrix} \\ &= \frac{2}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned}$$

which aside from the normalizing factor is the Jones vector for LHP light.

As a final example of the Jones vector, we show that the elliptically polarized light can be obtained by superposing two oppositely circularly polarized beams of unequal amplitude. The jones vector for two oppositely circular polarized beams can be represented by

$$E_+ = a \begin{pmatrix} 1 \\ \imath \end{pmatrix}, E_- = b \begin{pmatrix} 1 \\ -\imath \end{pmatrix}.$$

By the principle of coherent superposition we can then write

$$\begin{aligned} E &= E_+ + E_- \\ &= a \begin{pmatrix} 1 \\ \imath \end{pmatrix} + b \begin{pmatrix} 1 \\ -\imath \end{pmatrix} \\ &= \begin{pmatrix} a + b \\ \imath(a - b) \end{pmatrix}. \end{aligned} \tag{26}$$

In component form, Eq. (26) can be written as

$$E_x = a + b, E_y = (a - b)\exp \imath\frac{\pi}{2}. \tag{27}$$

The propagator is now restored to Eq. (27) and we now write

$$E_x(z,t) = (a + b)\exp \imath(\omega t - kz). \tag{28}$$

$$E_y(z,t) = (a - b)\exp \imath(\omega t - kz + \imath\frac{\pi}{2}). \tag{29}$$

Taking the real part of Eq. (28) and Eq. (29), we have

$$\begin{aligned} E_x(z,t) &= (a + b)\cos(\omega t - kz) \\ \frac{E_x(z,t)}{(a+b)} &= \cos(\omega t - kz). \end{aligned} \tag{30}$$

$$\begin{aligned} E_y(z,t) &= (a - b)\cos(\omega t - kz + \imath\frac{\pi}{2}) \\ &= -(a - b)\sin(\omega t - kz) \\ \frac{E_y(z,t)}{(a-b)} &= -\sin(\omega t - kz). \end{aligned} \tag{31}$$

Squarind and adding Eq. (30) and Eq. (31), we get

$$\frac{E_x^2(z,t)}{(a+b)^2} + \frac{E_y^2(z,t)}{(a-b)^2} = 1. \tag{32}$$

Eq. (32) is the equation of an ellipse in standard form whose major and minor axes lenghts are $a + b$ and $a - b$, respectively. Thus, the superposition of two oppositely circularly polarized beams of unequal magnitudes give rise to a (nonrotated) ellipse with it locus vector moving in a counterclockwise direction.

8 Jones matrices for the Polarizer, Waveplate, and Rotator

We now determine the Jones matrices for polarizers, retarders, and rotators. A 2×1 column matrix requires a 2×2 matrix in order to be transformed to another 2×1 column matrix. In order to do this, the Jones matrix calculus assumes that the components of a beam emerging from a polarizing element are linearly related to the components of the incident beam. This relation is then expressed as

$$\begin{aligned} E_x^{'} &= \jmath_{xx}E_x + \jmath_{xy}E_y. \\ E_y^{'} &= \jmath_{yx}E_x + \jmath_{yy}E_y. \end{aligned} \tag{33}$$

Before we proceed to determine the Jones matrices we should point out that if the polarization train of optical elements consists entirely of waveplates and rotators, then all the matrices are unitary. Since we shall be dealing only with square matrices with complex entities, the matrix is called unitary if

$$A^{-1} = A^{\dagger}.$$

Example

Show that the following (square) matrix is unitary.

$$A = \frac{1}{2}\begin{pmatrix} 1+\imath & 1+\imath \\ 1-\imath & -1+\imath \end{pmatrix}.$$

We first find the inverse matrix A^{-1}. This is found to be

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 1-\imath & 1+\imath \\ 1-\imath & -1-\imath \end{pmatrix} \tag{34}$$

After taking the complex transpose, we get

$$A^{\dagger} = \frac{1}{2}\begin{pmatrix} 1-\imath & 1+\imath \\ 1-\imath & -1-\imath \end{pmatrix} \tag{35}$$

Comparing Eq. (34) and Eq. (35), we see that the unitary condition is satisfied. So, A is indeed a unitary matrix.

