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Critical Scaling at the Anderson Localization Transition in the Strong Multifractality Regime

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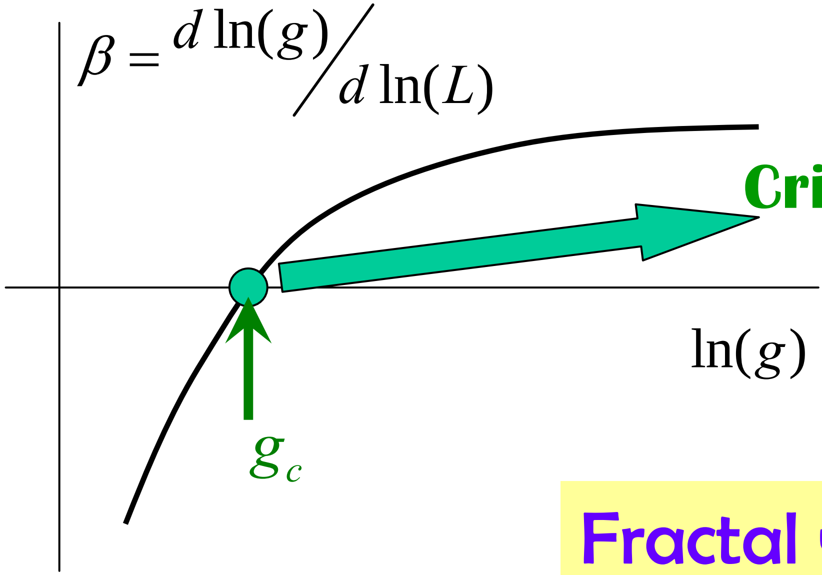
Outline of the talk

1. Introduction:
   - Disordered systems with fractal wavefunctions
   - Spatial correlations of fractal wavefunctions and the dynamical scaling hypothesis
   - How dynamical scaling shows up in the return probability
2. Strong multifractality regime:
   - Model (the Critical RMT) and method (the Virial Expansion)
3. Scaling exponents:
   - Leading logarithmic terms
   - When dynamical scaling exists?
4. Conclusions

Localization transition in disordered systems

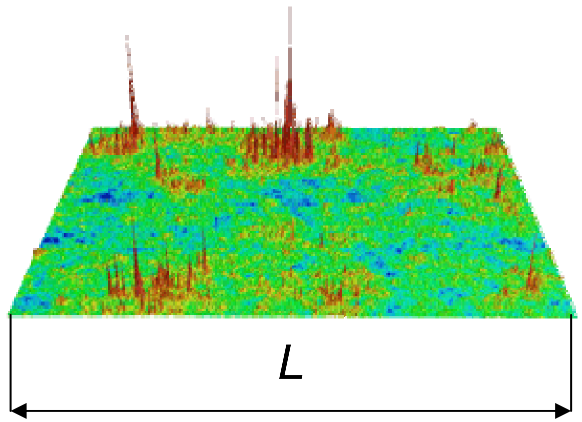


Anderson, 1958
the Gang of four, 1979

Criticality at the localization transition

Fractal wave-functions

(Wegner, 1980)



Inverse Participation Ratio

$$\mathcal{P}_q = \nu^{-1}\left\langle \sum_{n,\vec{r}} \left|\psi_n(\vec{r})\right|^{2q} \delta\left(E-\varepsilon_n\right) \right\rangle$$

$$\lim_{L\to\infty}\left(\mathcal{P}_q\right) \propto 1\big/L^{(q-1)d_q}$$

fractal dimension: $0 < d_q < d$ *Wavefunction occupies a fraction of space*

Correlations of the fractal wavefunctions

Two point correlation function:

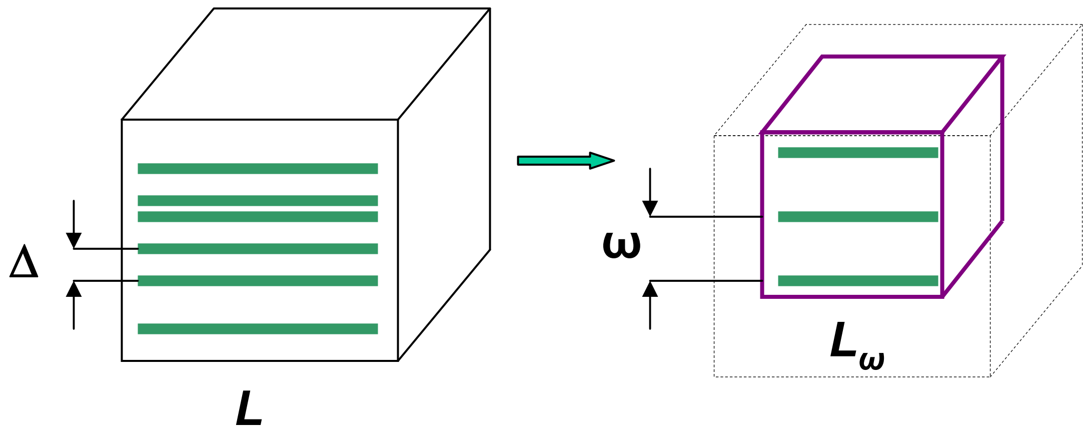
$$C_2(\omega, \mathbf{R}) \equiv \nu^{-1} \langle \sum_{\mathbf{p}} \sum_{m,n} \delta(\omega/2 - \xi_n)\delta(\omega/2 + \xi_m) |\psi_{\xi_n}(\mathbf{p})|^2 |\psi_{\xi_m}(\mathbf{p}+\mathbf{R})|^2 \rangle$$

