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Preparatory School to the Winter College on Optics and Energy

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FUNDAMENTALS OF FOURIER OPTICS

I. Ashraf Zahid
Quaid-I-Azam University
Pakistan

Strada Costiera 11, 34151 Trieste, Italy - Tel.+39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it

FUNDAMENTALS OF FOURIER OPTICS

IMRANA ASHRAF ZAHID
QUAID-I-AZAM UNIVERSITY
ISLAMABAD
PAKISTAN

LAYOUT

- **What is a Fourier Series?**
- **Fourier Series Expansions**
  - **Odd and Even Functions**
  - **Cosine and Sine Series**
- **Fourier Series to Fourier Transforms**
- **Fourier Transforms**
- **Properties of Fourier Transform**
- **The Convolution and Autocorrelation**
- **The Dirac Delta Function**
- **The Uncertainty Principle**
- **Examples**

What is a Fourier Series?

- A **Fourier series** decomposes a periodic function or periodic signal into a sum of simple oscillating functions, namely sines and cosines (or complex exponentials).

OR

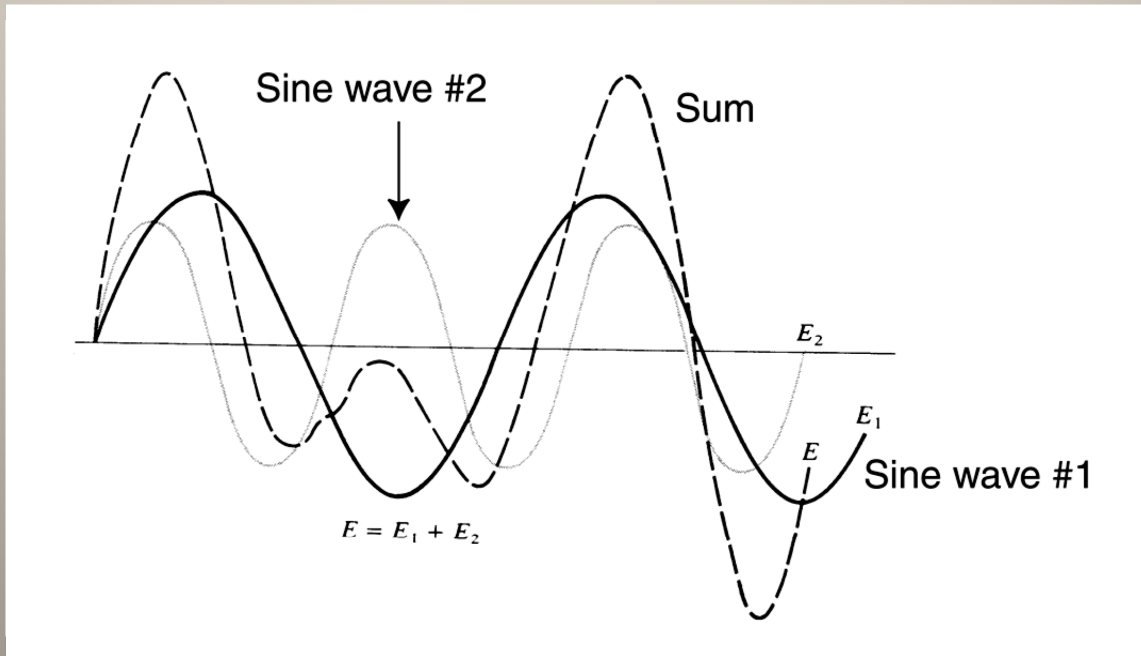
- The theory of Fourier series lies in the idea that most signals can be represented as a sum of sine waves-including square waves and triangle waves-they're possibly the most-used examples.

Sine Waves

- Sine waves have lots of interesting properties- many natural operations deal with a set of differing frequency sine waves
- Fourier series give a great picture of the kind of content of a signal.
- A sharp transition in data generally results from a high-frequency sine wave- only high-frequency sine waves have the fast-changing edge required.
- By cutting out the low frequencies- one can pick out the edges. This is particularly useful in image processing.

Anharmonic Waves are Sums of Sinusoids

Consider the sum of two sine waves (i.e., harmonic waves) of different frequencies:

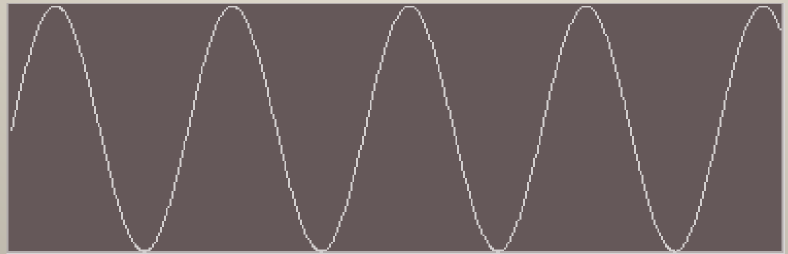


The resulting wave is periodic, but not harmonic.

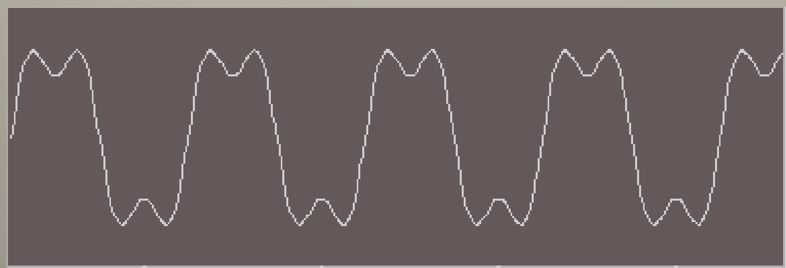
Essentially all waves are anharmonic.

Building a Square Wave

Let's see how a square wave is built up- start with a sine wave:

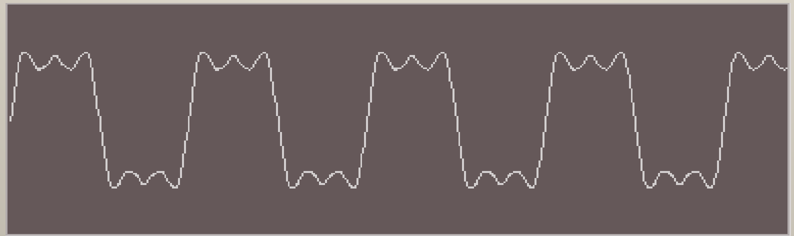


- Add another, with an amplitude 1/3 of the original and a frequency 3 times that of the first- 3rd harmonic

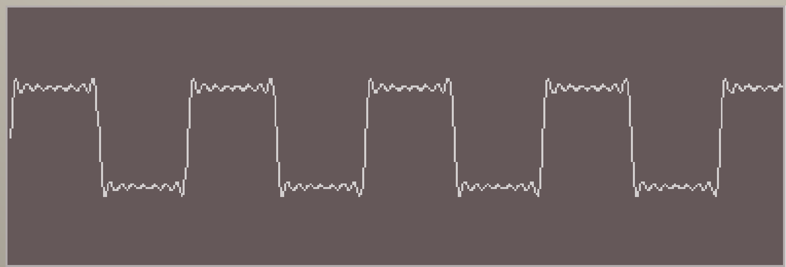


Building a Square Wave

Add another, with an amplitude 1/5 of the original and a frequency 5 times that of the first- *5th harmonic*



- If we carry on until the 15th harmonic- we should see a pattern emerging

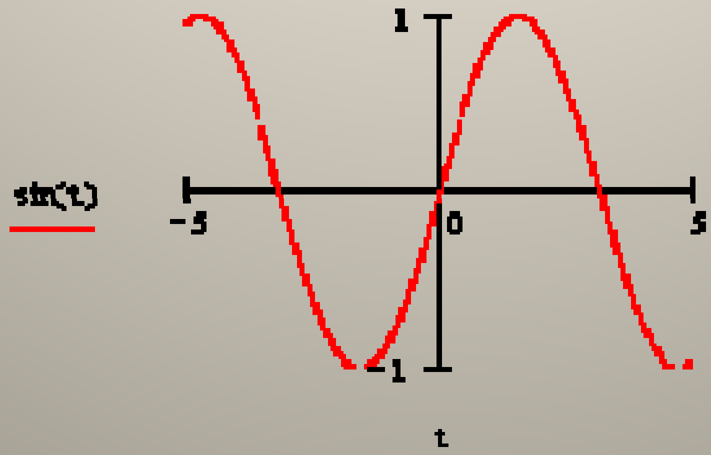


- It looks quite noisy- but bears resemblance to the square wave.
- If we add more and more harmonics,- we get closer and closer to a square wave.

Fourier Series Expansions

- **Odd and Even Functions:**

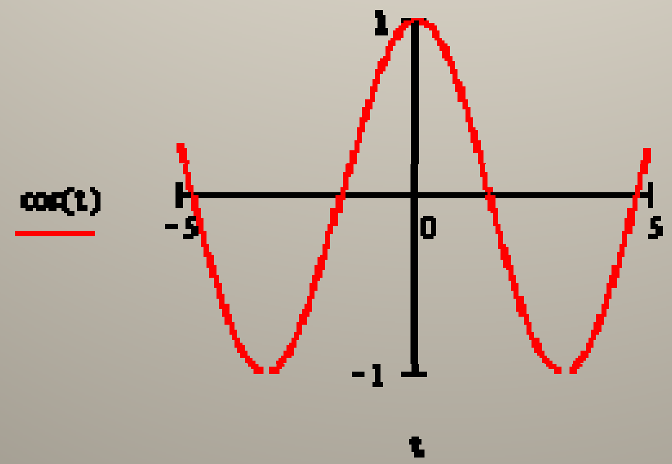
An important point is the difference between *odd* and *even* functions. For odd functions- on the other side of the *y*-axis- the function is inverted.



sin is an odd function. So, we can see that an odd function is made up of *sin* functions only. And any combination of *sin* functions will produce odd functions.

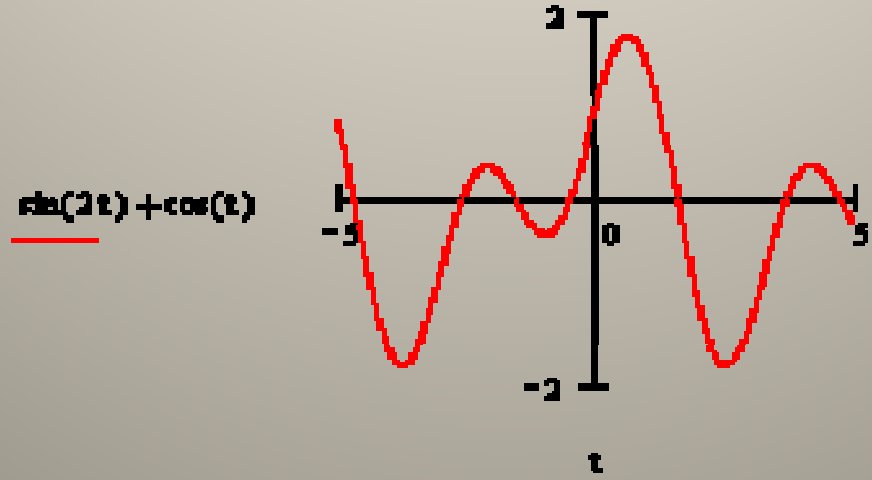
Even and Odd Functions

- **Even Function-** *cosine* is an even function-mirrored about the *y*-axis- and so combinations of *cosine* functions produce even functions.



Asymmetric Function

Some functions are neither wholly odd nor even- are asymmetric. Asymmetric functions are made up of both *sine* and *cosine* functions:



Fourier Cosine Series

Because cos(*mt*) is an even function (for all *m*), we can write an even function, *f*(*t*), as:

$$f(t) = \frac{1}{\pi} \sum_{m=0}^{\infty} F_m \cos(mt)$$

where the set $\{F_m; m = 0, 1, \ldots\}$ is a set of coefficients that define the series.