Unitary matrices play an important role in fibre optics. Some of their major properties are

(1) A matrix is unitary if and only if its column (or rows) form an orthonormal set of vectors.

(2) The prouct of unitary matrices of the same order is a unitary matrix.

(3) All of the eigenvalues of a unitary matrix have an absolute value of 1.

(4) The determinant of a unitary matrix has an absolute value of 1.

An orthogonal matrix is also a unitary matrix whose elements are all real. If P is orthogonal then $P^{-1} = P^{\dagger}$.

Eq. (33) can be written in a matrix form as

$$\begin{pmatrix} E_x^{'} \\ E_y^{'} \end{pmatrix} = \begin{pmatrix} \jmath_{xx} & \jmath_{xy} \\ \jmath_{yx} & \jmath_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix},$$

or as a symbolic matrix equation

$$E^{'} = J.E,$$

where J is the Jones matrix of the polarizing element,

$$J = \begin{pmatrix} \jmath_{xx} & \jmath_{xy} \\ \jmath_{yx} & \jmath_{yy} \end{pmatrix}.$$

We now determine the Jones matrices for a polarizer, waveplate, and rotator.

8.1 Jones Matrix for Polarizer

A polarizer is characterized by the relations

$$E_x^{'} = p_x E_x, E_y^{'} = p_x E_y, 0 \leq p_{x,y} \leq 1.$$

For a polarizer the Jones matrix equation is

$$J_{POL} = \begin{pmatrix} p_x & 0 \\ 0 & p_y \end{pmatrix}, 0 \leq p_{x,y} \leq 1. \tag{36}$$

For an ideal linear horizontal polarizer $p_x = 1$ and $p_y = 0$. So, Eq. (36) becomes

$$J_{LHP} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Similarly, for an ideal linear vertical polarizer, we have

$$J_{LVP} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Jones Matrix for a Waveplate

The next important polarizing element of importance is the waveplate. As before, the waveplate is characterized by a phase shift of $+\frac{\phi}{2}$ along the fast axis and a phase shif of -$\frac{\phi}{2}$ along the orthonormal slow y-axis. This behavior leads to the following relation between the output and input fields,

$$E_x^{'} = \exp \imath \left(\frac{\phi}{2}\right) E_x, E_y^{'} = \exp -\imath \left(\frac{\phi}{2}\right) E_y.$$

Then the Jones matrix for the waveplate is

$$J_{WP} = \begin{pmatrix} \exp \imath \left(\frac{\phi}{2}\right) & 0 \\ 0 & \exp -\imath \left(\frac{\phi}{2}\right) \end{pmatrix}.$$

The two most common types of waveplates are the quarter-waveplate and the half-waveplate. For these devices, $\phi = \frac{\pi}{2}$ and π, respectively.

The Jones matrix for quarter-waveplate is

$$\begin{aligned} J_{QWP} &= \begin{pmatrix} \exp \imath \left(\frac{\pi}{4}\right) & 0 \\ 0 & \exp -\imath \left(\frac{\pi}{4}\right) \end{pmatrix} \\ &= \exp \imath \left(\frac{\pi}{4}\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned}$$

and the Jones matrix for half-waveplate is

$$\begin{aligned} J_{HWP} &= \begin{pmatrix} \exp \imath \left(\frac{\pi}{2}\right) & 0 \\ 0 & \exp -\imath \left(\frac{\pi}{2}\right) \end{pmatrix} \\ &= \imath \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Finally, the Jones matrix for a rotator is obtained from the familiar equations for rotation,

$$\begin{aligned} E_x^{'} &= \cos\theta E_x + \sin\theta E_y \\ E_y^{'} &= -\sin\theta E_x + \cos\theta E_y, \end{aligned}$$

which leads immediately to the Jones matrix for the rotator

$$J_{ROT} = J(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

We now apply these results to determining the Jones matrices for different polarizer configurations. The most useful of these configurations is to know the Jones matrix for a linear polarizer rotated through an angle θ. This can be done by using the familiar rotation transformation

