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Workshop on Localization Phenomena in Novel Phases of Condensed Matter

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Topological Insulators: Disorder, Interaction and Quantum Criticality of Dirac Fermions

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Topological insulators:

Disorder, interaction, and quantum criticality of Dirac fermions

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PRB 74, 235443 (2006); PRL 98, 256801 (2007);

EPJ Special Topics 148, 63 (2007);

arXiv:0910.1338

Outline

- **Anderson localization theory: Symmetries and topologies**
- **Graphene and 2D Dirac fermions**
- **Conductivity at Dirac point:**
  Absence of localization for chiral disorder *and*
  topological delocalization for long-range disorder
- **Topological insulators (TIs): General classification**
- **2D and 3D $\mathbb{Z}_2$ TIs in time-reversal-invariant systems with spin-orbit interaction**
- **Coulomb interaction in TIs:**
  quantum criticality at the surface of 3D TI *and*
  quantum spin Hall transition 2D TI to normal insulator

50 years of Anderson localization

Philip W. Anderson

1958 "Absence of diffusion in certain random lattices"

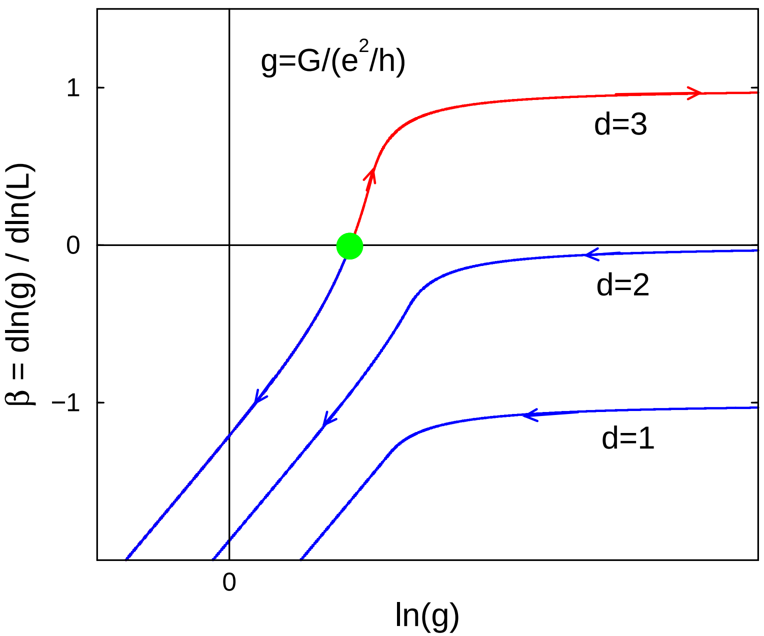
sufficiently strong disorder $\longrightarrow$ quantum localization

$\longrightarrow$ eigenstates exponentially localized, no diffusion

$\longrightarrow$ Anderson insulator

The Nobel Prize in Physics 1977

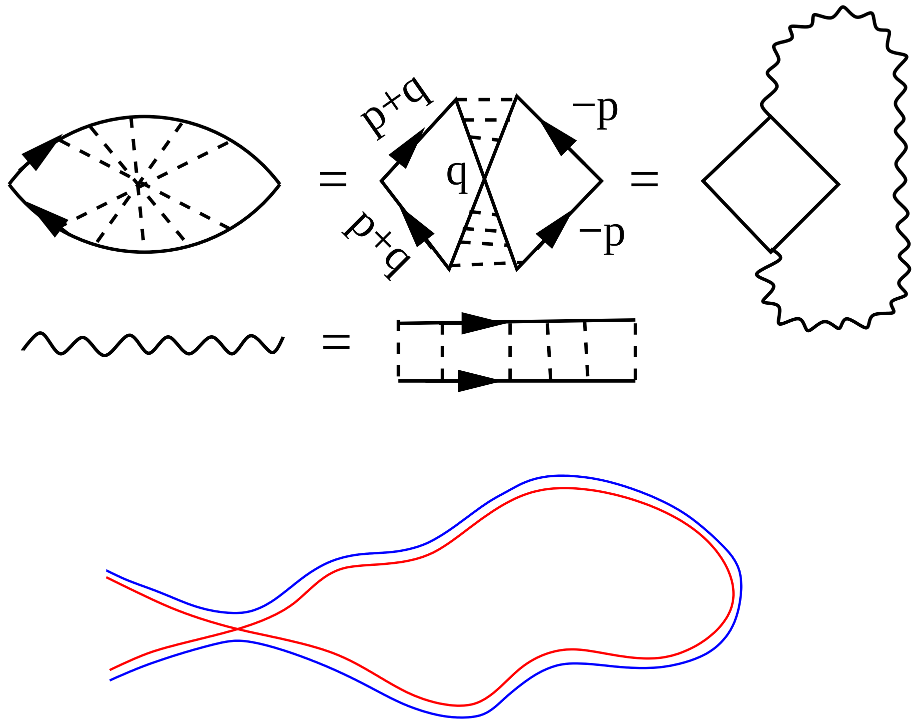
Anderson Insulators & Metals



Scaling theory of localization:
Abrahams, Anderson, Licciardello, Ramakrishnan '79

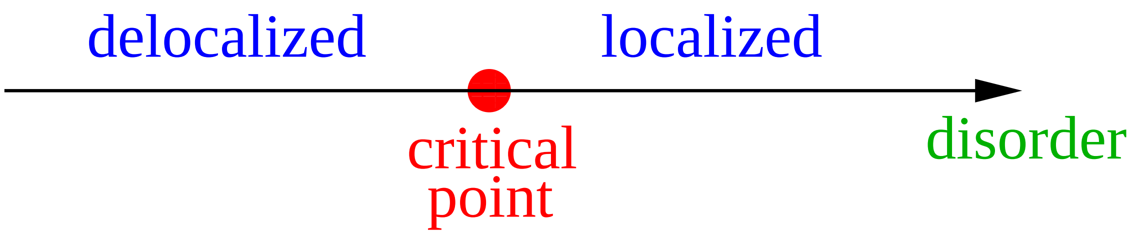
Modern approach:
RG for field theory (σ-model)

large g: weak localization



quasi-1D, 2D : all states are localized

$d > 2$: Anderson metal-insulator transition



review: Evers, ADM, Rev. Mod. Phys. 80, 1355 (2008)