And where we'll only worry about the function *f*(*t*) over the interval (–π,π).

The Kronecker delta function

$$\delta_{m,n} \equiv \begin{cases} 1 \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases}$$

Finding the coefficients, F_m, in a Fourier Cosine Series

Fourier Cosine Series: $$f(t) = \frac{1}{\pi}\sum_{m=0}^{\infty} F_m \cos(mt)$$

To find F_m, multiply each side by $\cos(m't)$, where m' is another integer, and integrate:

$$\int_{-\pi}^{\pi} f(t)\cos(m't)\,dt = \frac{1}{\pi}\sum_{m=0}^{\infty}\int_{-\pi}^{\pi} F_m \cos(mt)\cos(m't)\,dt$$

But: $$\int_{-\pi}^{\pi}\cos(mt)\cos(m't)\,dt = \begin{cases}\pi & if\ m = m' \\ 0 & if\ m \neq m'\end{cases} \equiv \pi\,\delta_{m,m'}$$

So: $$\int_{-\pi}^{\pi} f(t)\cos(m't)\,dt = \frac{1}{\pi}\sum_{m=0}^{\infty} F_m\,\pi\,\delta_{m,m'}$$ ← only the $m' = m$ term contributes

Dropping the $'$ from the m: $$F_m = \int_{-\pi}^{\pi} f(t)\cos(mt)\,dt$$

← yields the coefficients for any $f(t)$!

Fourier Sine Series

Because sin(*mt*) is an odd function (for all *m*), we can write any odd function, *f*(*t*), as:

$$f(t) = \frac{1}{\pi} \sum_{m=0}^{\infty} F'_m \sin(mt)$$

where the set $\{F'_m;\ m = 0, 1, \ldots\}$ is a set of coefficients that define the series.

where we'll only worry about the function *f*(*t*) over the interval (–π,π).

Finding the coefficients, F'_m, in a Fourier Sine Series

Fourier Sine Series:

$$f(t) = \frac{1}{\pi}\sum_{m=0}^{\infty} F'_m \sin(mt)$$

To find F_m, multiply each side by $\sin(m't)$, where m' is another integer, and integrate:

$$\int_{-\pi}^{\pi} f(t)\sin(m't)\,dt = \frac{1}{\pi}\sum_{m=0}^{\infty}\int_{-\pi}^{\pi} F'_m \sin(mt)\sin(m't)\,dt$$

But:

$$\int_{-\pi}^{\pi} \sin(mt)\sin(m't)\,dt = \begin{cases} \pi & \text{if } m = m' \\ 0 & \text{if } m \neq m' \end{cases} \equiv \pi\,\delta_{m,m'}$$

So:

$$\int_{-\pi}^{\pi} f(t)\sin(m't)\,dt = \frac{1}{\pi}\sum_{m=0}^{\infty} F'_m\,\pi\,\delta_{m,m'}$$

← only the $m' = m$ term contributes

Dropping the $'$ from the m:

$$F'_m = \int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

← yields the coefficients for any $f(t)$!

Fourier Series

So if $f(t)$ is a general function, neither even nor odd, it can be written:

$$f(t) = \frac{1}{\pi}\sum_{m=0}^{\infty} F_m \cos(mt) + \frac{1}{\pi}\sum_{m=0}^{\infty} F'_m \sin(mt)$$

even component odd component

where

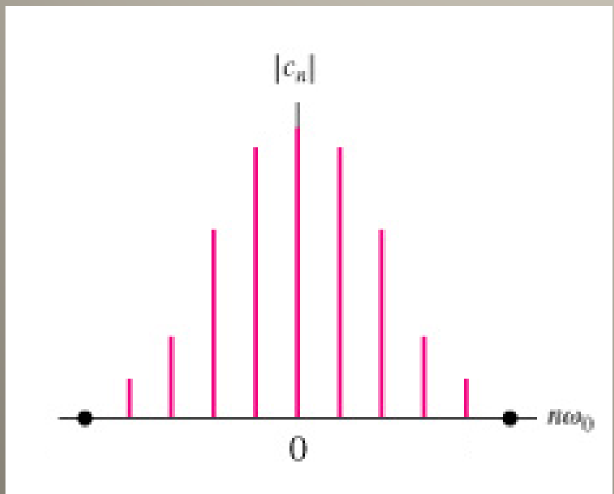
$$F_m = \int f(t)\cos(mt)\,dt \quad \text{and} \quad F'_m = \int f(t)\sin(mt)\,dt$$

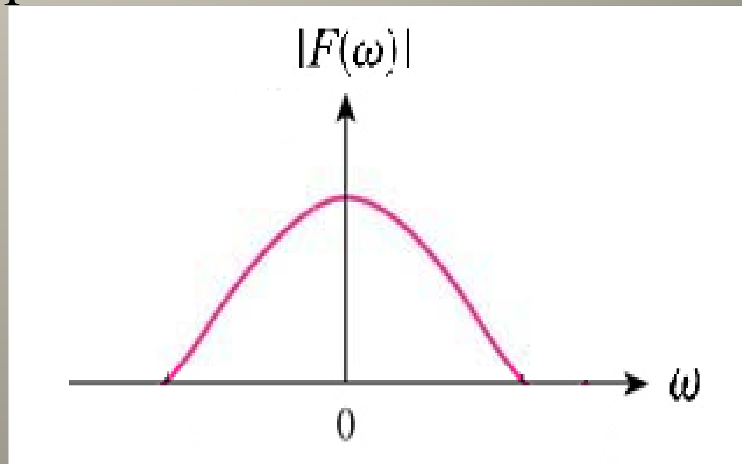
Fourier Series to Fourier Transform

- Periodic signals- represented by linear combinations of harmonically related complex exponentials
- To extend this to non-periodic signals, we need to consider a periodic signals with infinite period.
- As the period becomes infinite, the corresponding frequency components form a continuum and the Fourier series sum becomes an integral
- **Fourier transform** - is a **complex valued function** in the frequency domain

Comparison between Fourier series and Fourier transform

Fourier series	Fourier transform
• Support periodic function	• Support non-periodic function
• Discrete frequency spectrum	• Continuous frequency spectrum





The Fourier Transform

Consider the Fourier coefficients. Let's define a function $F(m)$ that incorporates both cosine and sine series coefficients, with the sine series distinguished by making it the imaginary component:

$$F(m) \equiv F_m - i\,F'_m = \int f(t)\cos(mt)\,dt \;-\; i\int f(t)\sin(mt)\,dt$$

Let's now allow $f(t)$ to range from $-\infty$ to ∞, so we'll have to integrate from $-\infty$ to ∞, and let's redefine m to be the "frequency," which we'll now call ω:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt$$

The Fourier Transform

$F(\omega)$ is called the Fourier Transform of $f(t)$. **It contains equivalent information to that in $f(t)$.** We say that $f(t)$ lives in the **time domain**, and $F(\omega)$ lives in the **frequency domain**. $F(\omega)$ is just another way of looking at a function or wave.

The Inverse Fourier Transform

The Fourier Transform takes us from $f(t)$ to $F(\omega)$.

Recall our formula for the Fourier Series of $f(t)$:

$$f(t) = \frac{1}{\pi}\sum_{m=0}^{\infty} F_m \cos(mt) + \frac{1}{\pi}\sum_{m=0}^{\infty} F'_m \sin(mt)$$

Now transform the sums to integrals from $-\infty$ to ∞, and again replace F_m with $F(\omega)$. Remembering the fact that we introduced a factor of i and included a factor of 2, we have:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(i\omega t)\,d\omega$$

Inverse Fourier Transform

The Fourier Transform and its Inverse

The Fourier Transform and its Inverse:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt$$

FourierTransform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(i\omega t)\,d\omega$$

Inverse Fourier Transform

So we can transform to the frequency domain and back. Interestingly, these transformations are very similar.

There are different definitions of these transforms. The 2π can occur in several places, but the idea is generally the same.

- Generally, the Fourier transform $F(\omega)$ exists when the Fourier integral converges
- A condition for a function $f(t)$ to have a Fourier transform is, $f(t)$ can be completely interable.
- This condition is sufficient but not necessary

$$\int_{-\infty}^{\infty} \left| f(t) \right| dt < \infty$$

Fourier Transform Notation

There are several ways to denote the Fourier transform of a function.

If the function is labeled by a lower-case letter, such as *f*, we can write:

$f(t)$ ® $F(\omega)$

If the function is already labeled by an upper-case letter, such as *E*, we can write:

$$E(t) \to \mathscr{F}\{E(t)\} \quad \text{or:} \quad E(t) \to \tilde{E}(\omega)$$

Properties of Fourier transform

1 **Linearity**:

For any constants *a, b* the following equality holds:

$$F\{af(t)+bg(t)\}=aF\{f(t)\}+bF\{g(t)\}=aF(\omega)+bG(\omega)$$

2 Scaling:

For any constant a, the following equality holds:

$$F\{f(at)\}=\frac{1}{|a|}F(\frac{\omega}{a})$$

Properties of Fourier transform 2

3 Time shifting:

$$F\{f(t-a)\} = e^{-i\omega a} F(\omega)$$

Proof:

$$F\{f(t-a)\} = \int_{-\infty}^{\infty} f(t-a) e^{-i\omega t} dt = e^{-i\omega a} \int_{-\infty}^{\infty} f(t_0) e^{-i\omega t_0} dt_0$$

- **Frequency shifting:**

$$F\{e^{i\omega t_0} f(t)\} = F(\omega - \omega_0)$$

Proof:

$$F\{e^{i\omega_0 t} f(t)\} = \int_{-\infty}^{\infty} e^{-i\omega t_0} f(t) e^{-i\omega t} dt = F(\omega - \omega_0)$$

Properties of Fourier transform 3

5. Symmetry (Duality):

$$F\{F(t)\} = 2\pi f(-\omega)$$

Proof:

The inverse Fourier transform is

$$f(t) = F^{-1}\{f(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

therefore

$$2\pi f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{i\omega t}dt = F\{F(t)\}$$

Properties of Fourier transform 4

6. Modulation:

$$F\{f(t)\cos(\omega_0 t)\} = \frac{1}{2}[F(\omega+\omega_0)+F(\omega-\omega_0)]$$

$$F\{f(t)\sin(\omega_0 t)\} = \frac{1}{2}[F(\omega+\omega_0)-F(\omega-\omega_0)]$$

Proof:

Using Euler formula, properties 1 (linearity) and 4 (frequency shifting):

$$F\{f(t)\cos(\omega_0 t)\} = \frac{1}{2}[F\{e^{i\omega_0 t} f(t)\}+F\{e^{-i\omega_0 t} f(t)\}]$$

$$= \frac{1}{2}[F(\omega+\omega_0)+F(\omega-\omega_0)]$$

Scale Theorem

The Fourier transform of a scaled function, $f(at)$:

$$\mathcal{F}\{f(at)\} = F(\omega/a) \,/\, |a|$$

Proof:

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} f(at)\exp(-i\omega t)\,dt$$