$$J_{POL}(\theta) = J(-\theta) J_{POL} J(\theta).$$

For a rotated polarizer, we have

$$\begin{aligned} J_{POL}(\theta) &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p_x & 0 \\ 0 & p_y \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} p_x\cos^2\theta + p_y\sin^2\theta & (p_x - p_y)\sin\theta\cos\theta \\ (p_x - p_y)\sin\theta\cos\theta & p_x\sin^2\theta + p_y\cos^2\theta \end{pmatrix}. \qquad (37) \end{aligned}$$

For an ideal linear horizontal polarizer, $p_x = 1$ and $p_y = 0$, So Eq. (37) becomes

$$J_{POL}(\theta) = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}. \qquad (38)$$

The Jones matrix for an ideal linear polarizer rotated through $+45°$ is

$$J_{LHP}(\frac{\pi}{4}) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

If the linear polarizer is not ideal, then

$$J_{POL}(\frac{\pi}{4}) = \frac{1}{2}\begin{pmatrix} p_x + p_y & p_x - p_y \\ p_x - p_y & p_x + p_y \end{pmatrix}.$$

For $\theta = 0$ or $\frac{\pi}{2}$, Eq. (38) gives the Jones matrices for an ideal linear horizontal and linear vertical polarizer, respectively.

Eq. (37) also describes a neutral densit (ND) filter and its effect on polarized light. The condition for a neutral density filter is $p_x = p_y = p$. So, Eq. (37) becomes

$$J_{ND}(\theta) = P\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 \leq p \leq 1.$$

Thus the neutral density does not affect the polarization state of the incident beam.

We now investigate the behavior of the waveplate. The Jones matrix for a rotated waveplate determined from

$$\begin{aligned} J_{WP}(\theta) &= J(-\theta) J_{WP} J(\theta) \\ &= \begin{pmatrix} \exp \imath \left(\frac{\phi}{2}\right) \cos^2\theta + \exp -\imath \left(\frac{\phi}{2}\right) \sin^2\theta & (\exp \imath \left(\frac{\phi}{2}\right) - \exp -\imath \left(\frac{\phi}{2}\right)) \sin\theta\cos\theta \\ (\exp \imath \left(\frac{\phi}{2}\right) - \exp -\imath \left(\frac{\phi}{2}\right)) \sin\theta\cos\theta & \exp \imath \left(\frac{\phi}{2}\right) \sin^2\theta + \exp -\imath \left(\frac{\phi}{2}\right) \cos^2\theta \end{pmatrix} \end{aligned} \tag{39}$$

Using the half angle formulas, Eq. (39) can be wirtten as

$$J_{WP}(\theta) = \begin{pmatrix} \cos\frac{\phi}{2} + \imath \sin\frac{\phi}{2} \cos 2\theta & 2\imath \sin\frac{\phi}{2} \sin 2\theta \\ 2\imath \sin\frac{\phi}{2} \sin 2\theta & \cos\frac{\phi}{2} - \imath \sin\frac{\phi}{2} \cos 2\theta \end{pmatrix}. \tag{40}$$

For a quarter- and a half-waveplate, Eq. (40) reduces, respectively to

$$J_{QWP}(\theta) = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 + \imath \cos 2\theta & \imath \sin 2\theta \\ \imath \sin 2\theta & 1 - \imath \cos 2\theta \end{pmatrix}.$$

and

$$J_{HWP}(\theta) = \imath \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}. \tag{41}$$

Eq. (41) looks very similar to a rotator. However, when we discussed the rotated half-waveplate in the Mueller-Stokes formulation, it is actually a matrix of a pseduo-rotator.