Field theory: non-linear σ-model

$$S[Q] = \frac{\pi\nu}{4} \int d^d\mathrm{r}\, \mathrm{Str}\, [-D(\nabla Q)^2 - 2i\omega\Lambda Q], \qquad Q^2(\mathrm{r}) = 1$$

Wegner'79 (replicas); Efetov'83 (supersymmetry)

σ-model manifold:

- unitary class:
  - fermionic replicas: $\mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n)$, $\quad n \to 0$
  - bosonic replicas: $\mathrm{U}(n,n)/\mathrm{U}(n) \times \mathrm{U}(n)$, $\quad n \to 0$
  - supersymmetry: $\mathrm{U}(1,1|2)/\mathrm{U}(1|1) \times \mathrm{U}(1|1)$
- orthogonal class:
  - fermionic replicas: $\mathrm{Sp}(4n)/\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)$, $\quad n \to 0$
  - bosonic replicas: $\mathrm{O}(2n,2n)/\mathrm{O}(2n) \times \mathrm{O}(2n)$, $\quad n \to 0$
  - supersymmetry: $\mathrm{OSp}(2,2|4)/\mathrm{OSp}(2|2) \times \mathrm{OSp}(2|2)$

in general, in supersymmetry:

$Q \in \{$"sphere" $\times$ "hyperboloid"$\}$ "dressed" by anticommuting variables

Disordered electronic systems: Symmetry classification

Altland, Zirnbauer '97

Conventional (Wigner-Dyson) classes

	T	spin rot.	chiral	p-h	symbol
GOE	+	+	−	−	AI
GUE	−	+/−	−	−	A
GSE	+	−	−	−	AII

Chiral classes

	T	spin rot.	chiral	p-h	symbol
ChOE	+	+	+	−	BDI
ChUE	−	+/−	+	−	AIII
ChSE	+	−	+	−	CII

$$\boldsymbol{H} = \begin{pmatrix} \mathbf{0} & \mathrm{t} \\ \mathrm{t}^{\dagger} & \mathbf{0} \end{pmatrix}$$

Bogoliubov-de Gennes classes

	T	spin rot.	chiral	p-h	symbol
	+	+	−	+	CI
	−	+	−	+	C
	+	−	−	+	DIII
	−	−	−	+	D

$$\boldsymbol{H} = \begin{pmatrix} \mathrm{h} & \boldsymbol{\Delta} \\ -\boldsymbol{\Delta}^{*} & -\mathrm{h}^{T} \end{pmatrix}$$

Disordered electronic systems: Symmetry classification

Ham. class	RMT	T	S	compact symmetric space	non-compact symmetric space	σ-model B|F	σ-model compact sector $\mathcal{M}_F$
Wigner-Dyson classes							
A	GUE	$-$	$\pm$	$U(N)$	$GL(N,\mathbb{C})/U(N)$	AIII|AIII	$U(2n)/U(n)\times U(n)$
AI	GOE	$+$	$+$	$U(N)/O(N)$	$GL(N,\mathbb{R})/O(N)$	BDI|CII	$Sp(4n)/Sp(2n)\times Sp(2n)$
AII	GSE	$+$	$-$	$U(2N)/Sp(2N)$	$U^*(2N)/Sp(2N)$	CII|BDI	$O(2n)/O(n)\times O(n)$
chiral classes							
AIII	chGUE	$-$	$\pm$	$U(p+q)/U(p)\times U(q)$	$U(p,q)/U(p)\times U(q)$	A|A	$U(n)$
BDI	chGOE	$+$	$+$	$SO(p+q)/SO(p)\times SO(q)$	$SO(p,q)/SO(p)\times SO(q)$	AI|AII	$U(2n)/Sp(2n)$
CII	chGSE	$+$	$-$	$Sp(2p+2q)/Sp(2p)\times Sp(2q)$	$Sp(2p,2q)/Sp(2p)\times Sp(2q)$	AII|AI	$U(n)/O(n)$
Bogoliubov - de Gennes classes							
C		$-$	$+$	$Sp(2N)$	$Sp(2N,\mathbb{C})/Sp(2N)$	DIII|CI	$Sp(2n)/U(n)$
CI		$+$	$+$	$Sp(2N)/U(N)$	$Sp(2N,\mathbb{R})/U(N)$	D|C	$Sp(2n)$
BD		$-$	$-$	$SO(N)$	$SO(N,\mathbb{C})/SO(N)$	CI|DIII	$O(2n)/U(n)$
DIII		$+$	$-$	$SO(2N)/U(N)$	$SO^*(2N)/U(N)$	C|D	$O(n)$

Symmetry alone is not always sufficient to characterize the system.

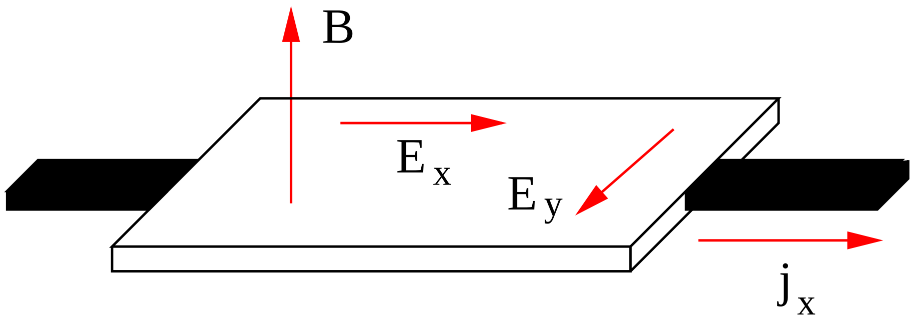
There may be also a non-trivial topology !