- For a disordered system at the critical point (fractal wavefunctions)

$$\boxed{C_2(\omega = 0, \mathbf{R}) \propto (L/|\mathbf{R}|)^{d-d_2}, \ \ |\mathbf{R}| \leq L}$$

(Wegner, 1985)

- If $\omega > \Delta$ then $L_\omega = L(\Delta/\omega)^{1/d}$ must play a role of L:



Correlations of the fractal wavefunctions

Two point correlation function:

$$C_2(\omega,\mathbf{R}) \equiv \nu^{-1}\langle \sum_{\mathbf{p}} \sum_{m,n} \delta(\omega/2-\xi_n)\delta(\omega/2+\xi_m)|\psi_{\xi_n}(\mathbf{p})|^2|\psi_{\xi_m}(\mathbf{p}+\mathbf{R})|^2 \rangle$$

For a disordered system at the critical point (fractal wavefunctions)

$$C_2(\omega=0,\mathbf{R}) \propto (L/|\mathbf{R}|)^{d-d_2}\,,\ \ |\mathbf{R}| \leq L$$ (Wegner, 1985)

Dynamical scaling hypothesis: $L^{d-d_2} \longrightarrow (L_\omega)^{d-d_2}$

$$\boxed{\Rightarrow C_2(\omega>\Delta,\mathbf{R}) \propto (L_\omega/|\mathbf{R}|)^{d-d_2}\,,\ l \leq |\mathbf{R}| \leq L_\omega < L}$$

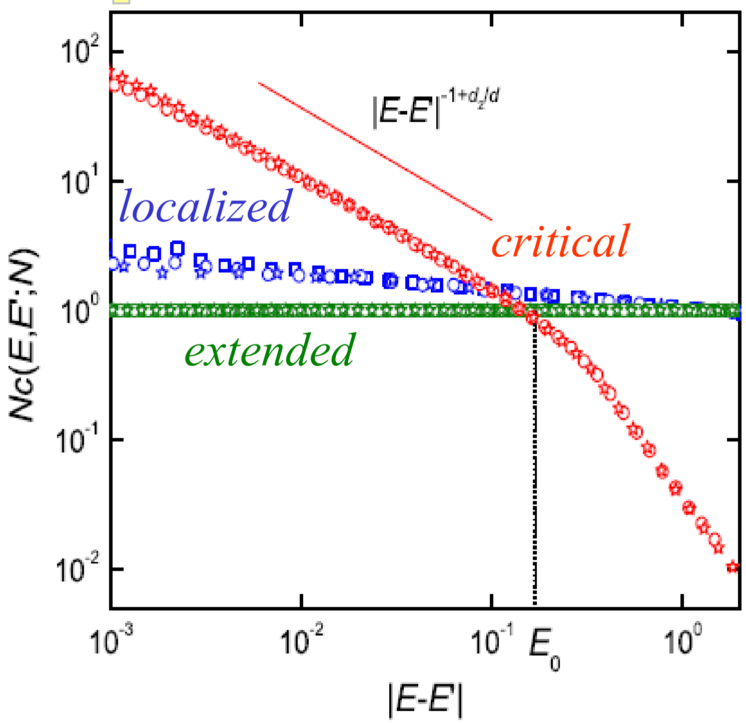
(Chalker, Daniel, 1988; Chalker, 1990)

d –space dimension, Δ - mean level spacing, l – mean free path, <…> - disorder averaging

Fractal enhancement of correlations

Dynamical scaling: $$C_2(\Delta < \omega < E_0, |\mathbf{R}| \ll l) \propto \left(\frac{E_0}{\omega}\right)^{1-d_2/d}$$

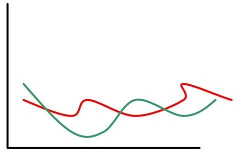
$\boxed{E_0/\omega > 1,\ \ 1 - d_2/d > 0}$ **- Enhancement of correlations**



(Cuevas , Kravtsov, 2007)

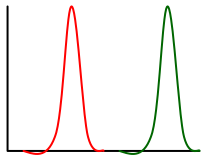
(The Anderson model: tight binding Hamiltonian)

Extended WF:



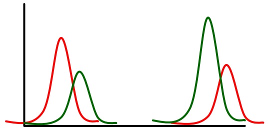
small amplitude
substantial overlap in space

Localized WF:



high amplitude
small overlap in space

Fractal WF:



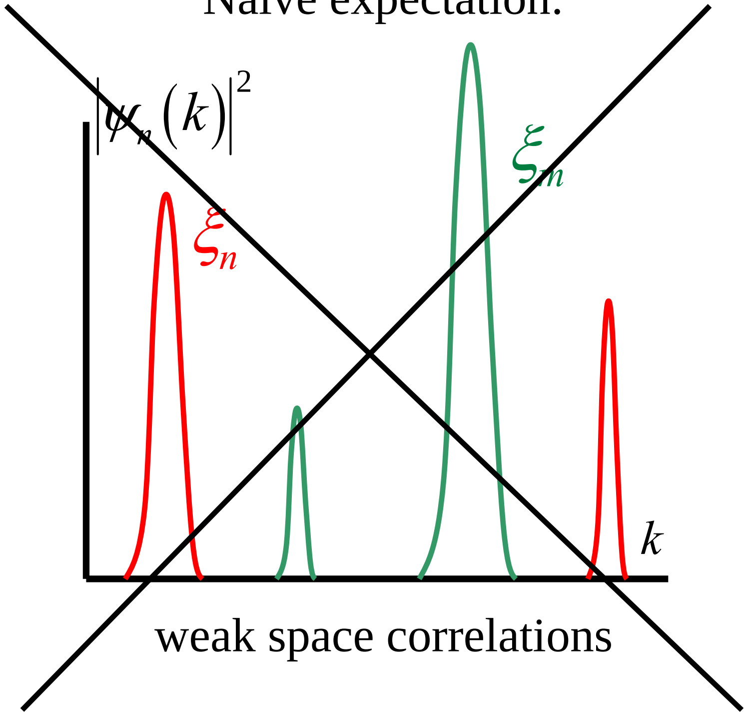
relatively high amplitude and

the fractal wavefunctions strongly overlap in space

Strong fractality regime: do WFs really overlap in space?