The final Jones matrix is to determine the effect of rotating a true rotator. We have the familiar transformation

$$J_{ROT}(\theta) = J(-\theta) J_{ROT} J(\theta), \tag{42}$$

where

$$J_{ROT} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}. \tag{43}$$

So, Eq. (42) becomes

$$J_{ROT}(\theta) = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} = J(\theta). \tag{44}$$

The polarization ellipse can only be rotated by an amount β as shown in Eq. (44). We conclude that the only way to rotate the polarization ellipse mechanically is to use a half-waveplate.

9 Geometrical and Wave Optics

Matrix formulation of geometrical optics with in the paraxial ray approximation.

9.1 Ray

By definition normal to the optical wave front is called a ray. Understanding of the ray behavior makes it possible to understand the behavior of complex optical waves passing through optical elements.

9.2 Paraxial Ray

Paraxial ray is a ray whose angular deviation from the cylinderical axis is small enough that the sine and tangent of the angle can be approximated by the angle itself.

10 Matrix Formulation of Geometrical Optics

Many important features of optical resonators and lens wave guide can be obtained by geometric or ray optics, neglecting diffraction effects.

Consider a ray of light that is either transmitted by or reflected from an optical element (e.g. a lens or a mirror).

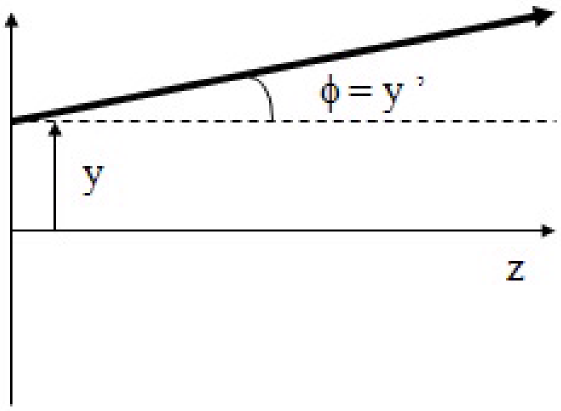

Figure 5:

If the ray is travelling approximately along the z-direction, then the ray vector at a given point is given by it radial displacement (lateral) r(z) from the z-axis and its angular displacement θ.

$$r^{'} = \frac{dr}{dz} = \tan\theta \simeq \theta.$$

Example

Consider the propagation of light ray in vacuum from $z = z_1$ to $z = z_2 = z_1 + L$. The ray displacement and slope at output plane are related to the input isplacement and slope at z_1 by

$$r_2 = r_1 + Lr_1^{'},$$

where $r_1^{'}$ is the slope at $z = z_1$. As in vacuum there is nothing to change the direction of a ray

$$r_2^{'} = r_1^{'}.$$

i.e.,

$$\begin{aligned} r_1^{'} &= \frac{dr_1}{dz_1} = \theta_1. \\ r_2^{'} &= \frac{dr_2}{dz_2} = \theta_2. \end{aligned}$$

In matrix notation, we write this equation as

$$\begin{pmatrix} r_2 \\ r_2^{'} \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_1^{'} \end{pmatrix}.$$

This implies that a ray is completely described by column (or 2×1) matrix.

The propagation of a ray through a general optical element is given by

$$\begin{aligned} r_2 &= Ar_1 + Br_1^{'}. \\ r_2^{'} &= Cr_1 + Dr_1^{'}. \end{aligned}$$

Or

$$\begin{pmatrix} r_2 \\ r_2^{'} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_1 \\ r_1^{'} \end{pmatrix}.$$

This is called ABCD matrix. The ABCD matrix completely characterizes the given optical element with in the paraxial ray approximation.

11 Free Space Propagation

Consider the free space propagation of a ray along a length $\Delta z = L$ of a given material with refractive index n. If the input and output planes lies just outside the medium, in a medium of refractive index equal to unity, then we have

$$\begin{aligned} r_2 &= r_1 + \frac{Lr_1^{'}}{n}, \\ r_2^{'} &= r_1^{'}. \end{aligned}$$

The corresponding ABCD matrix is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & \frac{L}{n} \\ 0 & 1 \end{pmatrix}.$$