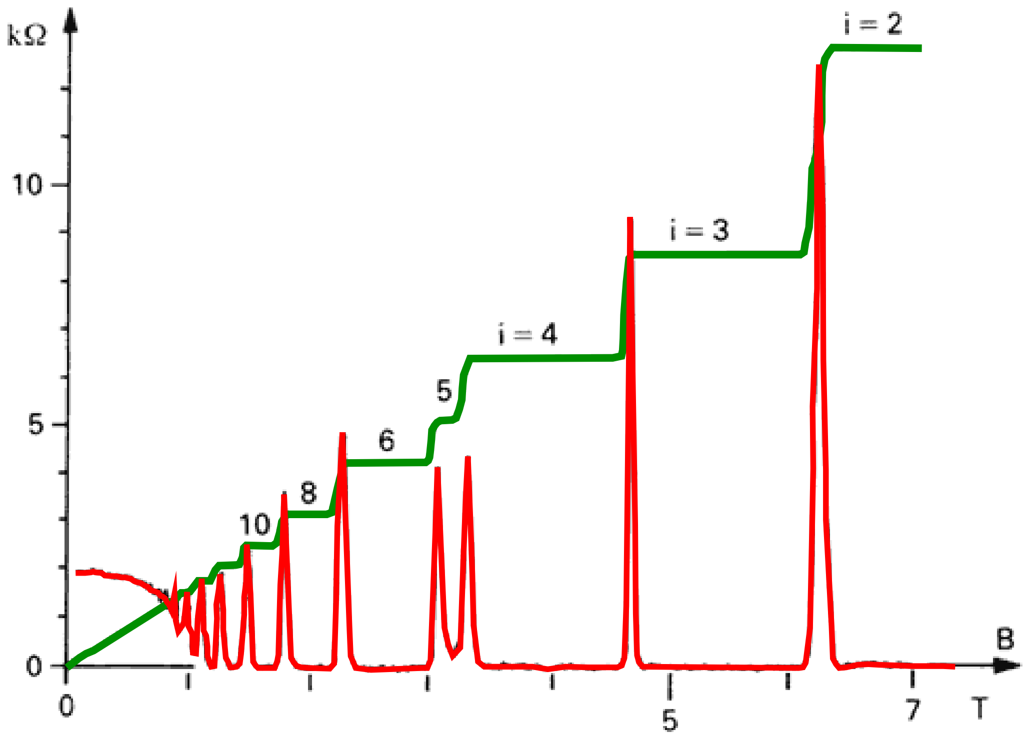
Magnetotransport in 2D: Integer Quantum Hall Effect

resistivity tensor:

$\rho_{xx} = E_x / j_x$

$\rho_{yx} = E_y / j_x$ **Hall resistivity**



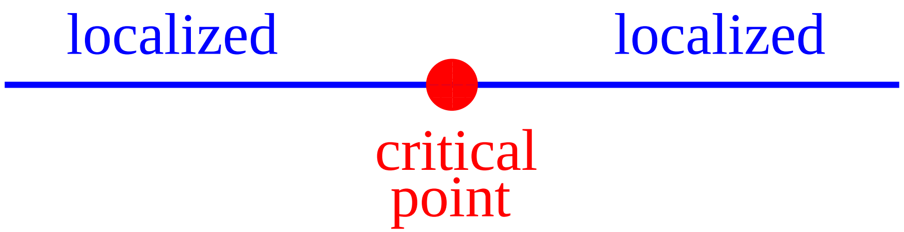


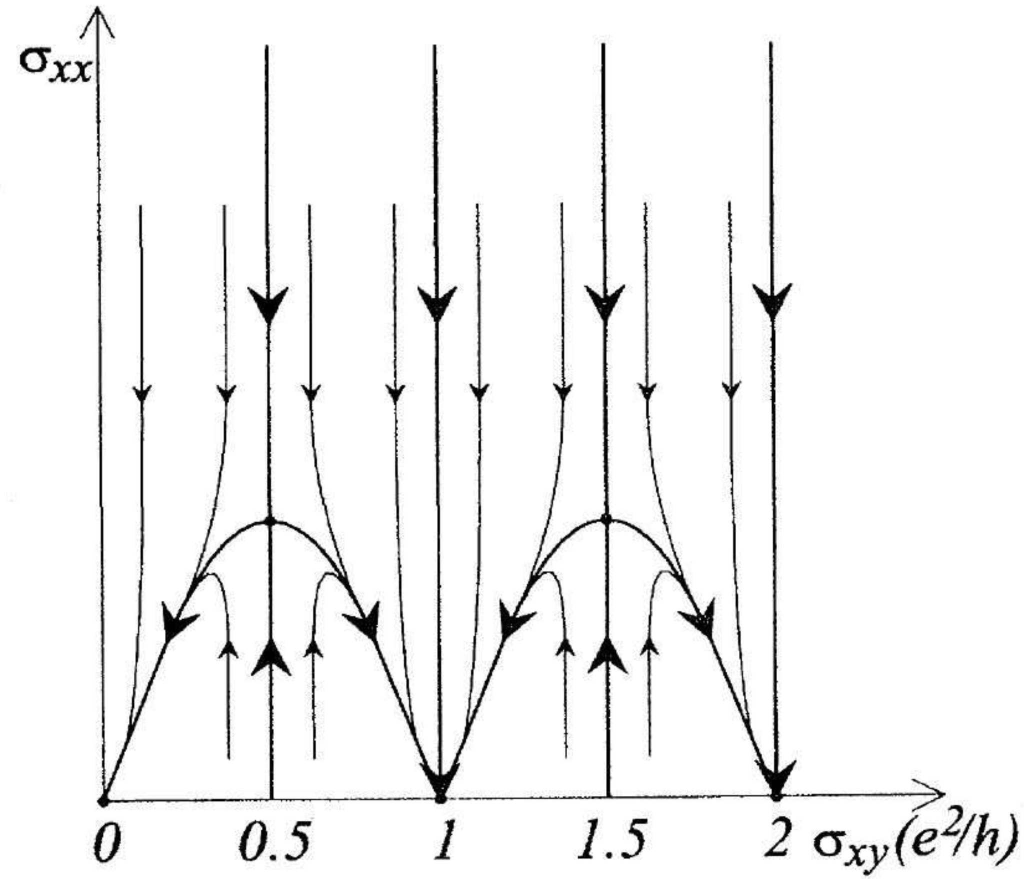
Klaus von Klitzing
Nobel Prize 1985

IQHE: $\mathbb{Z}$ topological insulator

IQHE flow diagram

Khmelnitskii' 83, Pruisken' 84





Field theory (Pruisken):