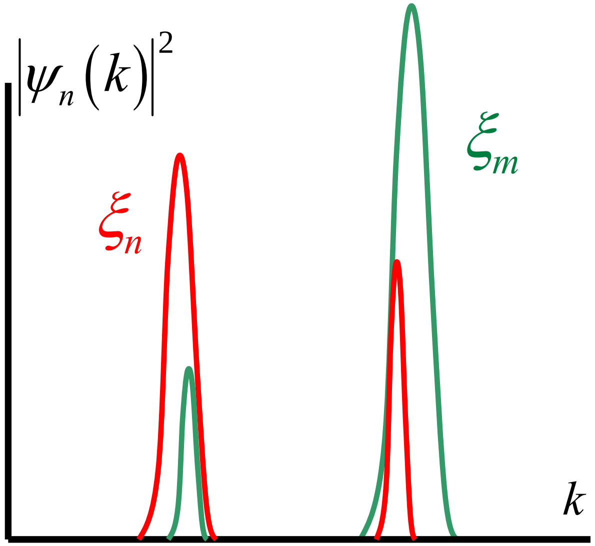
$0 < d_2 \ll d$ ***- sparse fractals,*** $\Delta \ll |\xi_m - \xi_n| \ll E_0$

Naïve expectation:



weak space correlations

A consequence of the dynamical scaling:



strong space correlations

So far, no analytical check of the dynamical scaling; just a numerical evidence

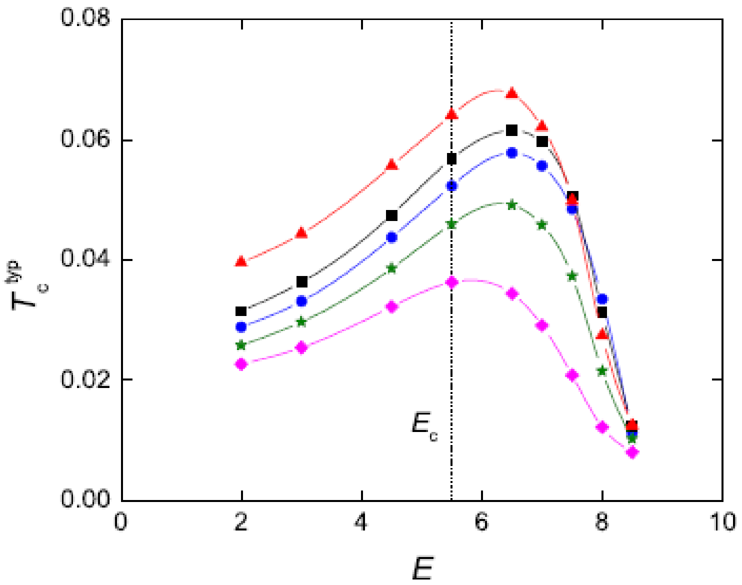
IQH WF: Chalker, Daniel (1988),
Huckestein, Schweitzer (1994), Prack, Janssen, Freche (1996)
Anderson transition in 3d: Brandes, Huckestein, Schweitzer (1996)
WF of critical RMTs: Cuevas, Kravtsov (2007)

Why dynamical scaling is important

$C_2(\omega, \mathbf{R})$ governs the interaction matrix elements $\Longrightarrow$

Critical correlations are important for many body systems with interactions:

"Multifractal superconductivity" (Feigel'man, Kravtsov, et.al., 2007-2010)



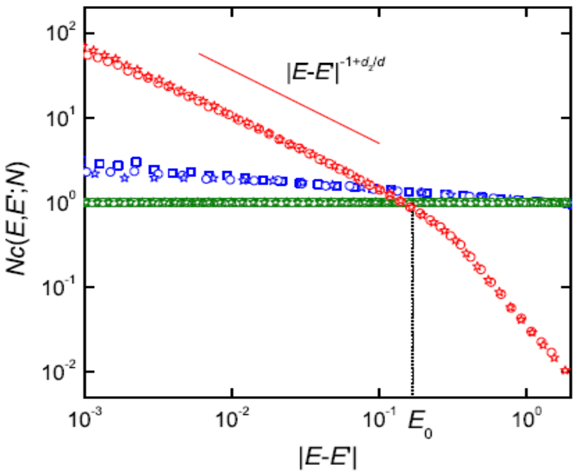
T_c – critical temperature

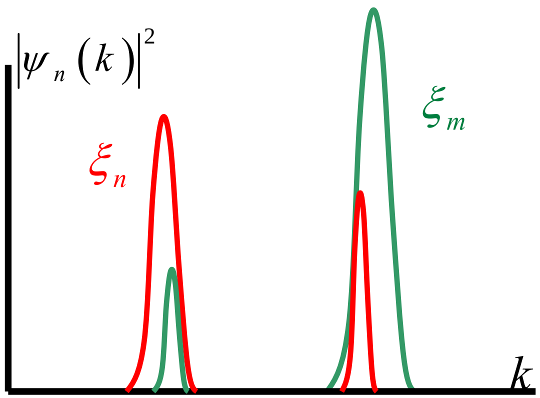
E_c – mobility edge

E – Fermi-level position

"Multifractal Stoner instability" (Kiselev, OY, Kravtsov, in progress)