Assuming $a > 0$, change variables: $t_0 = at$

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} f(t_0)\exp(-i\omega[t_0/a])\,dt_0 \,/\, a$$

$$= \int_{-\infty}^{\infty} f(t_0)\exp(-i\,[\omega/a]\,t_0)\,dt_0 \;/\; a$$

$$= F(\omega/a) \,/\, a$$

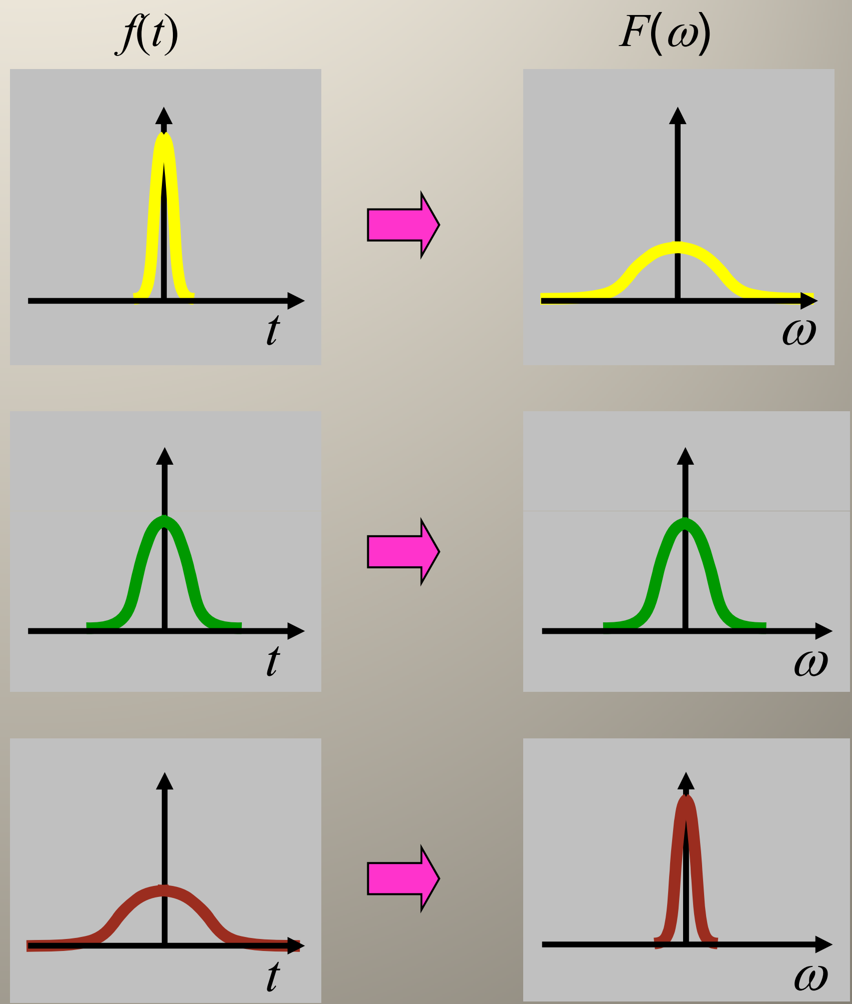
If $a < 0$, the limits flip when we change variables, introducing a minus sign, hence the absolute value.

The Scale Theorem in action

Short pulse

Medium-length pulse

Long pulse



The shorter the pulse, the broader the the spectrum!

This is the essence of the Uncertainty Principle!

Shift Theorem

The Fourier transform of a shifted function, $f(t-a)$:

Proof:

$$\mathcal{F}\left\{f(t-a)\right\} = \exp(-i\omega a)F(\omega)$$

$$\mathcal{F}\left\{f\left(t-a\right)\right\} = \int_{-\infty}^{\infty} f(t-a)\exp(-i\omega t)\,dt$$

Change variables: $t_0 = t - a$

$$\int_{-\infty}^{\infty} f(t_0)\exp(-i\omega[t_0 + a])\,dt_0$$

$$= \exp(-i\omega a)\int_{-\infty}^{\infty} f(t_0)\exp(-i\omega t_0)\,dt_0$$

$$= \exp(-i\omega a)F(\omega)$$

Parseval's Theorem

Parseval's Theorem says that the energy is the same, whether you integrate over time or frequency:

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$$

Proof:

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} f(t) f^*(t) \, dt =$$

Use ω', not ω, to avoid conflicts in integration variables.

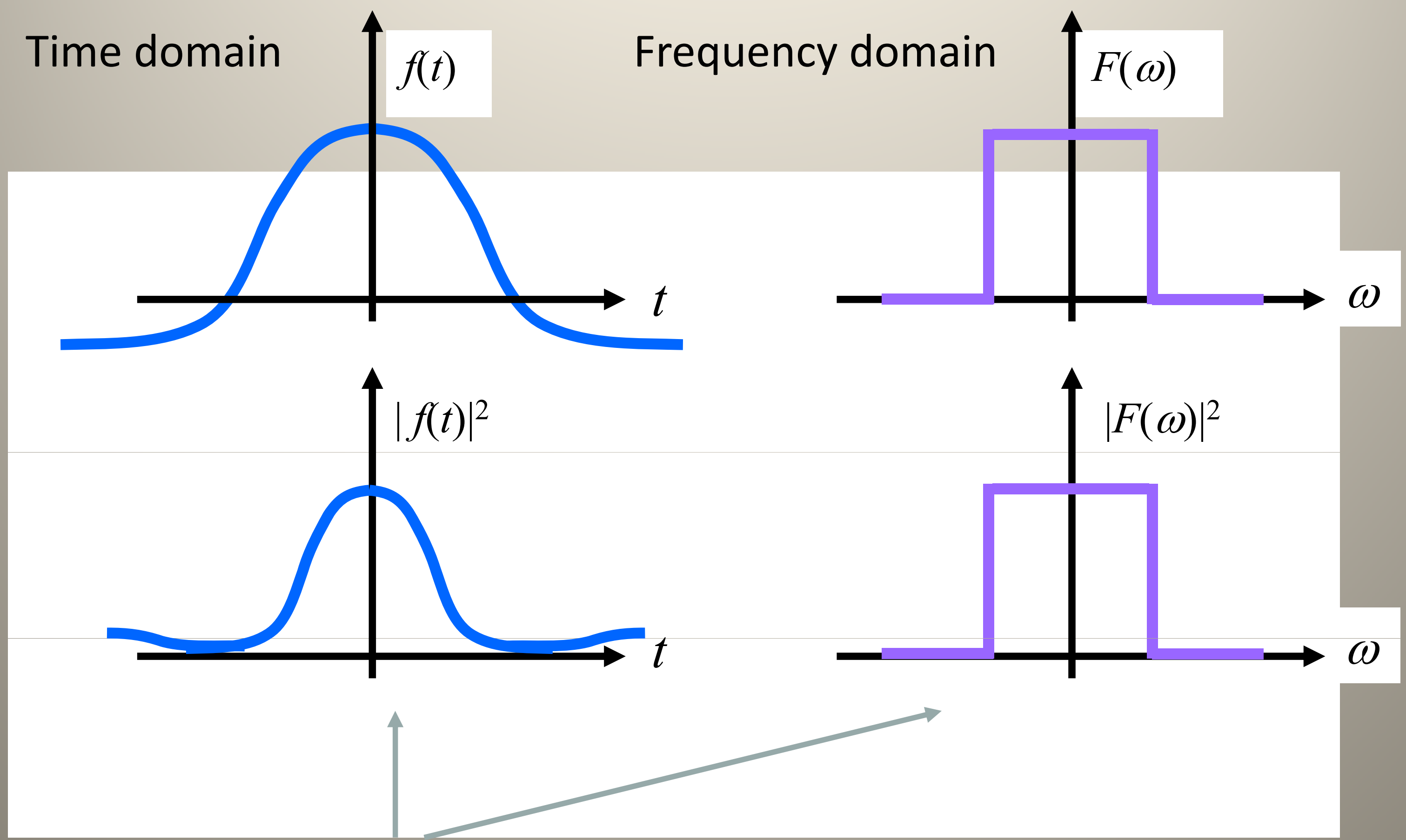
$$= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t) \, d\omega \right] \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega') \exp(-i\omega' t) \, d\omega' \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega') \left[\int_{-\infty}^{\infty} \exp(i[\omega - \omega']t) \, dt \right] d\omega' d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega') \, [2\pi\delta(\omega - \omega')] \, d\omega' d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \, F^*(\omega) \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Parseval's Theorem in action



The two areas (i.e., the light pulse energy) are the same.

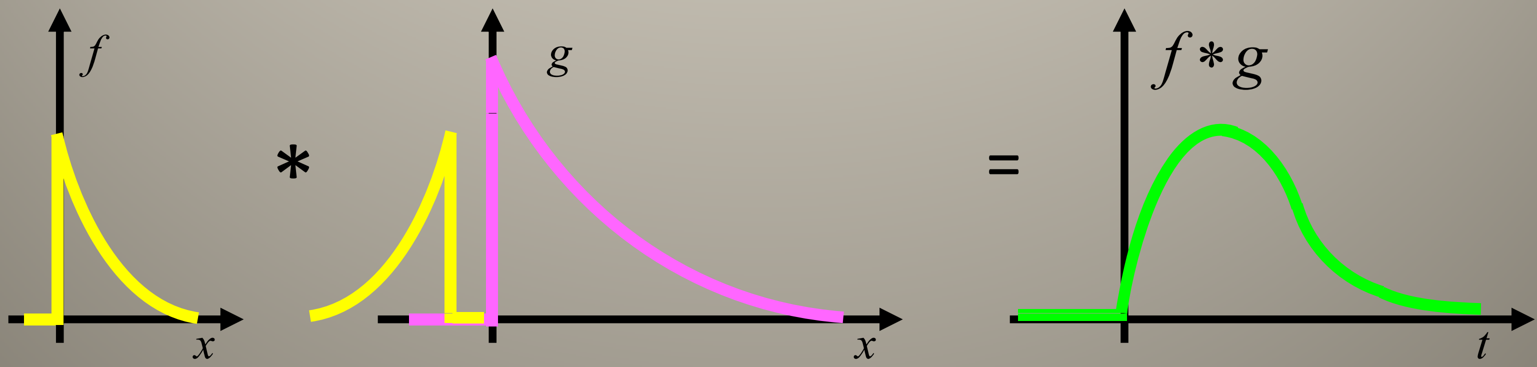
The Convolution

The convolution allows one function to smear or broaden another.

$$f(t) * g(t) \equiv f(t) \otimes g(t) \equiv \int_{-\infty}^{\infty} f(t_0)\, g(t-t_0)\, dt_0$$

$$= \int_{-\infty}^{\infty} f(t-t_0)\, g(t_0)\, dt_0$$

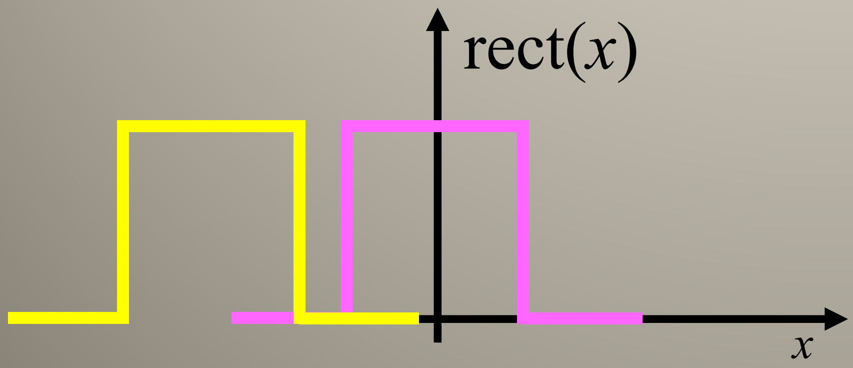
changing variables:
$t_0 \rightarrow t - t_0$

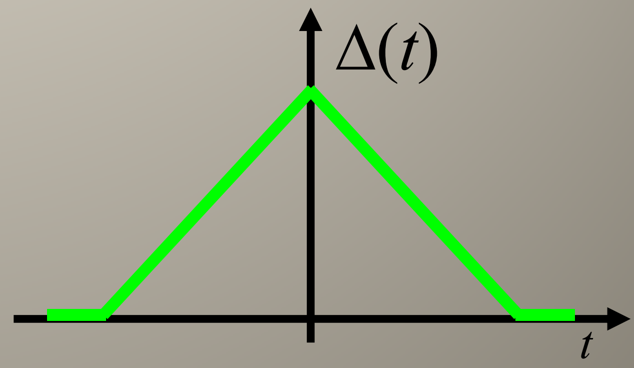


The convolution can be performed visually: rect * rect

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(t - t_0)\, g(t_0)\, dt_0$$

$$\mathrm{rect}(t) * \mathrm{rect}(t) = \Delta(t)$$





Convolution with a delta function

$$f * g = \int_{-\infty}^{\infty} f(t - t_0)\, g(t_0)\, dt_0$$

$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(t - t_0)\, \delta(t_0)\, dt_0$$

$$= f(t)$$

- Convolution with a delta function simply centers the function on the delta-function.
- This convolution does not smear out $f(t)$. Since a device's performance can usually be described as a convolution of the quantity it's trying to measure and some instrument response, a perfect device has a delta-function instrument response.