Proof

Using Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

where θ_1 is the angle of incident and θ_2 is the angle of refraction. For small angles sin$\theta = \theta$, so

$$n_1\theta_1 = n_2\theta_2,$$

where $n_1 = 1$ for air. When ray enters the meium of refractive index n. Using Snell's law, we get

$$\begin{aligned}\theta_1 &= n\theta_1^{'},\\ \theta_1^{'} &= \frac{\theta_1}{n} = \frac{r_1^{'}}{n}.\end{aligned}$$

When ray again crosses the boundary angle of incident is $\theta_1^{'}$ with refractive index n so using Snell's law again, we get

$$\begin{aligned}n\theta_1^{'} &= \theta_2,\\ n\frac{\theta_1}{n} &= \theta_2,\\ \theta_1 &= \theta_2.\end{aligned}$$

This implies that

$$r_2^{'} = r_1^{'}.$$

12 ABCD Matrix for Propagation Through Lens

Consider a thin convex lens of focal length f. If lens is thin then the input and output distance from the z-axis are equal. i.e.,

$$r_2 = r_1.$$

Using laws of geometrical optics that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}. \tag{45}$$

From the figure, we have

$$p = \frac{r_1}{r_1^{'}}, q = \frac{r_2}{r_2^{'}}$$

Using Eq.(45) and $r_2 = r_1$ we get

$$r_2^{'} = -\frac{r_1}{f} + r_1^{'}.$$

This implies that

$$\begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}.$$

Bit if a ray passes through the centre of lens then $r_1 = 0$ and slopes are

$$r_1^{'} = r_2^{'}.$$

If the ray enters the lens parallel to the axis at transverse distance r_1 then $r_1^{'} = 0$, as lens is convex then ray bends towards the focal point adn its final slope is

$$r_2^{'} = -\frac{r_1}{f}.$$

This implies that

$$\begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 0 \end{pmatrix}.$$

For concave lens replace f with -f , therefore it has negative focal point.

12.1 ABCD Matrix for Spherical Mirror

Now consider the reflection of a ray by a spherical mirror of radius of curvature R (R is positive for a concave mirror and negative for a convex mirror).

In this case the z_1 and z_2 planes are taken to be coincient and to be placed just infront of the mirror. The displacement of the ray is the same immediately before and after reflection from the mirror.

$$r_2 = r_1.$$

The ray matrix of a concave mirror of curvature R and hence focal length f = $\frac{R}{2}$ becomes identical to that of a positive lens of focal length f.

$$r_2^{'} = r_1^{'} - \frac{2}{R}r_1.$$

The slope $r^{'}$ is $r^{'} > 0$ if r is increasing with propagation, otherwise $r^{'} < 0$.

Radius of curvature R is positive for a concave mirror and negative for a convex mirror. Similarly, focal length f is positive for convex (converging) lens and negative for concave (diverging) lens. So a concave mirror becomes identical to a convex lens.

The ray matrix is therefore

$$\begin{pmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{pmatrix}.$$

An important property of ABCD matrix is its determinant is equal to 1 for optical element.

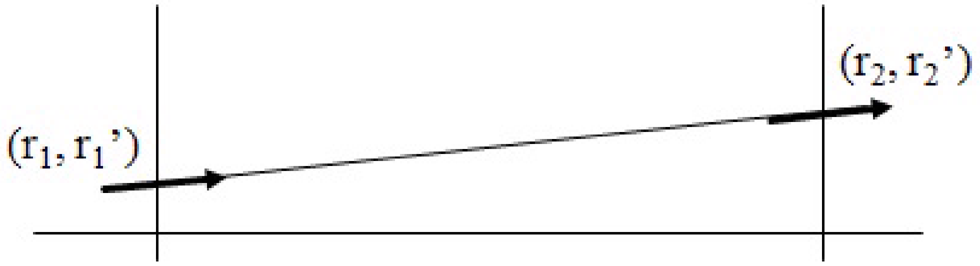

Figure 6:

13 Ray Matrices

(i) Straight section of length L.
The ray matrix is

$$\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{aligned} r_2 &= r_1 + Lr_1^{'}. \\ r_2^{'} &= r_1^{'}. \end{aligned}$$