Statement of problem





We address two questions:

- **Does the dynamical scaling hypothesis hold true in the strong fractality regime?**
- **Which conditions are necessary for the existence of dynamical scaling?**

Coinciding space point: scaling in energy-domain

$\mathbf{R} = 0,$ energy representation

$$C_2(\omega,0) \equiv \nu^{-1} \langle \sum_{\mathbf{p}} \sum_{m,n} \delta\left(\frac{\omega}{2} - \xi_n\right) \delta\left(\frac{\omega}{2} + \xi_m\right) |\psi_{\xi_n}(\mathbf{p})|^2 |\psi_{\xi_m}(\mathbf{p})|^2 \rangle = C_{2,\mathrm{diag}} + C_{2,\mathrm{off-diag}}$$

Diagonal part

$$C_{2,\mathrm{diag}} = \delta(\omega) \underbrace{\left\langle \nu^{-1} \sum_{\mathbf{p}} \sum_{m} \delta(\xi_m) |\psi_{\xi_m}(\mathbf{p})|^4 \right\rangle}_{\mathrm{IPR}, \mathcal{P}_2} \propto \delta(\omega) V^{-d_2/d}, \qquad V = L^d$$

(space scaling)

Off-diagonal part

$$C_{2,\mathrm{off-diag}}|_{\Delta \ll \omega \ll E_0} \propto \omega^{-\mu}$$

(dynamical scaling)

$$\boxed{1 - \mu = d_2/d}$$

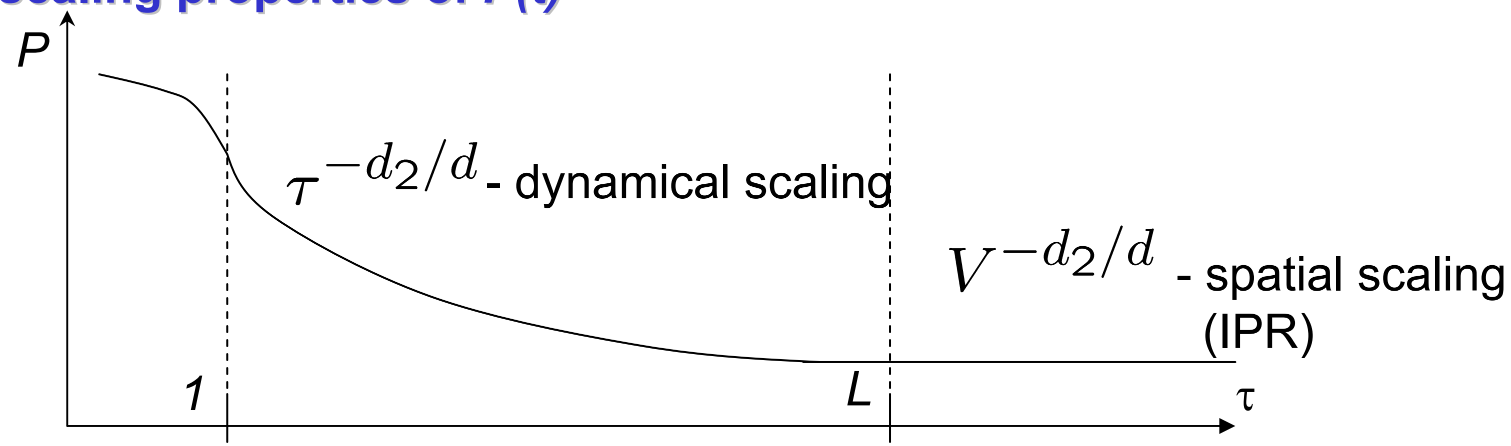
Coinciding space point: scaling in time-domain

$\mathbf{R} = 0,$ time representation

Fourier transform of $C_2(\omega,0)$: $$P(t) = \int C_2(\omega,0)e^{-\imath\omega t}\frac{d\omega}{2\pi}$$

– averaged return probability for a wave packet

Expected scaling properties of *P(t)*



(τ - scaled time)

P(t) **is more convenient for the further analysis**

Scaling of the return probability

$\Upsilon = \min(\tau, L)$ - IR cutoff of the theory

Expected behaviour of *P(t) in the long time limit*

$$P(t) = A(\Upsilon)\left(\frac{B(\Upsilon)}{\Upsilon}\right)^{\kappa(\Upsilon)}$$

Universal exponent

$$\log(P) = \log(A) + \kappa\Big(\log(B) - \log(\Upsilon)\Big)$$

can be non-universal → let's eliminate them

Scaling of the return probability

$\Upsilon = \min(\tau, L)$ - IR cutoff of the theory

<u>Expected</u> behaviour of *P(t) in the long time limit*

$$P(t) = A(\Upsilon)\left(\frac{B(\Upsilon)}{\Upsilon}\right)^{\kappa(\Upsilon)}$$