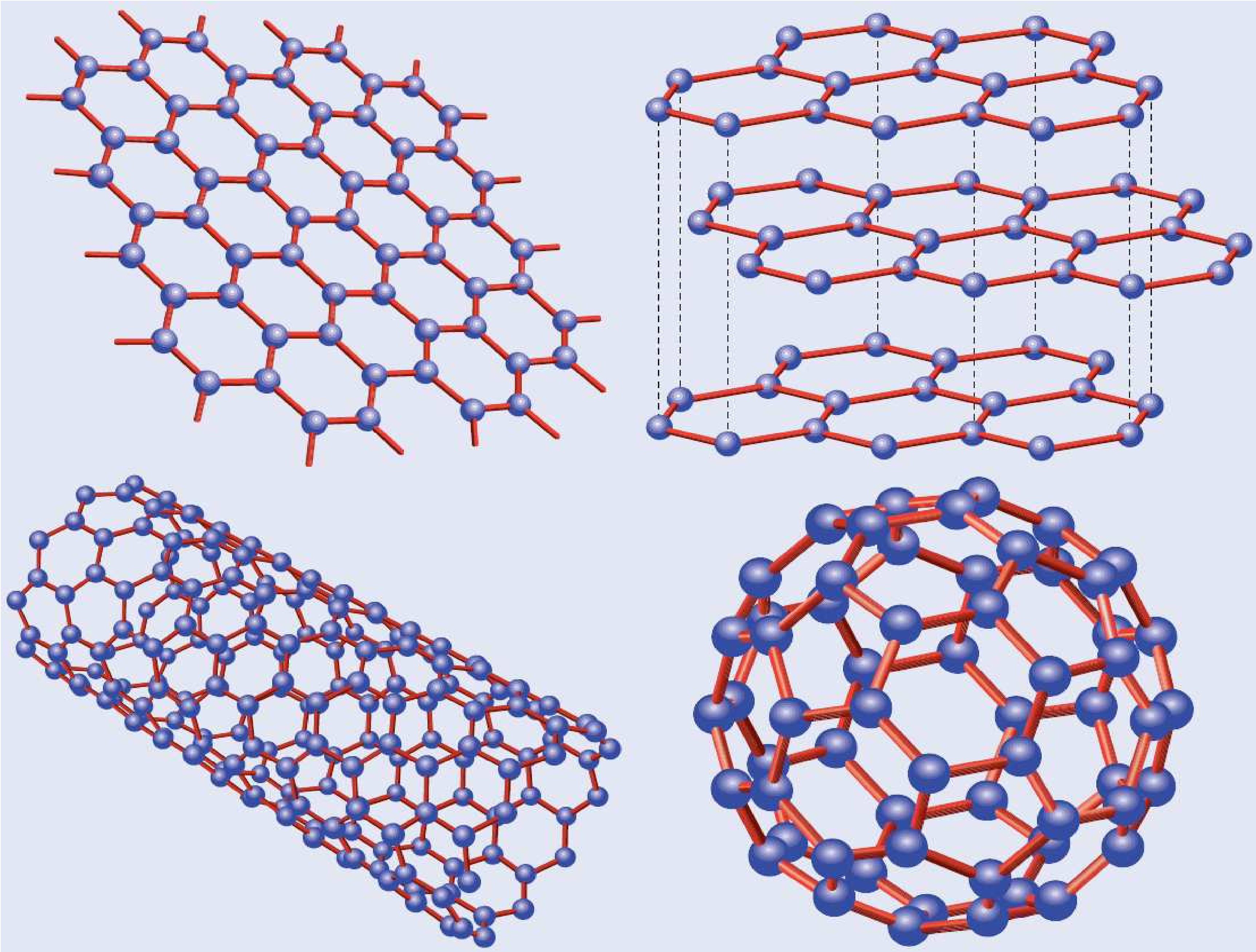
***σ*-model with topological term**

$$S = \int d^2 r \left\{ -\frac{\sigma_{xx}}{8} \mathrm{Tr}(\partial_\mu Q)^2 + \frac{\sigma_{xy}}{8} \mathrm{Tr}\epsilon_{\mu\nu} Q \partial_\mu Q \partial_\nu Q \right\}$$

QH insulators $\longrightarrow$ $n = \ldots, -2, -1, 0, 1, 2, \ldots$ **edge states**