The Convolution Theorem

The Convolution Theorem turns a convolution into the inverse FT of the product of the Fourier Transforms:

$$\mathcal{F}\{f(t) * g(t)\} = F(\omega)\,G(\omega)$$

Proof:

$$\mathcal{F}\{f(t)*g(t)\} = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(t_0)\,g(t-t_0)\,dt_0\right\}\exp(-i\omega t)\,dt$$

$$= \int_{-\infty}^{\infty} f(t_0)\left\{\int_{-\infty}^{\infty} g(t-t_0)\exp(-i\omega t)\,dt\right\}dt_0$$

$$= \int_{-\infty}^{\infty} f(t_0)\{G(\omega)\exp(-i\omega t_0)\}\,dt_0$$

$$= \int_{-\infty}^{\infty} f(t_0)\exp(-i\omega t_0)\,dt_0\,G(\omega) = F(\omega)G(\omega)$$

The Autocorrelation

The convolution of a function $f(x)$ with itself (the **auto- convolution**) is given by:

$$f * f = \int_{-\infty}^{\infty} f(t_0) f(t - t_0)\, dt_0$$

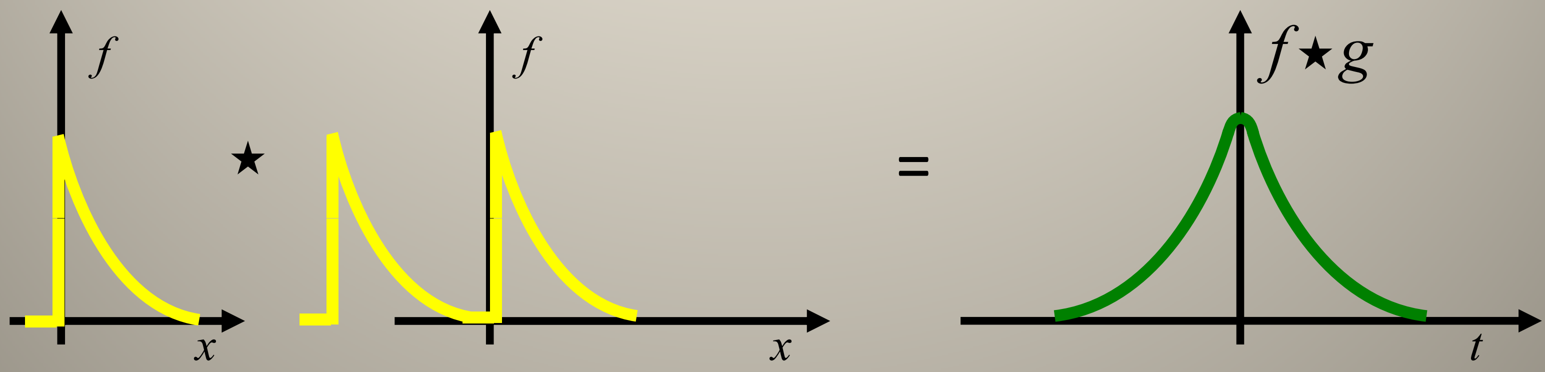
Suppose that we don't negate any of the arguments, and we complex-conjugate the 2nd factor. Then we have the **autocorrelation**:

$$f \star f \equiv \int_{-\infty}^{\infty} f(t_0) f^*(t_0 - t)\, dt_0$$

The autocorrelation plays an important role in optics.

The Autocorrelation

As with the convolution, we can also perform the autocorrelation graphically. It's similar to the convolution, but without the inversion.



Like the convolution, the autocorrelation also broadens the function in time. For real functions, the autocorrelation is symmetrical (even).

The Autocorrelation Theorem

The Fourier Transform of the autocorrelation is the spectrum!

$$\mathscr{F}\left\{\int_{-\infty}^{\infty} f(t_0)\, f^*(t_0 - t)\, dx\right\} = \left|\mathscr{F}\{f(t)\}\right|^2$$

Proof:

$$\mathscr{F}\left\{\int_{-\infty}^{\infty} f(t_0)\, f^*(t_0 - t)\, dx\right\} = \int_{-\infty}^{\infty} \exp(-i\omega t) \int_{-\infty}^{\infty} f(t_0)\, f^*(t_0 - t)\, dt_0\, dt$$

$$= \int_{-\infty}^{\infty} f(t_0) \left[\int_{-\infty}^{\infty} \exp(i\omega t)\, f(t_0 - t)\, dt\right]^* dt_0$$

$t' = -t$

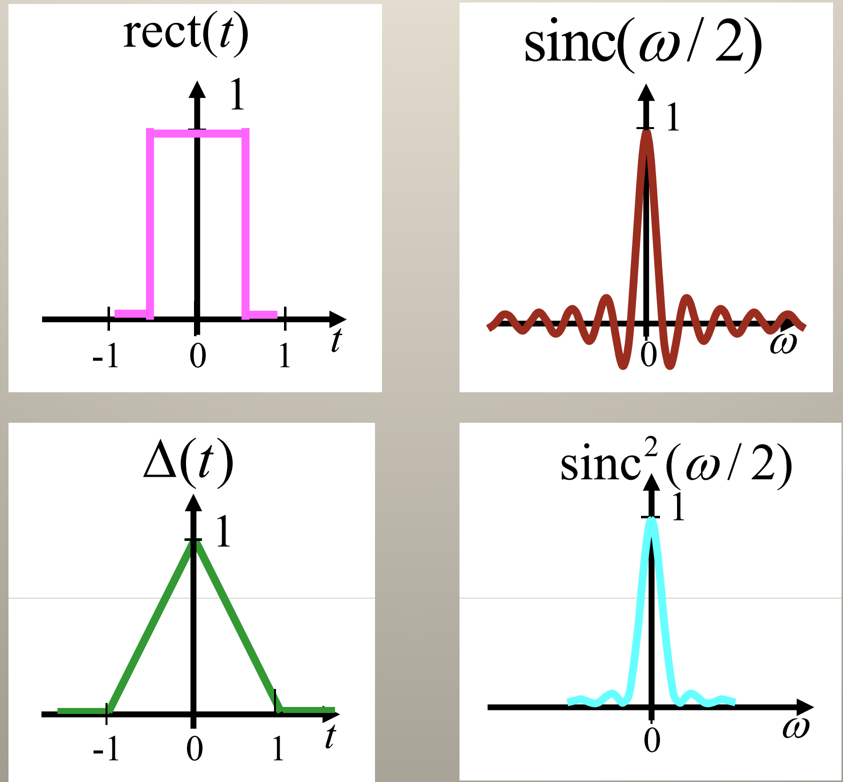
$$= \int_{-\infty}^{\infty} f(t_0) \left[\int_{-\infty}^{\infty} \exp(-i\omega t')\, f(t_0 + t')\, dt'\right]^* dt_0 = \int_{-\infty}^{\infty} f(t_0) \left[F(\omega) \exp(i\omega t_0)\right]^* dt_0$$

$$= \int_{-\infty}^{\infty} f(t_0) \exp(-i\omega t_0)\, dt_0\, F^*(\omega) = F(\omega) F^*(\omega) = \left|F(\omega)\right|^2$$

The Autocorrelation Theorem in action

$$\mathcal{F}\left\{\text{rect}(t)\right\} = \text{sinc}(\omega/2)$$

$$\text{rect}(t) \star \text{rect}(t) = \Delta(t)$$



$$\text{sinc}(\omega/2) \times \text{sinc}(\omega/2) = \text{sinc}^2(\omega/2)$$

$$\mathcal{F}\left\{\Delta(t)\right\} = \text{sinc}^2(\omega/2)$$

The Modulation Theorem:
The Fourier Transform of $E(t)\cos(w_0\, t)$

$$\mathcal{F}\left\{E(t)\cos(\omega_0 t)\right\} = \int_{-\infty}^{\infty} E(t)\cos(\omega_0 t)\exp(-i\,\omega\,t)\,dt$$

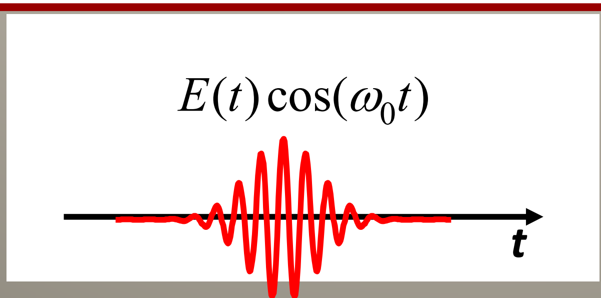
$$= \frac{1}{2}\int_{-\infty}^{\infty} E(t)\left[\exp(i\,\omega_0 t) + \exp(-i\,\omega_0 t)\right]\exp(-i\,\omega\,t)\,dt$$

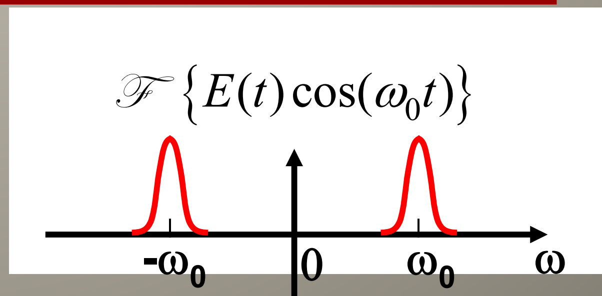
$$= \frac{1}{2}\int_{-\infty}^{\infty} E(t)\exp(-i[\omega-\omega_0]t)\,dt \quad + \quad \frac{1}{2}\int_{-\infty}^{\infty} E(t)\exp(-i[\omega+\omega_0]t)\,dt$$

$$\boxed{\mathcal{F}\left\{E(t)\cos(\omega_0 t)\right\} = \frac{1}{2}\tilde{E}(\omega-\omega_0) \quad + \quad \frac{1}{2}\tilde{E}(\omega+\omega_0)}$$

Example:

$E(t) = \exp(-t^2)$





The Fourier Transform of 1

- Using integration, the Fourier transform of 1 is

$$\mathcal{F}\{1\} = \int_{-\infty}^{\infty} 1 e^{-i\omega t} dt = \left[\frac{e^{-i\omega t}}{-i\omega} \right]_{-\infty}^{\infty} = \frac{e^{-i\infty} - e^{+i\infty}}{-i\omega} = ?$$

- At first, we may conclude that 1 has no Fourier transform, but in fact, it can be found using the principle of duality!

*Note that $e^{-i\infty}$ is neither 0 nor ∞ since

$$e^{-i\infty} = \cos\infty - i\sin\infty = ??$$

where cos ∞ and sin ∞ do not converge and |cos ∞| ≤ 1 and |sin ∞| ≤ 1.

But $e^{-(a+i\omega)\infty}$ equal to 0 since

$$e^{-(a+i\omega)\infty} = e^{-\infty}e^{-i\infty} = 0 \times e^{-i\infty} = 0$$

with condition a is real and $a > 0$.