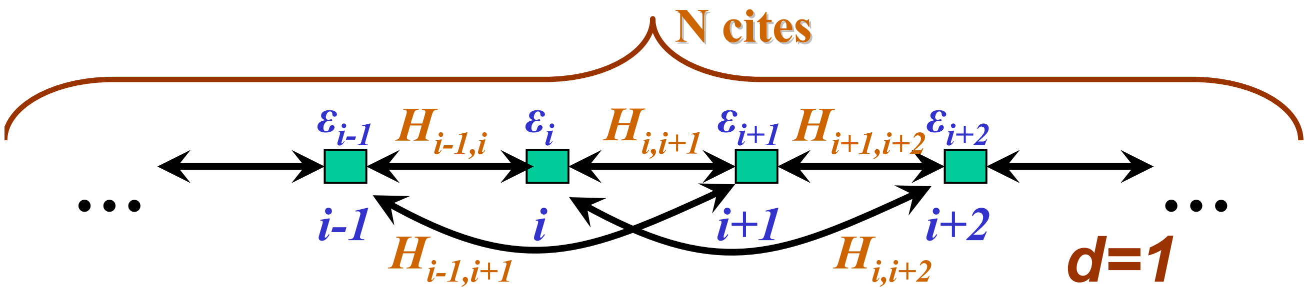
$$\partial_{\log(\Upsilon)} \log(P) = \cancel{\log(A)} + \kappa \cancel{\log(B)} - \partial_{\log(\Upsilon)}[\kappa \log(\Upsilon)]$$

IF the dynamical scaling hypothesis holds true then

1) $\partial_{\log(\Upsilon)} \log(P) = -\kappa(\Upsilon)$ **- Equation for κ**

2) $\kappa|_{V \gg \tau} = \kappa|_{\tau \gg V} = d_2/d$ **- Universality of κ**

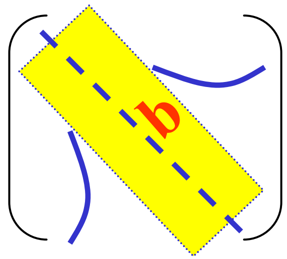
Model: MF RMT (Power-Law-Banded Random Matrices)



1d chain with random long-range hopping

$\hat{H}$ - *NxN* Hermitian BRM

b is the bandwidth



$$\langle |H_{i,j}|^2 \rangle \sim \begin{cases} 1, & |i-j| < b \\ \left(\frac{1}{i-j} \right)^{2\alpha}, & |i-j| > b \end{cases}$$

$\alpha=1$: RMT with multifractal eignestates at any band-width

(Mirlin, Fyodorov et. al., 1996, Mirlin, Evers, 2000)

$2\pi b \gg 1$

$$d_2 \approx 1 - \frac{\text{const}}{2\pi b}$$

$1-d_2 \ll 1$ – regime of **weak multifractality**

$b<1$

$$d_2 \approx \text{const}\; b$$

$d_2 \ll 1$ – regime of **strong multifractality**

Universality of critical correlations: MF RMT vs. the Anderson model

MF (critical) RMT, bandwidth ***b*** ⟷ **Anderson model at criticality (MF eigenstates)**, dimension ***d***

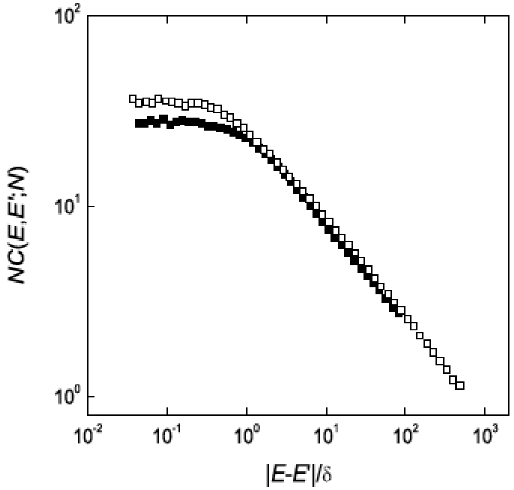


FIG. 5: Two-eigenfunction correlation function for the 3D Anderson model (orthogonal symmetry class) with a triangular distribution of random on-site energies (solid symbols) and the critical PLBRM Eq.(3) with $\beta = 1$ and $b = 0.42$ (open symbols). The energy difference $\omega = |E - E'|$ is measured in units of mean level spacing. The insert shows the mean density of states; the mobility edge corresponds to $\varepsilon = \pm 3.5$. The

"□" – MF PLBRMT, β=1, *b* = 0.42

"■" – 3d Anderson model from orthogonal class with MF eigenstates (at the mobility edge, E=3.5)

$$C_2(\Delta \ll \omega, 0) \propto 1/\omega^{\mu}\,,$$

(Cuevas, Kravtsov, 2007)

Variance of matrix elements for almost diagonal MF RMT

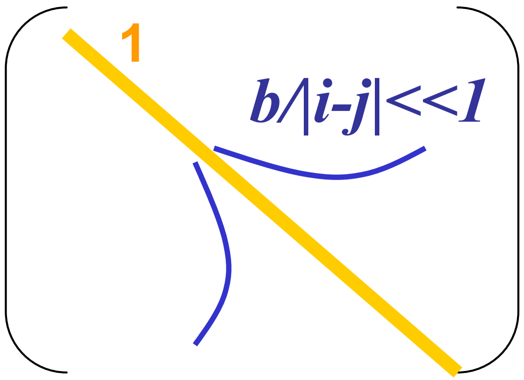
Almost diagonal MF RMT from the GUE symmetry class

$$\langle \varepsilon_i^2 \rangle = \frac{1}{2},$$

$$\langle |H_{i \neq j}|^2 \rangle = \frac{1}{2} \frac{1}{1 + |i-j|^{2(1-\epsilon)}/b^2} \simeq \frac{b^2/2}{|i-j|^{2(1-\epsilon)}}$$