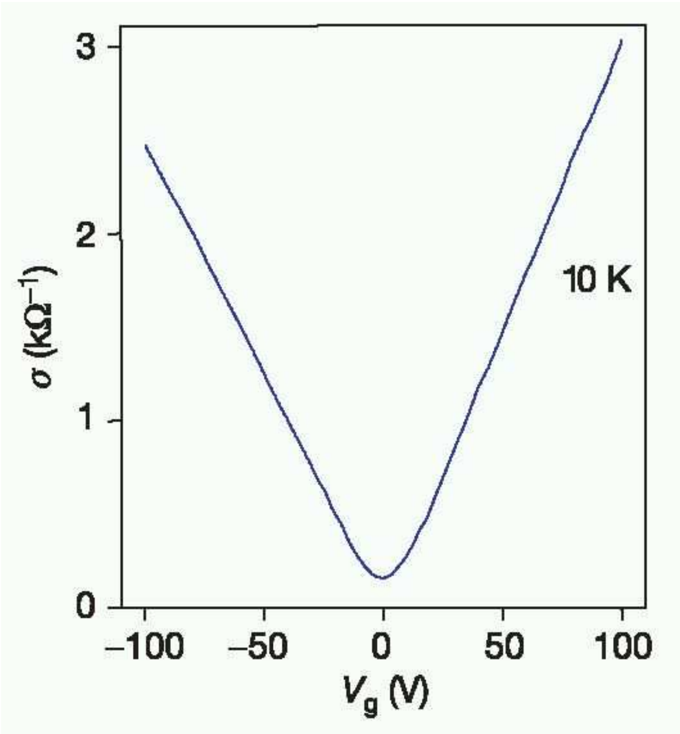
$\longrightarrow$ $\mathbb{Z}$ **topological insulator**