The Spectrum

We define the spectrum, $S(\omega)$, of a wave $E(t)$ to be:

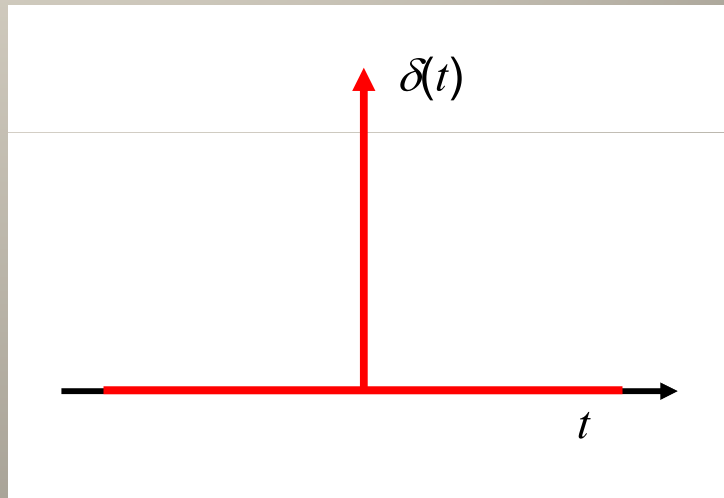
$$S(\omega) \equiv \left| \mathcal{F}\{E(t)\} \right|^2$$

This is the measure of the frequencies present in a light wave.

The Dirac delta function

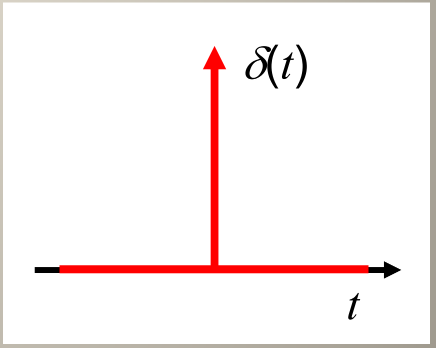
Unlike the Kronecker delta-function, which is a function of two integers, the Dirac delta function is a function of a real variable, *t*.

$$\delta(t) \equiv \begin{cases} \infty \text{ if } t = 0 \\ 0 \text{ if } t \neq 0 \end{cases}$$



Dirac δ–function Properties

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\, dt = \int_{-\infty}^{\infty} \delta(t-a) f(a)\, dt = f(a)$$

$$\int_{-\infty}^{\infty} \exp(\pm i\omega t)\, dt = 2\pi\, \delta(\omega)$$

$$\int_{-\infty}^{\infty} \exp[\pm i(\omega-\omega')t]\ \ dt = 2\pi\, \delta(\omega-\omega')$$

Example

Using the definition, find the Fourier transform of $\delta(t)$.

Then deduce the Fourier transform of 1.

Solution

*Recall the sifting property: $\int_{-\infty}^{\infty} \delta(t-a) f(t) dt = f(a)$

The Fourier transform of $f(t) = \delta(t)$ is

$$F(\omega) = \mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt = e^{-i\omega 0} = e^{0} = 1$$

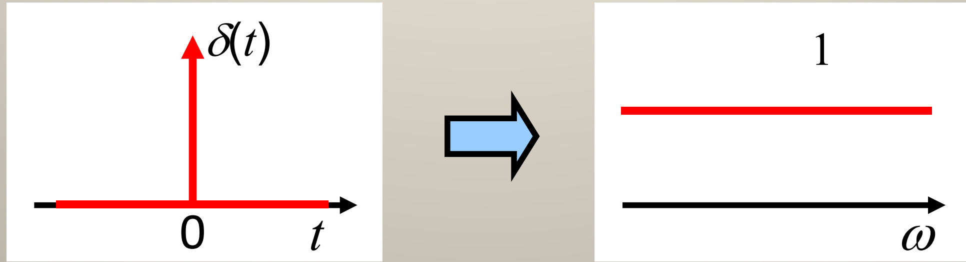
Then, using the duality principle, the Fourier transform of 1 is

$$\mathcal{F}\{1\} = 2\pi f(-\omega) = 2\pi\delta(-\omega) = 2\pi\delta(\omega)$$

since $\delta(-\omega) = \delta(\omega)$

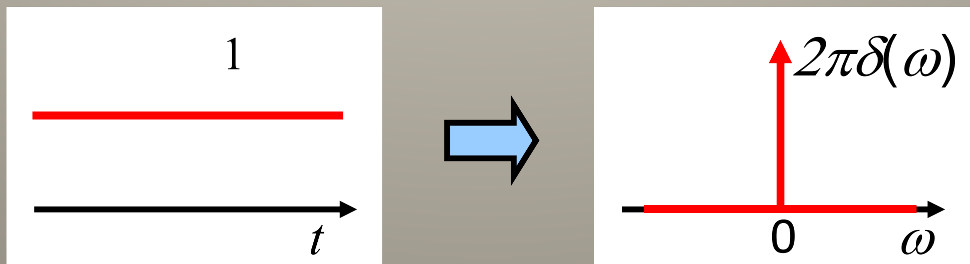
The Fourier Transform of $\delta(t)$ is 1

$$\int_{-\infty}^{\infty} \delta(t)\exp(-i\omega t)\,dt = \exp(-i\omega[0]) = 1$$



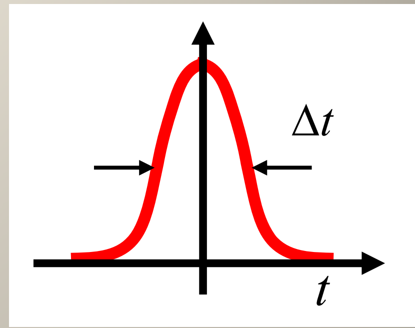
The Fourier Transform of 1 is $2\pi\delta(\omega)$:

$$\int_{-\infty}^{\infty} 1\exp(-i\omega t)\,dt = 2\pi\,\delta(\omega)$$



The Pulse Width

There are many definitions of the "width" or "length" of a wave or pulse.

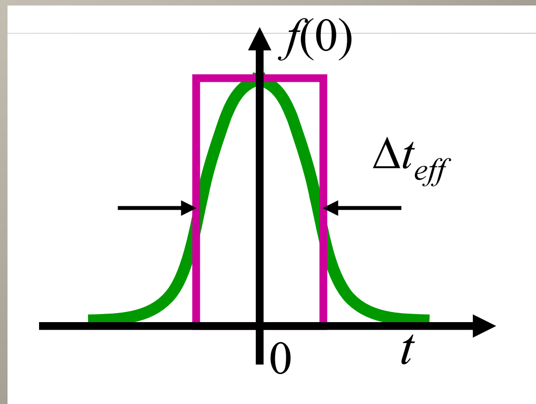


The **effective width** is the width of a rectangle whose ***height*** and ***area*** are the same as those of the pulse.

Effective width ≡ Area / height:

$$\Delta t_{eff} \equiv \frac{1}{f(0)} \int_{-\infty}^{\infty} \left| f(t) \right| dt$$

(Abs value is unnecessary for intensity.)



Advantage: It's easy to understand.

Disadvantages: The Abs value is inconvenient.

We must integrate to ± ∞.

The Uncertainty Principle

The Uncertainty Principle says that the product of a function's widths in the time domain (Δt) and the frequency domain ($\Delta\omega$) has a minimum.

Define the widths assuming $f(t)$ and $F(\omega)$ peak at 0:

$$\Delta t \equiv \frac{1}{f(0)} \int_{-\infty}^{\infty} \left| f(t) \right| dt \qquad \Delta\omega \equiv \frac{1}{F(0)} \int_{-\infty}^{\infty} \left| F(\omega) \right| d\omega$$

$$\Delta t \geq \frac{1}{f(0)} \int_{-\infty}^{\infty} f(t)\, dt = \frac{1}{f(0)} \int_{-\infty}^{\infty} f(t) \exp(-i[0]t)\, dt = \frac{F(0)}{f(0)}$$

$$\Delta\omega \geq \frac{1}{F(0)} \int_{-\infty}^{\infty} F(\omega)\, d\omega = \frac{1}{F(0)} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega[0])\, d\omega = \frac{2\pi f(0)}{F(0)}$$

Combining results:

$$\Delta\omega\, \Delta t \geq 2\pi \frac{f(0)\; F(0)}{F(0)\; f(0)}$$

(Different definitions of the widths and the Fourier Transform yield different constants.)

or: $\Delta\omega\, \Delta t \geq 2\pi$ $\quad$ $\Delta\nu\, \Delta t \geq 1$

Some common Fourier transform pairs

function	$f(x)$	$F(k) = \mathcal{F}_x[f(x)](k)$
Fourier transform--1	1	$\delta(k)$
Fourier transform--Cosine	$\cos(2\pi k_0 x)$	$\frac{1}{2}[\delta(k-k_0)+\delta(k+k_0)]$
Fourier transform--Delta function	$\delta(x-x_0)$	$e^{-2\pi i k x_0}$
Fourier transform--Exponential function	$e^{-2\pi k_0 \lvert x \rvert}$	$\frac{1}{\pi}\frac{k_0}{k^2+k_0^2}$
Fourier transform--Gaussian	$e^{-a x^2}$	$\sqrt{\frac{\pi}{a}}\, e^{-\pi^2 k^2/a}$
Fourier transform--Heaviside step function	$H(x)$	$\frac{1}{2}\left[\delta(k) - \frac{i}{\pi k}\right]$
Fourier transform--Inverse function	$-PV\frac{1}{\pi x}$	$i[1-2H(-k)]$
Fourier transform--Lorentzian function	$\frac{1}{\pi}\frac{\frac{1}{2}\Gamma}{(x-x_0)^2+(\frac{1}{2}\Gamma)^2}$	$e^{-2\pi i k x_0 - \Gamma\pi \lvert k \rvert}$
Fourier transform--Ramp function	$R(x)$	$\pi i \delta'(2\pi k) - \frac{1}{4\pi^2 k^2}$
Fourier transform--Sine	$\sin(2\pi k_0 x)$	$\frac{1}{2} i[\delta(k+k_0)-\delta(k-k_0)]$

Source: http://mathworld.wolfram.com/FourierTransform.html

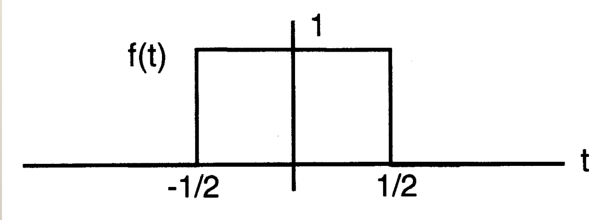
Example: the Fourier Transform of a rectangle function: rect(t)

$$F(\omega) = \int_{-1/2}^{1/2} \exp(-i\omega t)dt = \frac{1}{-i\omega}[\exp(-i\omega t)]_{-1/2}^{1/2}$$