$b \ll 1$ - small band width → **almost diagonal MF RMT**

$\epsilon \longrightarrow +0$ - UV regularizing parameter

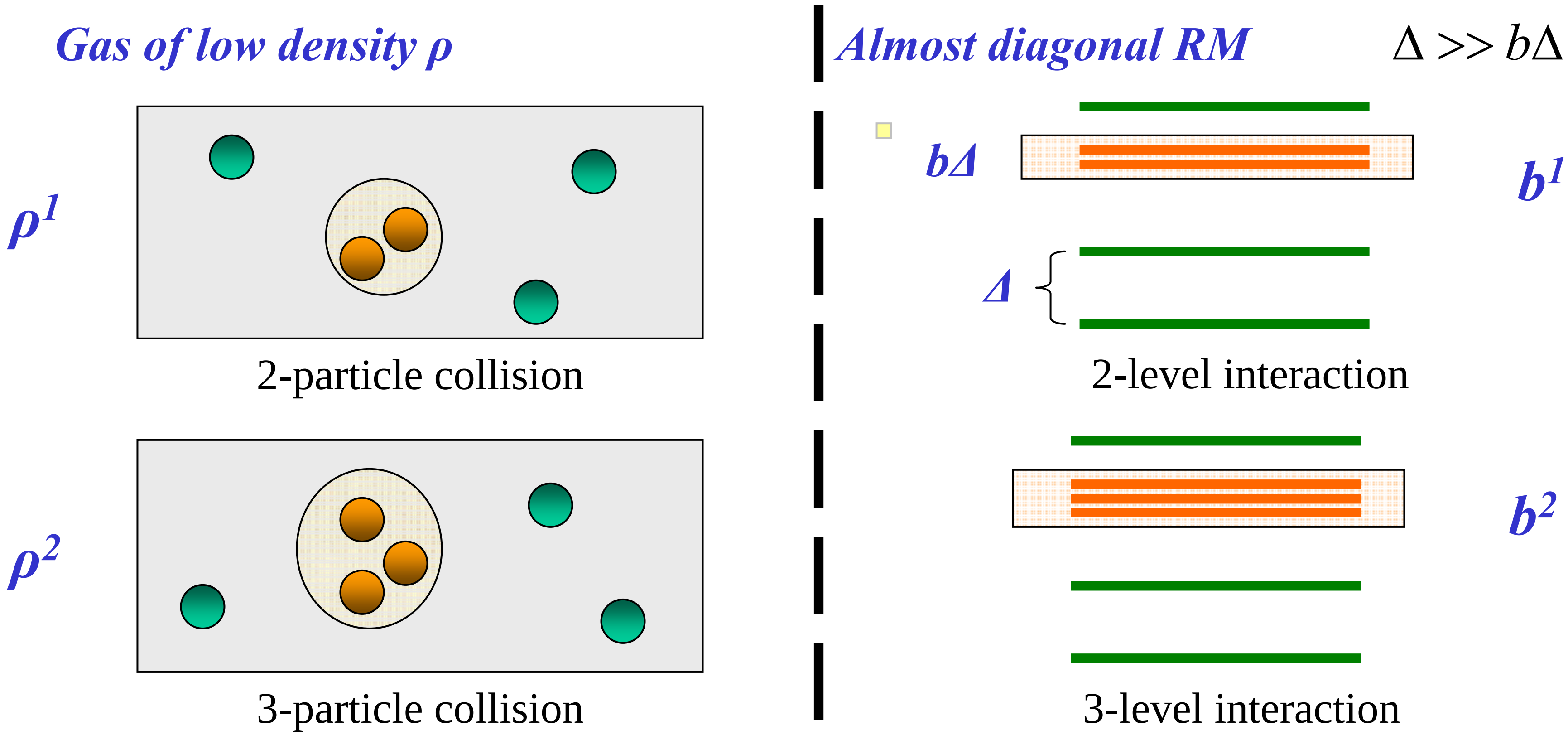


Method: The virial expansion

As an alternative to the *σ-model*, we use

Note: **a field theoretical machinery of the σ –model cannot be used in the case of the almost diagonal RMT**

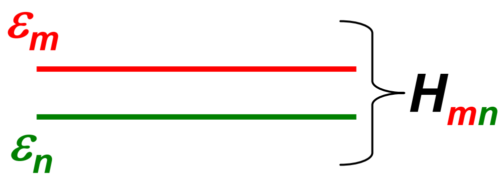
the virial expansion in the number of interacting energy levels.



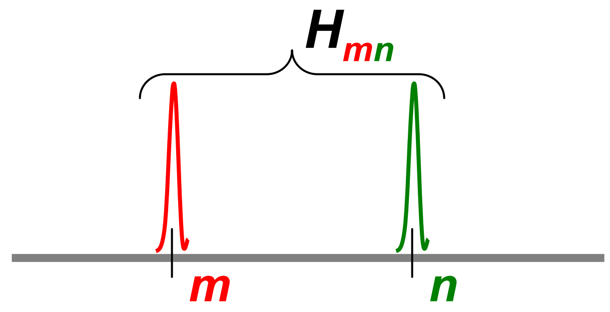
VE allows one to expand correlations functions in powers of $b<<1$

SuSy virial expansion

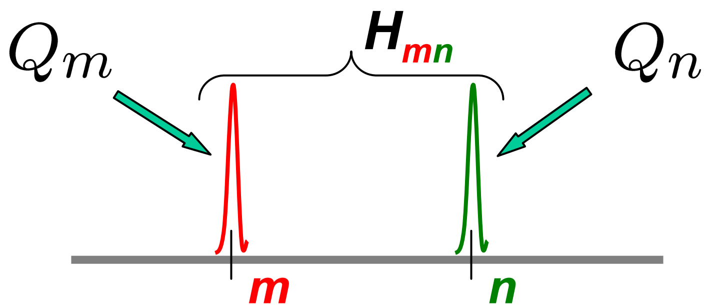
Interaction of energy levels ⇔ **Hybridization of localized stated**