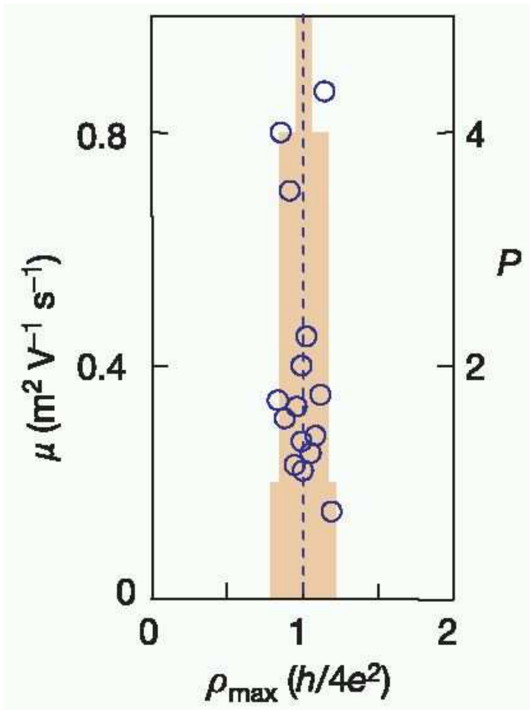
Graphene: monoatomic layer of carbon

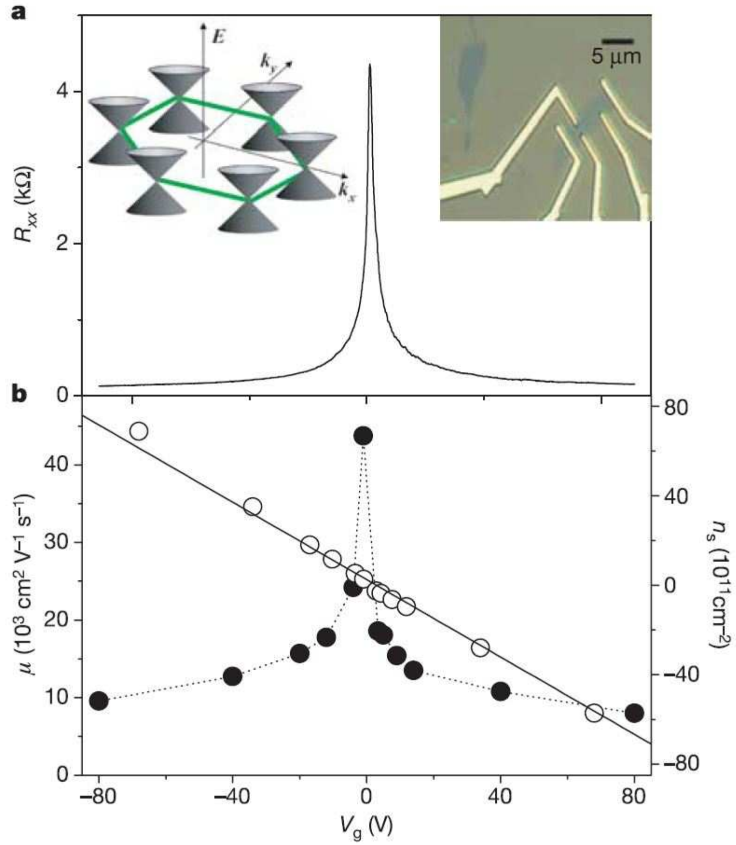


Experiments on transport in graphene

Novoselov, Geim et al; Zhang, Tan, Stormer, and Kim; Nature 2005





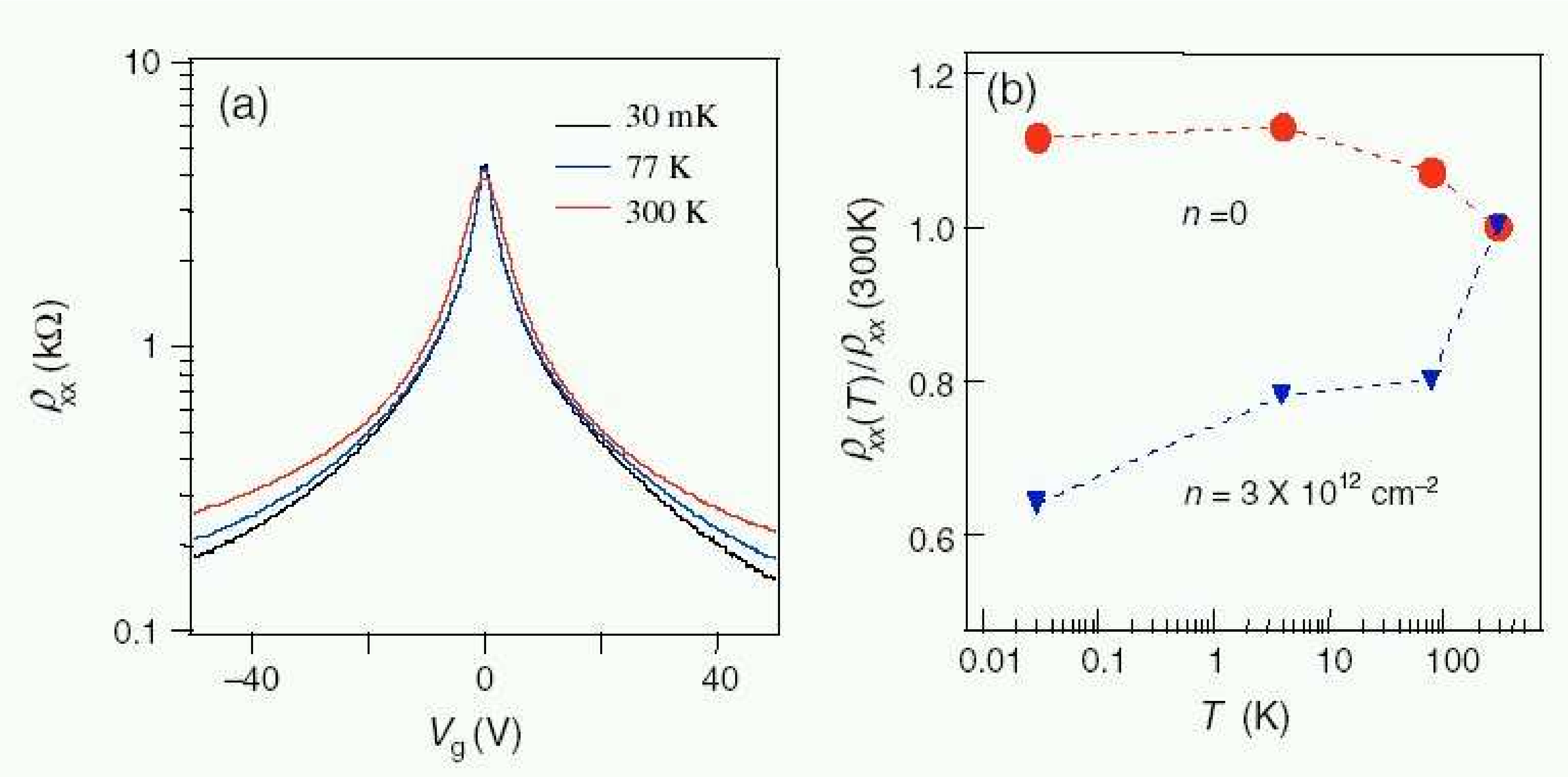


- linear dependence of conductivity on electron density ($\propto V_g$)
- minimal conductivity $\sigma \approx 4e^2/h$ ($\approx e^2/h$ per spin per valley)

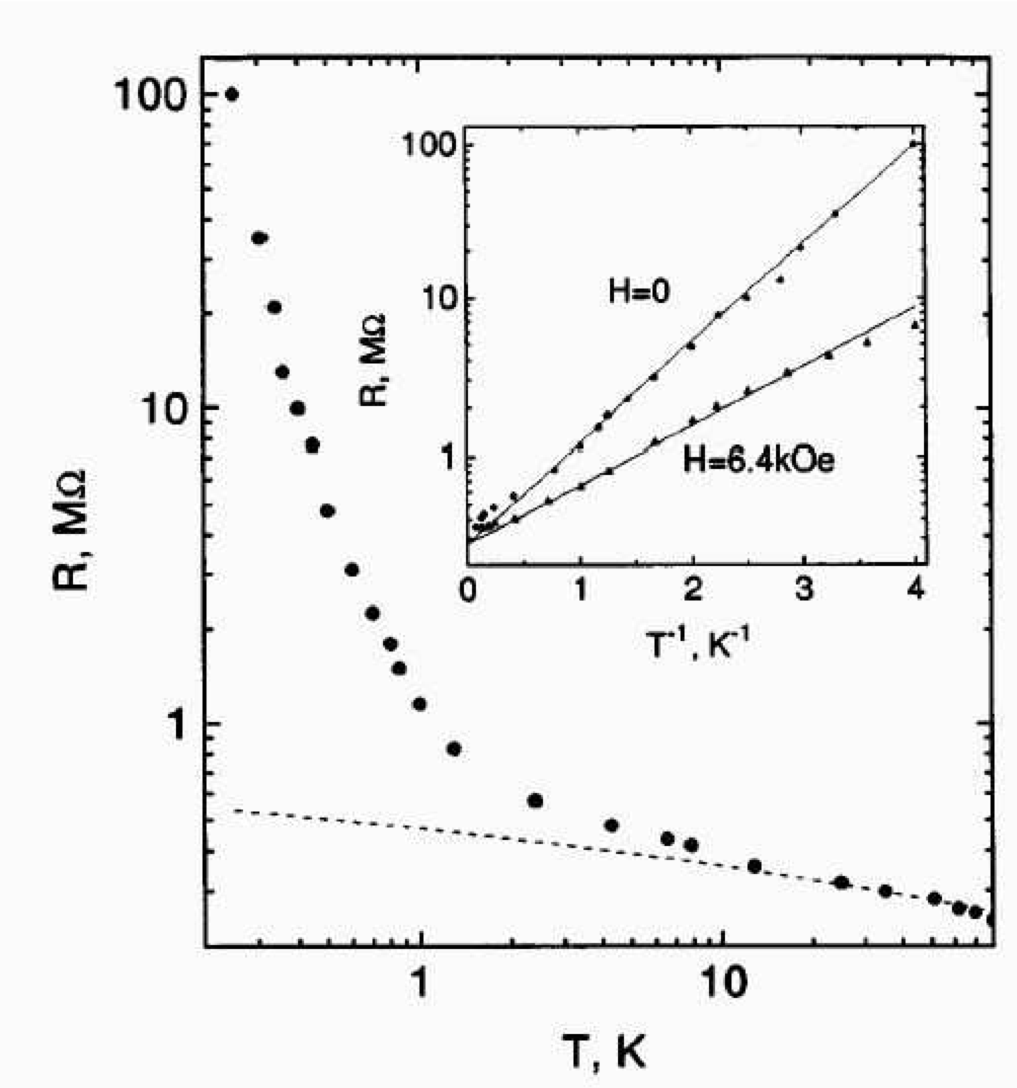
  T–independent in the range $T = 30$ mK $\div$ 300 K