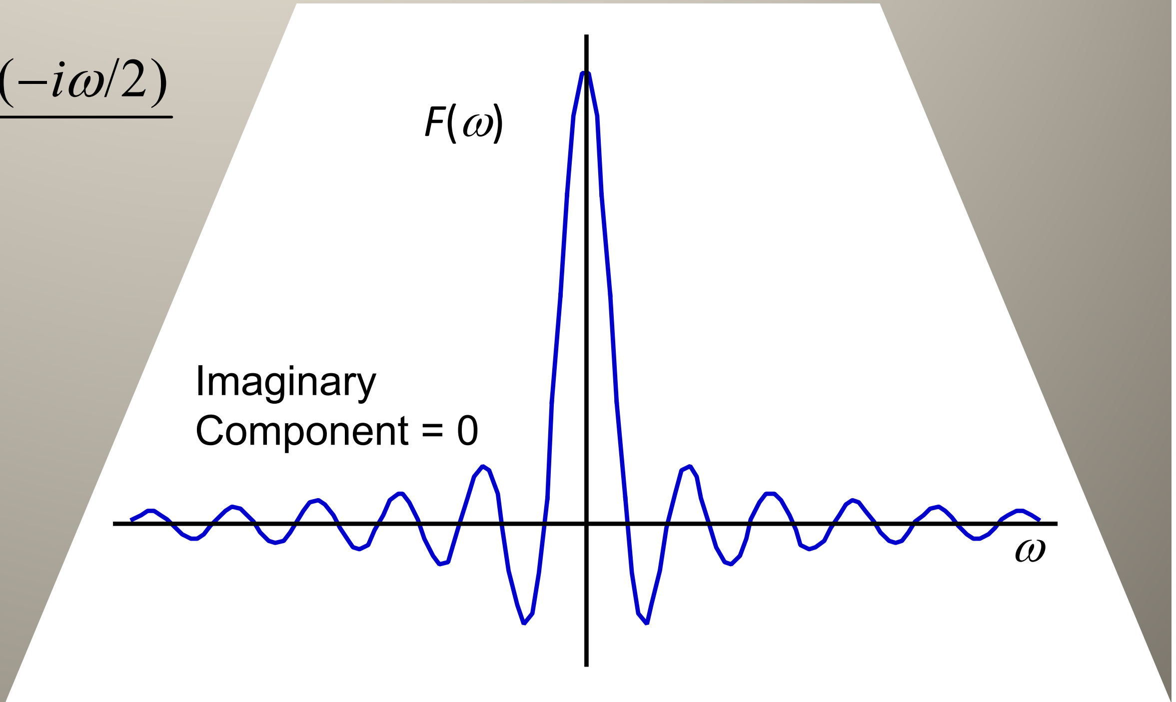
$$= \frac{1}{-i\omega}[\exp(-i\omega/2) - \exp(i\omega/2)]$$

$$= \frac{1}{(\omega/2)} \frac{\exp(i\omega/2) - \exp(-i\omega/2)}{2i}$$

$$= \frac{\sin(\omega/2)}{(\omega/2)}$$



$$F(\omega) = \text{sinc}(\omega/2)$$

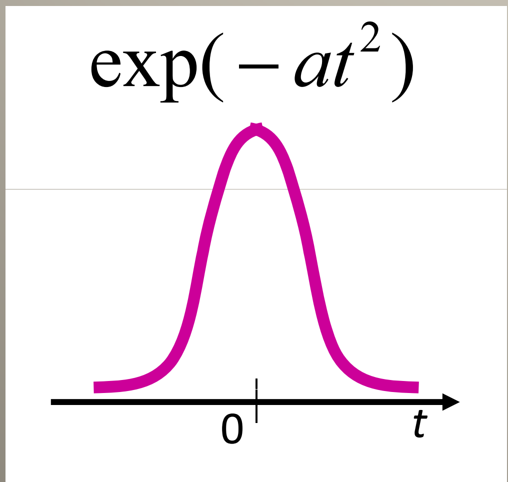


Example: the Fourier Transform of a Gaussian, exp(-at^2), is itself!

$$\mathcal{F}\{\exp(-at^2)\} = \int_{-\infty}^{\infty} \exp(-at^2)\exp(-i\omega t)\,dt$$

$$\propto \exp(-\omega^2 / 4a)$$

The details are a HW problem!



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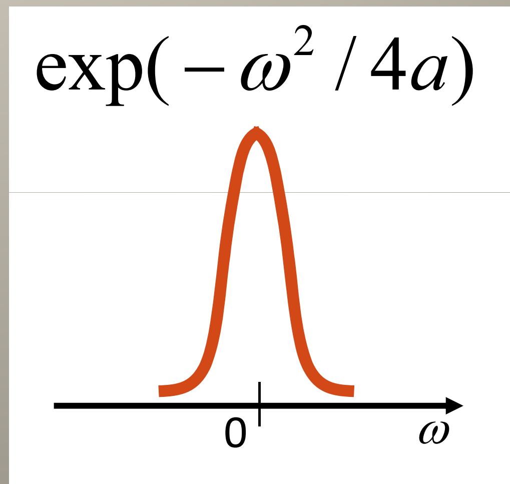
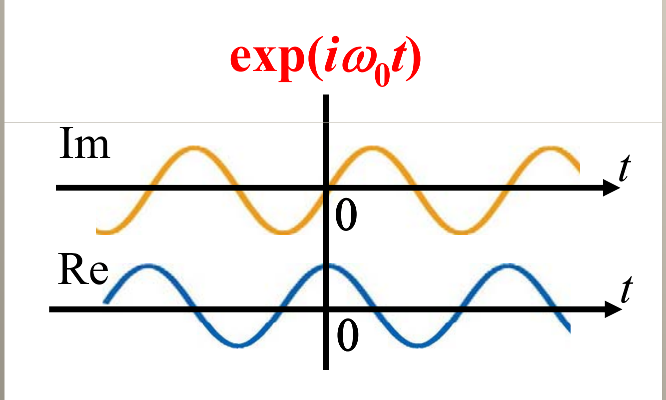
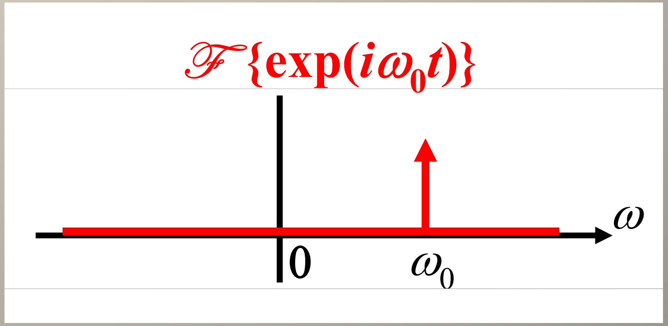


The Fourier transform of $\exp(i\omega_0 t)$

$$\mathcal{F}\{\exp(i\omega_0 t)\} = \int_{-\infty}^{\infty} \exp(i\omega_0 t)\exp(-i\omega t)\,dt$$

$$= \int_{-\infty}^{\infty} \exp(-i[\omega-\omega_0]t)\,dt = 2\pi\,\delta(\omega-\omega_0)$$





The function $\exp(i\omega_0 t)$ is the essential component of Fourier analysis. It is a pure frequency.

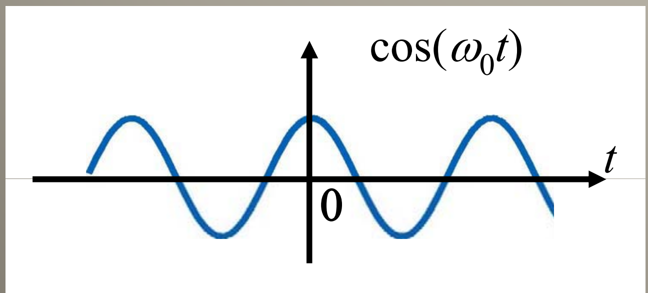
The Fourier transform of $\cos(\omega_0 t)$

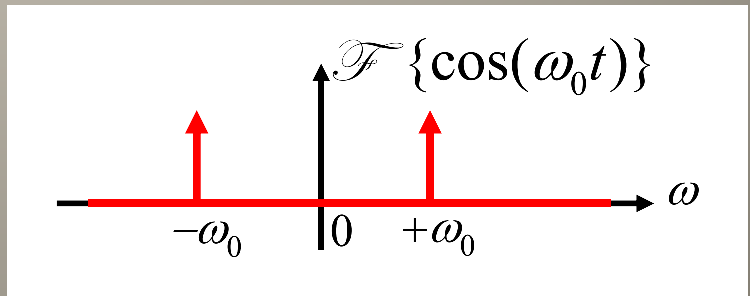
$$\mathcal{F}\{\cos(\omega_0 t)\} = \int_{-\infty}^{\infty} \cos(\omega_0 t)\exp(-i\,\omega\, t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left[\exp(i\,\omega_0\, t) + \exp(-i\,\omega_0\, t)\right]\exp(-i\,\omega\, t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \exp(-i[\omega-\omega_0]t)\,dt \quad + \quad \frac{1}{2}\int_{-\infty}^{\infty} \exp(-i[\omega+\omega_0]t)\,dt$$

$$= \pi\,\delta(\omega-\omega_0) \quad + \quad \pi\,\delta(\omega+\omega_0)$$



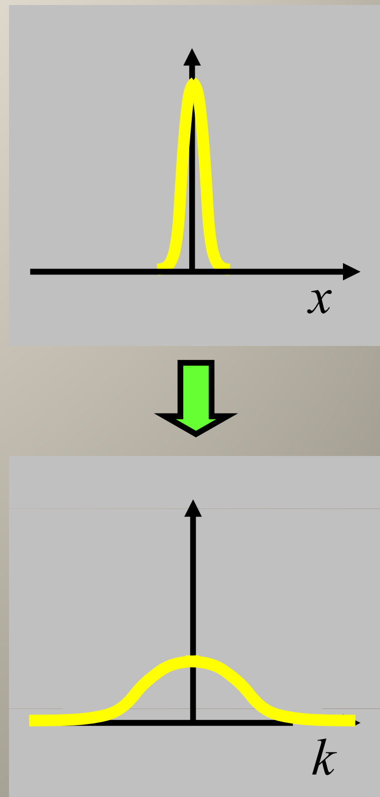


Fourier Transform with respect to space

- If $f(x)$ is a function of position,

$$F(k) = \int_{-\infty}^{\infty} f(x) \exp(-ikx)\, dx$$

$$\mathscr{F}\{f(x)\} = F(k)$$

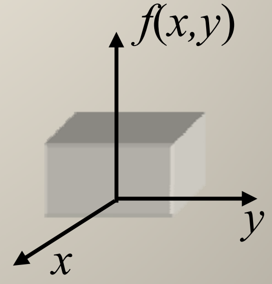


- We refer to k as the **spatial frequency**.
- Everything we've said about Fourier transforms between the t and ω domains also applies to the x and k domains.

The 2D Fourier Transform

$$\mathcal{F}^{(2)}\{f(x,y)\} = F(k_x,k_y)$$

$$= \iint f(x,y) \exp[-i(k_x x+k_y y)]\, dx\, dy$$

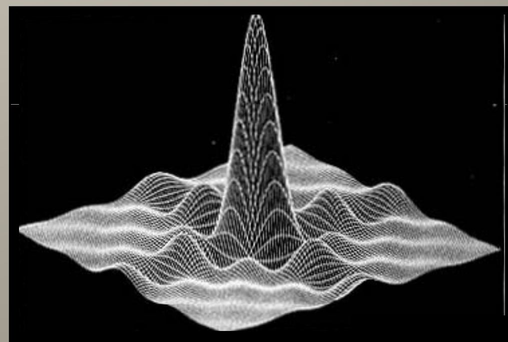


If $\quad f(x,y) = f_x(x)\, f_y(y),$

then the 2D FT splits into two 1D FT's.

But this doesn't always happen.

$\mathcal{F}^{(2)}\{f(x,y)\}$



Fourier Optics Layout

- Fourier Optics
- Wave Equations and Spectrum
- Separation of Variables
- The Superposition Integral
- Fourier Transform Pairs
- Optical Systems: General Overview
- Abbe Sine Condition
- 2D Convolution against Impulse Response Function
- Applications of Fourier Optics

Fourier Optics

- **Fourier optics** is the study of classical optics using techniques involving Fourier transforms and can be seen as an extension of the Huygens-Fresnel principle.
- Any wide wave which moves forward can actually be thought of as an infinite amount of wave points, all of which could move relatively independently of each other. The theorem basically says square objects can be made by combining an infinite amount of curved objects.
- If a wave is far enough away from something that it can be simplified to a square block moving forward, a Fraunhofer diffraction would be created.

Fourier Optics 1

- When the wave is close enough than more attention must be paid to the individual wave points and the wave can only be simplified to a round ball instead of a square block, a Fresnel diffraction would be created.
- Fourier optics forms much of the theory behind image processing techniques, as well as finding applications where information needs to be extracted from optical sources such as in quantum optics.
- Fourier optics makes use of the spatial frequency domain (k_x, k_y) as the conjugate of the spatial (x,y) domain.

Foundations of Scalar Diffraction Theory

- Diffraction plays an important role in the branches of physics and engineering that deals with wave propagation.
- To fully understand the properties of optical imaging and data processing system- it is essential that diffraction and its limitations on system performance be appreciated.
- First step to scalar diffraction theory is - Maxwell's Equations

From Vector to Scalar Theory

- **The Maxwell's equations in free space are written as**

$$\vec{\nabla} \bullet \varepsilon \vec{E} = 0$$

$$\vec{\nabla} \bullet \vec{H} = 0$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

From Vector to Scalar Theory

- Where ε and μ are permittivity and permeability of the medium in which wave is propagating.
- We assume that wave is propagating in a linear, homogenous, isotropic and non-dispersive dielectric medium.
- Isotropic –properties are independent of direction of polarization of the wave.
- Homogenous-permittivity is constant throughout the region of propagation.
- Non-dispersive - permittivity is independent of wavelength over the wavelength region occupied by the propagating wave.
- Non-magnetic-magnetic permeability is always equal to free space permeability.