⇔



The SuSy trick is used to average over disorder (OY, Ossipov, Kronmüller, 2007-2009)



$\longrightarrow$

Coupling of supermatrices

$$\exp\left(-\langle |H_{mn}|^2\rangle \mathrm{Str}(Q_m Q_n)\right)$$

Summation over all possible configurations

$$\sim \int \mathrm{d}x\, \overline{\exp\left(-\langle |H_{mn}|^2\rangle \mathrm{Str}(Q_m Q_n)\right) \underbrace{\mathcal{F}[Q]}_{\text{SuSy breaking factor}}}\,, \qquad x \equiv m - n$$

Application of the virial expansion

Expected behavior: $P \propto \Upsilon^{-\kappa}\,; \qquad \Upsilon = \min(\tau, N)\,, \;\; \kappa \sim b \ll 1$

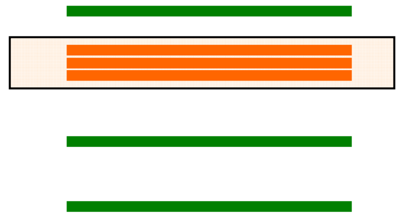
VE for the return probability: $P = 1 + \sum_{j=2} P_j\,, \qquad P_j \sim O\Big((b \log(\Upsilon))^{j-1} \Big)$

Pertubation theory for the scaling exponent

$$\kappa = -\partial_{\log(\Upsilon)} \log(1+P_2+P_3+\ldots) = \kappa_2+\kappa_3+\ldots$$

$$\underbrace{\kappa_2 \;\equiv\; -\partial_{\log(\Upsilon)} P_2}_{\text{2 level contribution } \sim O(b^1)}$$

$$\underbrace{\kappa_3 \;\equiv\; -\partial_{\log(\Upsilon)} \left(P_3 - \frac{1}{2} P_2^2 \right)}_{\text{3 level contribution } \sim O(b^2)}$$



What shall we calculate and check?

1) *Dynamical scaling:*

$$P \propto \Upsilon^{\kappa}\,; \qquad \kappa_2(\Upsilon) \equiv -\partial_{\log(\Upsilon)} P_2\,,\; \kappa_3(\Upsilon) \equiv -\partial_{\log(\Upsilon)} \left(P_3 - \frac{1}{2} P_2^2\right)$$

- **a) *Log-behavior of P_j:***

$$P_2 \sim b \log(\Upsilon)\,,\;\; P_3 \sim (b \log(\Upsilon))^2$$

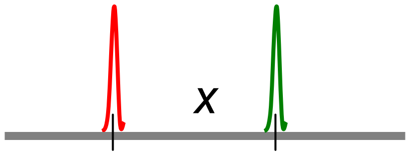
- **b) *Smallness of corrections*** $\kappa_{2,3}(\Upsilon)$

$\Longrightarrow$ ***log²(Υ)*** **must cancel out in** ***P_3- $(P_2)^2/2$***

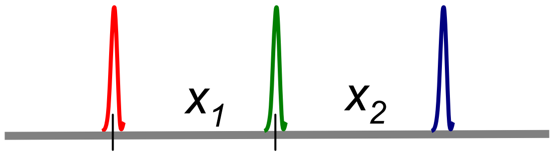
- **2) Universality: *scaling* exponent is cut-off independent**

$$\kappa_j|_{\Upsilon=N} = \kappa_j|_{\Upsilon=\tau}\,, \qquad j = 2, 3$$

Regular contributions to the scaling exponents



$$P_2|_{\tau,N\gg 1} = \int_0^\tau \frac{\mathrm{d}\beta}{\beta} \int_0^N \frac{\mathrm{d}x}{\beta}\ \mathcal{F}_2\left(\frac{\beta}{x^{1-\epsilon}}\right)\ , \qquad \tau \equiv bt$$



$$|x_1| \sim |x_2| \sim |x_1 - x_2|$$

$$P_3^{(reg)}|_{\tau,N\gg 1} \simeq \int_0^\tau \frac{\mathrm{d}\beta}{\beta} \iint_0^N \frac{\mathrm{d}x_1}{\beta}\frac{\mathrm{d}x_2}{\beta}\ \mathcal{F}_3^{(reg)}\left(\frac{\beta}{x_1^{1-\epsilon}}, \frac{\beta}{x_2^{1-\epsilon}}, \frac{\beta}{|x_1 - x_2|^{1-\epsilon}}\right)$$

Homogeneity at $\varepsilon \to 0$

Measure $\quad \mathrm{d}\beta/\beta,\ \ \mathrm{d}x/\beta$ - general property

Argument $\quad \beta/x^{1-\epsilon}|_{\epsilon\to+0} = \beta/x$ - property of the **critical** RMT