T-independent minimal conductivity in graphene

Tan, Zhang, Stormer, Kim '07 $\quad T = 30\ \text{mK} \div 300\ \text{K}$

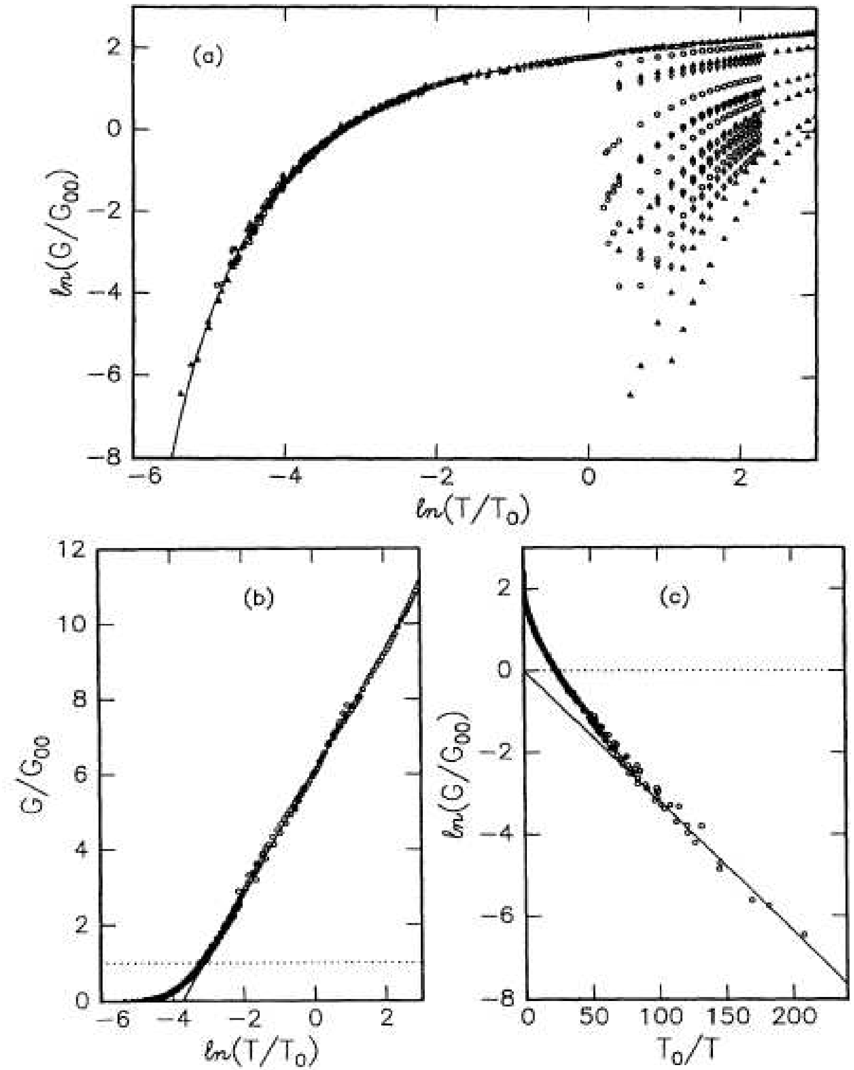


To compare: Disordered semiconductor systems: From metal to insulator with lowering T



quasi-1D geometry (long wires)