From Vector to Scalar Theory

Electromagnetic Wave Equation

- Electric and magnetic fields satisfy wave equations in free space

$$\nabla^2 \vec{E} - \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\nabla^2 \vec{B} - \mu_o \varepsilon_o \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

From Vector to Scalar Theory

- Since the vector wave equation is obeyed by both **E** and **B** – an identical scalar wave equation is obeyed by all the components of those vectors.
- For example x-component of **E** obeys the equation

$$\nabla^2 E_x - \mu_o \varepsilon_o \frac{\partial^2 E_x}{\partial t^2} = 0$$

It is possible to summarize the behavior of all components of E and B through a single scalar wave equation

Propagation of Light

An optical field (light) can be described as a waveform propagating through free space (vacuum) or a material medium (such as air or glass) - the amplitude of the wave is represented by a scalar wave function u that depends on both space and time. i.e.

$$u = u(r, t)$$

Where

$$r = (x, y, z)$$

represents position in three dimensional space, and t represents time.

Scalar Wave Equation

- In a dielectric medium that is, linear, isotropic , homogeneous, and non-dispersive- all components of the electric and magnetic field behaves identically and described by single scalar wave equation.

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u(r,t) = 0$$

- where $u(r,t)$ - represents any of the scalar field components in free space.
- Fourier optics begins with scalar wave equation

The Helmholtz Equation

- One possible solution of scalar wave equation for monochromatic fields can be

$$u(r,t) = u(r)e^{i\omega t}$$

where

$$u(r) = a(r)e^{i\phi(r)}$$

- By substituting this in the wave equation- the time-independent form of the wave equation may be derived- known as the Helmholtz equation.

$$\left(\nabla^2 + k^2\right)u(r) = 0$$

The Helmholtz Equation 1

- Where

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

- is the wave number, i is the imaginary unit and $u(r)$ is the time-independent- complex valued component of the propagating wave.
- The propagation constant k, and the frequency ω are linearly related to one another, a typical characteristic of transverse electromagnetic (TEM) waves.

The Paraxial Wave Equation

- An elementary solution to the Helmholtz equation takes the form

$$u(r) = u_0(k)e^{-ik.r}$$

Where

$$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$$

is the wave vector and

$$k = \|k\| = \sqrt{k_x{}^2 + k_y{}^2 + k_z{}^2} = \frac{\omega}{c}$$

is the wave number.

The Paraxial Wave Equation 1

- Using the paraxial approximation, it is assumed that

$$k_x{}^2 + k_y{}^2 \ll k_z{}^2$$

Or equivalently

$$\sin\theta \approx \theta$$

where θ is the angle between the wave vector k and the z-axis.
As a result

$$k_z = k\cos\theta \approx k(1 - \theta^2/2)$$

and

$$u(r) = u_0(k)e^{-i(k_x x + k_y y)} e^{ikz\theta^2/2} e^{-ikz}$$

The Paraxial Wave Equation 2

- Substituting $u(r)$ into the Helmholtz equation - the Paraxial wave equation is given by

$$\nabla_T^{\,2} u_0 - 2ik \frac{\partial u_0}{\partial z} = 0$$

where

$$\nabla_T^{\,2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Is the transverse Laplacian operator in cartesian coordinates.

Diffraction

Diffraction: The bending of light around the edges or some obstacle is called diffraction.

Two types of diffractions

1. **Fresnel diffraction-** Near Field diffraction
2. **Fraunhoffer Diffraction**- Far field diffraction

Huygens's Principle: Every point on a wave front can be considered as a point source for a spherical wave

Fresnel Diffraction

- Fresnel diffraction- is a process of diffraction that occurs when wave passes through an aperture and diffracts in the near field.
- Any diffraction pattern observed is different in size and shape- depending on the distance between the aperture and observation plane.
- It occurs due to the short distance in which the diffracted waves propagates- results in a Fresnel number greater than 1

$$F = \frac{a^2}{\lambda L} \geq 1$$

The Fraunhofer Diffraction

- Fraunhofer diffraction- is a form of wave diffraction that occurs when field waves are passed through an aperture or slit causing only the size of an observed aperture image to change.
- It is due to the far field location of observation and increasingly planar nature of outgoing diffracted waves passing through the aperture.

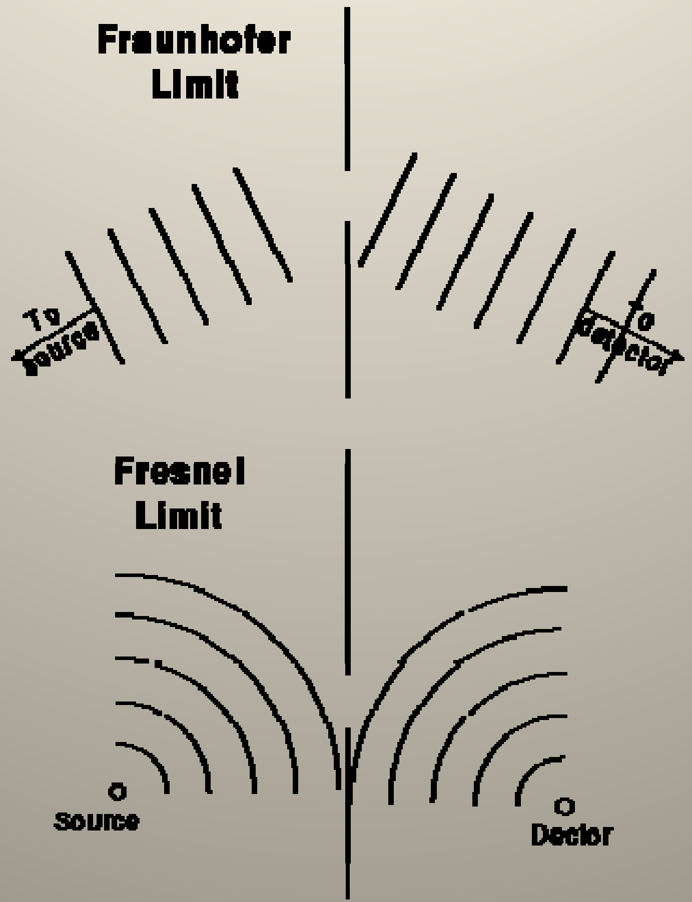
The Fraunhofer Diffraction

- If a light source and observation screen are effectively far enough from a diffraction aperture/slit- the wave fronts arriving at the aperture and the screen can be considered to be collimated or planar.

- Fraunhofer diffraction occurs when the Fresnel number is less than 1.

$$F = \frac{a^2}{\lambda L} \ll 1$$

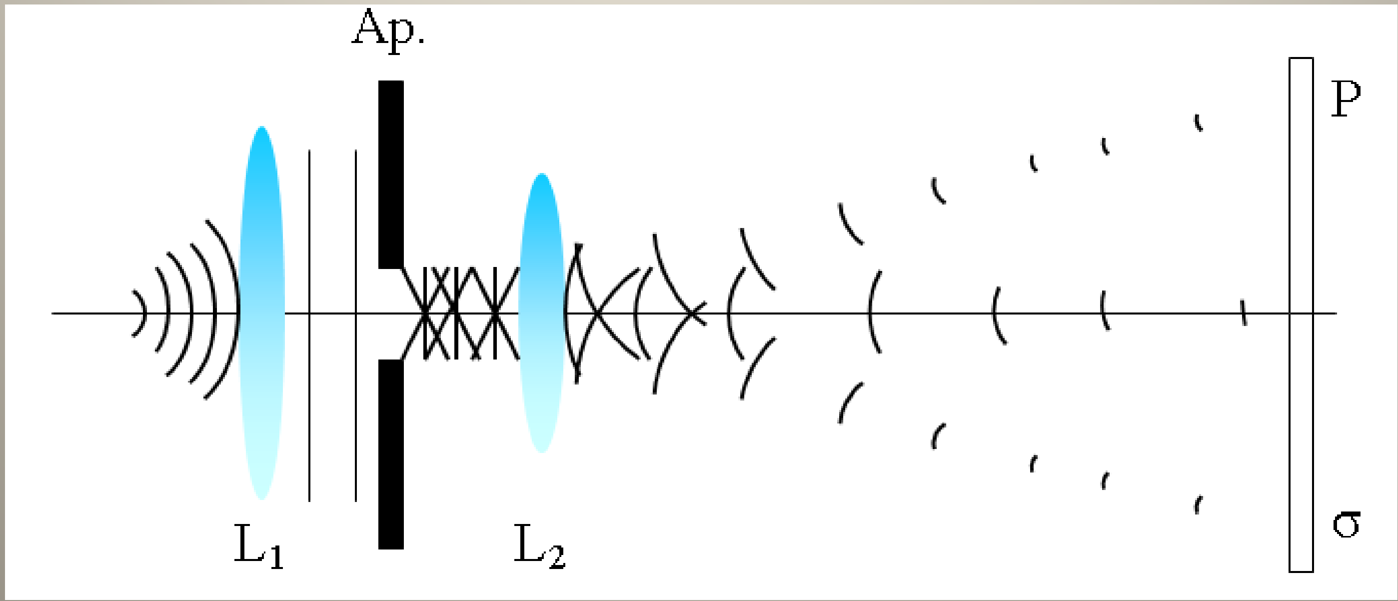
Comparison of Fresnel and Fraunhofer Diffraction



Fraunhofer Diffraction Using lenses

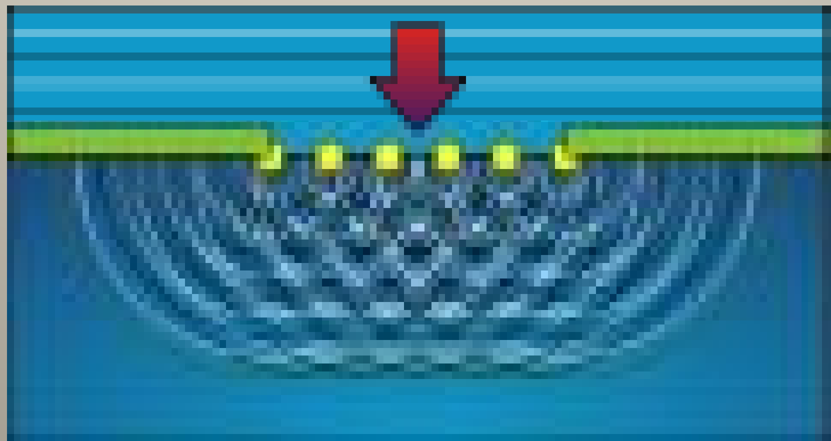
- Using a point like source for light and collimating lens it is possible to make parallel light.
- This light will then be passed through the slit.
- Another lens will focus the parallel light on observation plane.
- The same setup with multiple slits can also be used- creating a different diffraction pattern.

Fraunhofer Diffraction Using lenses



Huygens's Principle

- **Huygens's Principle:** Every point on a wave front can be considered as a point source for a spherical wave



The Huygens-Fresnel Principle

The Huygens-Fresnel Principle describes the value of field $U(P_0)$ as a superposition of diverging spherical waves originating from the secondary sources located at each and every point P_1 with in the aperture Σ.