(ii) Free space propagation of a wave through a medium of lehgth L and refractive index n.
The ray matrix is

$$\begin{pmatrix} 1 & \frac{L}{n} \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{aligned} r_2 &= r_1 + L\frac{r_1^{'}}{n}. \\ r_2^{'} &= r_1^{'}. \end{aligned}$$

(iii) Thin lens focal length is positive for converging lens and negative for iverging lens.The ray matrix is

$$\begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix},$$

and

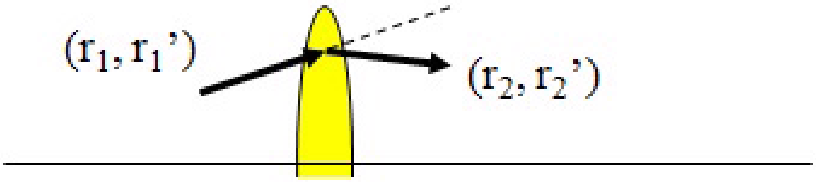

Figure 7:

$$
\begin{aligned}
r_2 &= r_1. \\
r_2^{'} &= -\frac{r_1}{f} + r_1^{'}.
\end{aligned}
$$

(iv) For spherical mirror, radius of curvature is positive for concave and negative for convex.

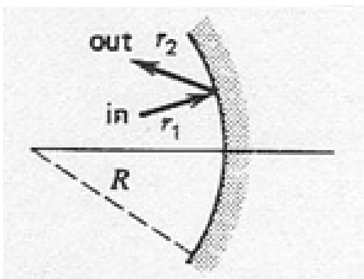

Figure 8:

The ray matrix is

$$
\begin{pmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{pmatrix}.
$$

and

$$
\begin{aligned}
r_2 &= r_1. \\
r_2^{'} &= -\frac{2}{R} + r_1^{'}.
\end{aligned}
$$

(v) Flat dielectric interface of refractive index n_1, n_2.The ray matrix is

$$
\begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix}.
$$

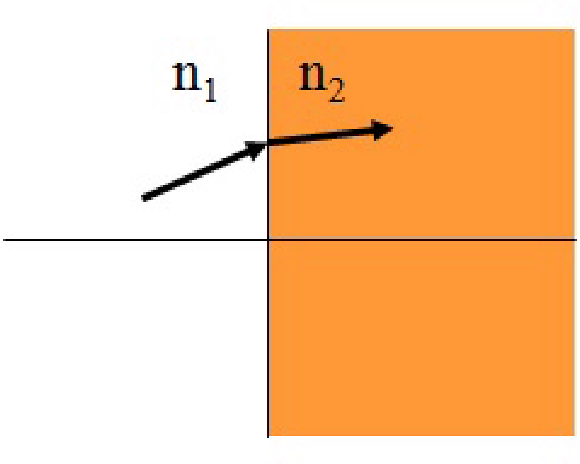

Figure 9:

and

$$\begin{aligned} r_2 &= r_1. \\ r_2' &= r_1' \frac{n_1}{n_2}. \end{aligned}$$

(vi) For curved dielectric interfaceThe ray matrix is

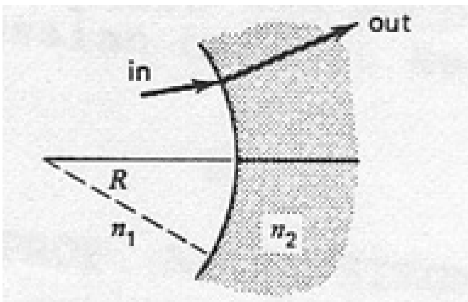

Figure 10:

$$\begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{n_2 R} & \frac{n_1}{n_2} \end{pmatrix}.$$

and

$$\begin{aligned} r_2 &= r_1. \\ r_2' &= \left(\frac{n_1 - n_2}{n_2 R}\right) r_1 + r_1' \frac{n_1}{n_2}. \end{aligned}$$