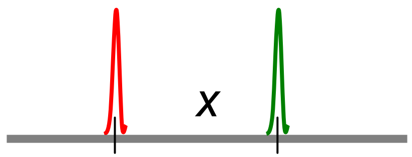
Role of criticality

Homogeneity results in equivalence of time/space coordinates at criticality

- $$\Rightarrow \partial_{\log(\tau)} P_j^{(reg)}\Big|_{\substack{N\to\infty \\ \epsilon\to 0}} = \partial_{\log(N)} P_j^{(reg)}\Big|_{\substack{\tau\to\infty \\ \epsilon\to 0}} = \int_0^\infty \mathrm{d}\{x_m\}\mathcal{F}_j^{(reg)}\left(\left\{\frac{1}{x_m}\right\}\right)$$

  $$j = 2,3\,; \quad m = 1\ldots j-1$$

- $j = 2$



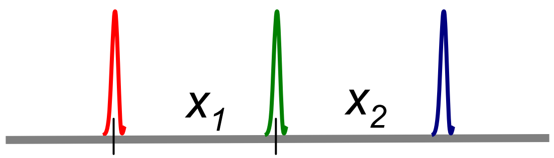
$$\boxed{\kappa_2(\Upsilon) = \frac{b\pi}{\sqrt{2}}}$$ **- universal**

Assumptions about

1) dynamical scaling and 2) universality of κ

hold true up to O(b)

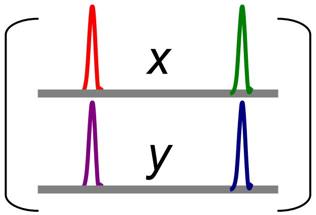
- $j = 3$



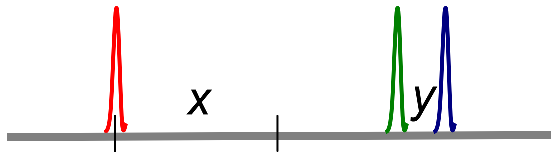
$$\boxed{\kappa_3^{(1)}(\Upsilon) = 0.141 \times \frac{(b\pi)^2}{2}}$$ **- universal**

Singular contributions

$$(P_2)^2$$



$$P_3^{(sing)}$$



$$|x| \gg |y|$$

Contribution to κ: $\kappa_3^{(2)}(\Upsilon) \equiv -\lim_{\epsilon\to 0}\left(\partial_{\log(\Upsilon)}\left(P_3^{(sing)} - \frac{1}{2}P_2^2\right)\right)$

$$\begin{cases} \frac{1}{2}\partial_{\log_\Upsilon} P_2^2 & = & \frac{\Upsilon^{2\epsilon}}{\epsilon}(A_2 + B_2\epsilon) \\ \partial_{\log_\Upsilon} P_3^{(sing)} & = & \frac{\Upsilon^{2\epsilon}}{\epsilon}(A_3 + B_3\epsilon) \end{cases}$$

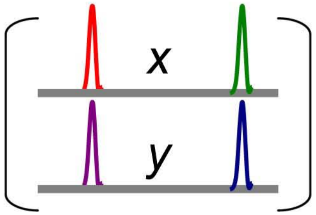
- 1/ε singularities

We cannot put ε → 0 and use homogeneity property before solving uncertainties

$$\kappa_3^{(2)}(\Upsilon) \equiv \lim_{\epsilon\to 0}\left(\frac{\Upsilon^{2\epsilon}}{\epsilon}([A_2 - A_3] + \epsilon[B_2 - B_3])\right)$$

Singular contributions

$(P_2)^2$



$P_3^{(sing)}$

$$|x| \gg |y|$$

$$\kappa_3^{(2)} = \frac{(\pi b)^2}{2}[\log(2) - 1]$$

Assumptions about

1) dynamical scaling and 2) universality of κ

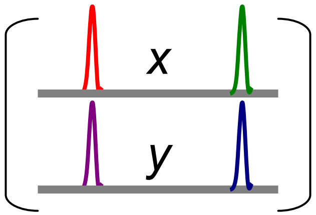
hold true up to O(b²) IF

1) $1/\varepsilon$ terms cancel out: $A_2 = A_3$;
2) subleading terms are universal $B_2 - B_3 =$ const.

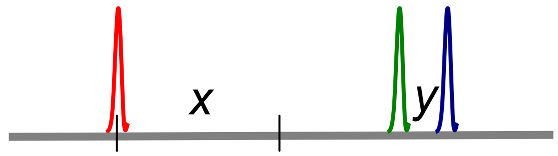
- **Results of $1/\varepsilon$-expansion:** $A_2 = A_3\,; \quad B_3 - B_2 = \frac{(\pi b)^2}{2}[\log(2) - 1]$
- $$\kappa_3^{(2)}(\Upsilon) \equiv \lim_{\epsilon \to 0} \left(\frac{\quad}{\epsilon}([A_2 - A_3] + \epsilon[B_2 - B_3] \quad \right)$$

Singular contributions

$$(P_2)^2$$



$$P_3^{(sing)}$$



$$|x| \gg |y|$$

Assumptions about

1) dynamical scaling and 2) universality of κ

hold true up to O(b^2) IF

1) $1/\varepsilon$ terms cancel out: $A_2 = A_3$;
2) subleading terms are universal $B_2 - B_3 =$ const.

$$\boxed{\kappa_3^{(2)} = \frac{(\pi b)^2}{2}[\log(2) - 1]}$$ **- small and universal**

Conclusions

$$\boxed{1-\mu = d_2 \simeq \kappa_2 + \kappa_3^{(1)} + \kappa_3^{(2)} = \frac{\pi b}{\sqrt{2}} + 0.818 b^2}$$

- Using the model of the **of the almost diagonal RMT with multifractal eigenstates in the strong fractality regime** we have shown that:

  - assumptions about the dynamical scaling and the relation $\mu = 1 - d_2$ hold true up to the leading and the subleading terms of the VE IF:

    - the system is critical
    - anomalous part of the scaling exponent is regular and universal

= =

Open question

- are anomalous contributions always regular and universal at criticality?