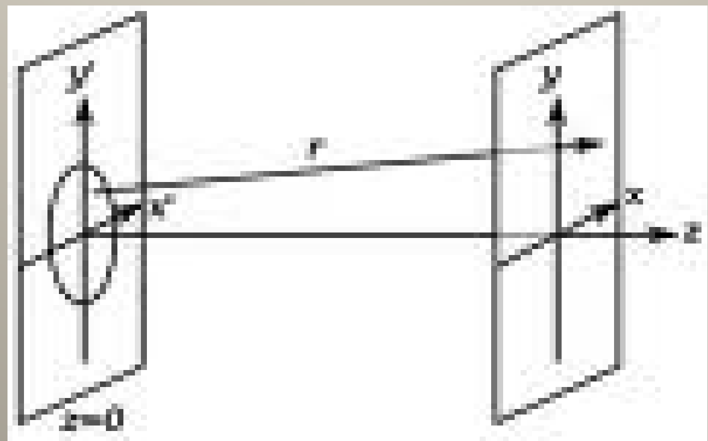
$$U(P_0)=\frac{1}{i\lambda}\iint_s U(P_1)\frac{\exp(ikr_{01})}{r_{01}}\cos\theta ds$$

Where θ is the angle between element of area and displacement vector between two point

The Huygens-Fresnel Principle

- According to the Huygens- Fresnel Principle every point in the plane (x',y',0) will act as a point source for spherical wave of type

$$U(r) = \frac{A}{r} e^{ikr}$$



The Fresnel Diffraction Integral

- The field diffraction pattern at a point (x,y,z) is given by

$$U(x,y,z)=\frac{z}{i\lambda}\iint U(x',y',0)\frac{e^{ikr}}{r}dx'dy'$$

where

$$r=\sqrt{(x-x')^2+(y-y')^2+z^2}$$

The Fresnel Approximation

- Under

$$r \approx z + \frac{(x-x')^2 + (y-y')^2}{2z}$$

The Fresnel diffraction integral can be written as

$$U(x,y,z) = \frac{e^{ikz}}{i\lambda z} \iint U(x',y',0) e^{\frac{ik}{2z}\left[(x-x')^2 + (y-y')^2\right]} dx'dy'$$

It shows that the propagating field is a spherical wave- orginating at the aperature and moving along z-axis.

This integral modulates the amplitude and phase of the spherical wave.

The Fraunhofer Diffraction Integral

- In scalar diffraction theory, the Fraunhofer approximation is a far field approximation made to the Fresnel diffraction integral.

- Under this condition the quadratic phase factor under the integral sign is approximately unit over the entire aperture.

The Fraunhofer Diffraction Integral

The Fraunhofer diffraction occurs if

$$k(x'^2 + y'^2) \ll z$$

Using this approximation in Fresnel diffraction integral we get

$$U(x,y) = \frac{e^{ikz} e^{\frac{ik}{2z}(x^2+y^2)}}{i\lambda z} \iint U(x',y') e^{-i\frac{2\pi}{kz}(x'x+y'y)} dx' dy'$$

The Plane Wave Spectrum

- The plane wave spectrum concept is the basic foundation of Fourier Optics.
- The plane wave spectrum is a continuous spectrum of *uniform* plane waves and there is one plane wave component in the spectrum for every tangent point on the far-field phase front.
- The amplitude of that plane wave component would be the amplitude of the optical field at that tangent point.
- In the far field defined as

$$\text{Range} = 2D^2/\lambda$$

- D is the maximum linear extent of the optical sources and λ is the wavelength.

The Plane Wave Spectrum 1

- The plane wave spectrum is often regarded as being discrete for certain types of periodic gratings.
- In reality- the spectra from gratings are continuous as well -since no physical device can have the infinite extent required to produce a true line spectrum.
- For optical systems- bandwidth is a measure of how far a plane wave is tilted away from the optic axis.
- This type of bandwidth is often referred to as angular bandwidth or spatial bandwidth.

The Plane Wave Spectrum 2

- The plane wave spectrum arises as the solution of the homogeneous electromagnetic wave equation in rectangular coordinates.
- In the frequency domain, the homogeneous electromagnetic wave equation or the Helmholtz equation takes the form

$$\nabla^2 E_r + k^2 E_r = 0$$

where r= (x, y, z) and $k = \frac{2\pi}{\lambda}$ is the wave number of the medium.

Separation of Variables

- Solution to the homogeneous wave equation-in rectangular coordinates can be found by using the principle of separation of variables for partial differential equations.
- This principle says that in separable orthogonal coordinates, an elementary product solution may be constructed to this wave equation of the following form

$$E_r(x, y, z) = f_x(x) \times f_y(y) \times f_z(z)$$

- i.e. a solution which is expressed as the product of a function of x times a function of y times a function of z.

Separation of Variables 1

- Putting this elementary product solution into the wave equation - using the scalar Laplacian in rectangular coordinates

$$\nabla^2 E_r = \frac{\partial^2 E_r}{\partial x^2} + \frac{\partial^2 E_r}{\partial y^2} + \frac{\partial^2 E_r}{\partial z^2}$$

we obtained

$$f_x''(x) f_y(y) f_z(z) + f_x(x) f_y''(y) f_z(z) + f_x(x) f_y(y) f_z''(z) + k^2 f_x(x) f_y(y) f_z(z) = 0$$

Rearranging

$$\frac{f_x''(x)}{f_x(x)} + \frac{f_y''(y)}{f_y(y)} + \frac{f_z''(z)}{f_z(z)} + k^2 = 0$$

Separation of Variables 2

- Three ordinary differential equations for the f_x, f_y and f_z, along with one separable condition are

$$\frac{d^2}{dx^2} f_x(x) + k_x{}^2 f_x(x) = 0$$

$$\frac{d^2}{dy^2} f_y(y) + k_y{}^2 f_y(y) = 0$$

$$\frac{d^2}{dz^2} f_z(z) + k_z{}^2 f_z(z) = 0$$

where

$$k_x{}^2 + k_y{}^2 + k_z{}^2 = k^2$$

Separation of Variables 3

- Each of these differential equations has the same solution- a complex exponential- so that the elementary product solution for E_r is

$$E_r(x,y,z) = e^{-i(k_x x + k_y y + k_z z)} = e^{-i(k_x x + k_y y)} e^{-ik_z z}$$

$$= e^{-i(k_x x + k_y y)} e^{\pm iz\sqrt{k^2 - k_x{}^2 + k_y{}^2}}$$

- This represents a propagating or exponentially decaying uniform plane wave solution to the homogeneous wave equation.
- The - sign is used for a wave propagating or decaying in the +z direction and the + sign is used for a wave propagating or decaying in the -z direction.

The Superposition Integral

- A general solution to the homogeneous electromagnetic wave equation in rectangular coordinates is formed as a weighted superposition of all possible elementary plane wave solutions as

$$E_r(x,y) = \iint E_r(k_x,k_y) e^{i(k_x x + k_y y)} e^{\pm iz\sqrt{k^2 - k_x{}^2 + k_y{}^2}} dk_x dk_y$$

- This integral extend from minus infinity to infinity.
- This plane wave spectrum representation of the electro-magnetic field is the basic foundation of Fourier Optics.
- When z=0, the equation above simply becomes a Fourier transform (FT) relationship between the field and its plane wave content.

The Superposition Integral

- All spatial dependence of the individual plane wave components is described explicitly via the exponential functions.

- The coefficients of the exponentials are only functions of spatial wave-numbers kx and ky just as in ordinary Fourier analysis and Fourier transforms.

Free Space as a Low-pass Filter

- If

$$k_x^{\,2} + k_y^{\,2} \succ k_z^{\,2}$$

- Then the plane waves are evanescent (decaying)- so that any spatial frequency content in an object plane transparency which is finer than one wavelength will not be transferred over to the image plane- simply because the plane waves corresponding to that content cannot propagate.
- In connection with lithography of electronic components, this phenomenon is known as the diffraction limit and is the reason why light of progressively higher frequency (smaller wavelength) is required for etching progressively finer features in integrated circuits.

Fourier Transform Pairs

- The Analysis equation is

$$f(k_x, k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y) e^{-i(k_x x + k_y y)} dxdy$$

- The Synthesis equation is

$$f(x, y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_x, k_y) e^{i(k_x x + k_y y)} dk_x dk_y$$

- The normalizing factor of $\frac{1}{4\pi^2}$ is present whenever angular frequency (radians) is used, but not when ordinary frequency (cycles) is used.

Optical systems: General Overview

- An optical system is consists of an input plane and output plane.
- A set of components that transforms the image **f** formed at the input into a different image **g** formed at the output.
- The output image is related to the input image by convolving the input image with the optical impulse response ***h*** - known as the, *point-spread function* - for focused optical systems.
- The impulse response uniquely defines the input-output behavior of the optical system.
- The optic axis of the system is taken as the z-axis. As a result, the two images and the impulse response are all functions of the transverse coordinates - x and y.

Optical systems: General overview 1

- Optical systems typically fall into one of two different categories.
- The first is the ordinary focused optical imaging system- wherein the input plane is called the object plane and the output plane is called the image plane.
- The field in the image plane is desired to be a high-quality reproduction of the field in the object plane.
- The impulse response of the optical system is desired to approximate a 2D delta function- at the same location in the output plane corresponding to the location of the impulse in the input plane.

Optical systems: General overview 2

- The second type is the optical image processing system- in which a significant feature in the input plane field is to be located and isolated.

- The impulse response of the system is desired to be a close replica of that feature which is being searched for in the input plane field- so that a convolution of the feature against the input plane field will produce a bright spot.

Input Plane and Output Plane

- The input plane is defined as the locus of all points such that $z = 0$. The input image f is therefore

$$f(x, y) = U(x, y, z)\big|_{z=0}$$

- The output plane is defined as the locus of all points such that $z = d$. The output image g is therefore

$$g(x, y) = U(x, y, z)\big|_{z=d}$$

The convolution of input function against impulse response function

The convolution of input function with input response function is

$$g(x,y) = h(x,y) * f(x,y)$$

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-x^{'}, y-y^{'}) f(x^{'}, y^{'}) dx^{'} dy^{'}$$

- The integral below tacitly assumes that the impulse response is not a function of the position (x', y') of the impulse of light in the input plane.
- This property is known as *shift invariance.*

The convolution of input function against impulse response function

This equation assumes unit magnification. If magnification is present then above eqn. becomes

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_M(x - Mx', y - My') f(x', y') dx' dy'$$

- It translates the impulse response function from x' to x=Mx'. The relation between magnified and unmagnified response function is

$$h_M = h\left(x/M , y/M \right)$$

Derivation of the convolution equation

- The convolution representation of the system response requires representing the input signal as a weighted superposition over a train of impulse functions by using the *shifting property* of Dirac delta functions.

$$f(t) = \int_{-\infty}^{\infty} \delta(t - t^{'}) f(t^{'}) dt^{'}$$

- The system under consideration is *linear*- that is the output of the system due to two different inputs is the sum of the individual outputs of the system to the two inputs - when introduced individually.

Derivation of the convolution equation 1

- The output of the system is then simplified to a single delta function input- which would be the *impulse response* of the system h(t - t'). Thus, the output of the linear system to a general input function *f*(*t*) is

$$Output(t) = \int_{-\infty}^{\infty} h(t - t^{'}) f(t^{'}) dt^{'}$$

- The convolution equation is useful because it is often much easier to find the response of a system to a delta function input - and then perform the convolution above to find the response to an arbitrary input - than find the response to the arbitrary input directly.

System Transfer Function

- The Fourier transform of above equation becomes

$$Output(\omega) = H(\omega)F(\omega)$$

where

- $Output(\omega)$ is the spectrum of the output signal.
- $H(\omega)$ is the system transfer function.
- $F(\omega)$ is the spectrum of the input signal.

Thus, the input-plane plane wave spectrum is transformed into the output-plane plane wave spectrum through the multiplicative action of the system transfer function.

Applications of Fourier optics

- Fourier optics is used in the field of optical information processing.
- The Fourier transform properties of a lens provide numerous applications in optical signal processing such as spatial filtering, optical correlation and computer generated holograms.
- Fourier optical theory is used in interferometry, optical tweezers, atom traps, and quantum computing.
- Concepts of Fourier optics are used to reconstruct the phase of light intensity in the spatial frequency plane

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THANK YOU