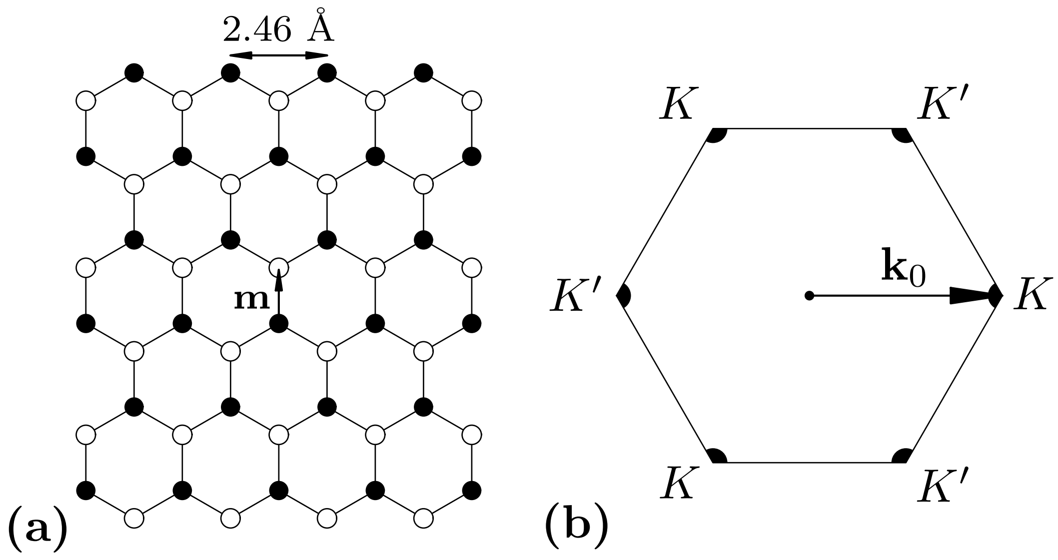
Gershenson et al ’97

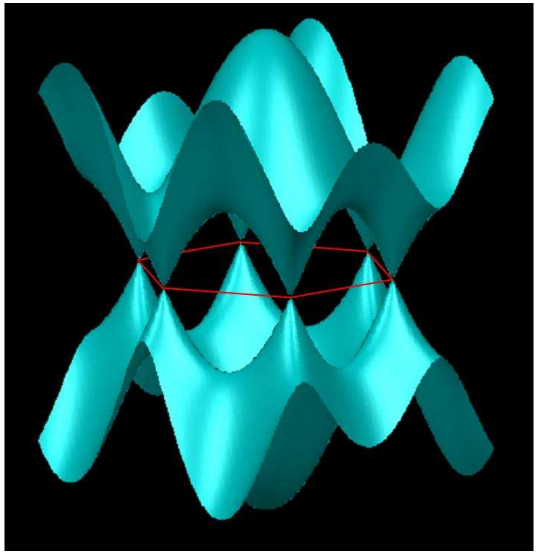


2D geometry

Hsu, Valles ’95

Graphene dispersion: 2D massless Dirac fermions





Two sublattices: A and B **Hamiltonian:** $H = \begin{pmatrix} 0 & t_{\mathrm{k}} \\ t^*_{\mathrm{k}} & 0 \end{pmatrix}$

$t_{\mathrm{k}} = t\left[1 + 2e^{i(\sqrt{3}/2)k_y a}\cos(k_x a/2)\right]$ **Spectrum** $\varepsilon^2_{\mathrm{k}} = |t_{\mathrm{k}}|^2$

The gap vanishes at 2 points, $K, K' = (\pm k_0, 0)$, where $k_0 = 4\pi/3a$.

In the vicinity of K, K': massless Dirac-fermion Hamiltonian:

$$H_K = v_0(k_x\sigma_x + k_y\sigma_y), \qquad H_{K'} = v_0(-k_x\sigma_x + k_y\sigma_y)$$

$v_0 \simeq 10^8$ cm/s – effective "light velocity", sublattice space $\longrightarrow$ isospin

Graphene: Disordered Dirac-fermion Hamiltonian

Hamiltonian $\longrightarrow$ 4×4 matrix operating in:

AB space of the two sublattices ($\boldsymbol{\sigma}$ Pauli matrices),

$\boldsymbol{K}$–$\boldsymbol{K'}$ space of the valleys ($\boldsymbol{\tau}$ Pauli matrices).

Four-component wave function:

$$\Psi = \{\phi_{AK}, \phi_{BK}, \phi_{BK'}, \phi_{AK'}\}^T$$

Hamiltonian:

$$H = -iv_0\tau_z(\sigma_x\nabla_x + \sigma_y\nabla_y) + V(x,y)$$

Disorder:

$$V(x,y) = \sum_{\mu,\nu=0,x,y,z} \sigma_\mu\tau_\nu V_{\mu\nu}(x,y)$$

Clean graphene: symmetries

Space of valleys K–K': Isospin $\Lambda_x = \sigma_z\tau_x$, $\Lambda_y = \sigma_z\tau_y$, $\Lambda_z = \sigma_0\tau_z$.

Time inversion	Chirality
$T_0: \quad H = \sigma_x\tau_x H^T \sigma_x\tau_x$	$C_0: \quad H = -\sigma_z\tau_0 H \sigma_z\tau_0$

Combinations with $\Lambda_{x,y,z}$

$T_x: \quad H = \sigma_y\tau_0 H^T \sigma_y\tau_0$	$C_x: \quad H = -\sigma_0\tau_x H \sigma_0\tau_x$
$T_y: \quad H = \sigma_y\tau_z H^T \sigma_y\tau_z$	$C_y: \quad H = -\sigma_0\tau_y H \sigma_0\tau_y$
$T_z: \quad H = \sigma_x\tau_y H^T \sigma_x\tau_y$	$C_z: \quad H = -\sigma_z\tau_z H \sigma_z\tau_z$

Spatial isotropy $\Rightarrow$ $T_{x,y}$ and $C_{x,y}$ occur simultaneously $\Rightarrow$ $T_\perp$ and $C_\perp$

Conductivity at $\mu = 0$

Drude conductivity (SCBA = self-consistent Born approximation):

$$\sigma = -\frac{8e^2v_0^2}{\pi\hbar}\int \frac{d^2k}{(2\pi)^2}\frac{(1/2\tau)^2}{[(1/2\tau)^2 + v_0^2k^2]^2} = \frac{2e^2}{\pi^2\hbar} = \frac{4}{\pi}\frac{e^2}{h}$$

BUT: For generic disorder, the Drude result $\sigma = 4 \times e^2/\pi h$ at $\mu = 0$ does not make much sense: Anderson localization will drive $\sigma \to 0$.

Experiment: $\sigma \approx 4 \times e^2/h$ independent of T

Can one have non-zero σ (i.e. no localization) in the theory?

Yes, if disorder either

(i) preserves one of chiral symmetries

or

(ii) is of long-range character (does not mix the valleys)

Absence of localization of Dirac fermions in graphene with chiral or long-range disorder

Disorder	Symmetries	Class	Conductivity
Vacancies	C_z, T_0	BDI	$\approx 4e^2/\pi h$
Vacancies + RMF	C_z	AIII	$\approx 4e^2/\pi h$
$\sigma_z\tau_{x,y}$ disorder	C_z, T_z	CII	$\approx 4e^2/\pi h$
Dislocations	C_0, T_0	CI	$4e^2/\pi h$
Dislocations + RMF	C_0	AIII	$4e^2/\pi h$
random v, resonant scatterers	C_0, Λ_z, $T_\perp$	2×DIII	$4e^2/\pi h \times \{1, \log L\}$
Ripples, RMF	C_0, Λ_z	2×AIII	$4e^2/\pi h$
Charged impurities	Λ_z, $T_\perp$	2×AII	$(4e^2/\pi h) \log L$
random Dirac mass: $\sigma_z\tau_{0,z}$	Λ_z, $CT_\perp$	2×D	$4e^2/\pi h$
Charged imp. + RMF/ripples	Λ_z	2×A	$4\sigma_U^*$

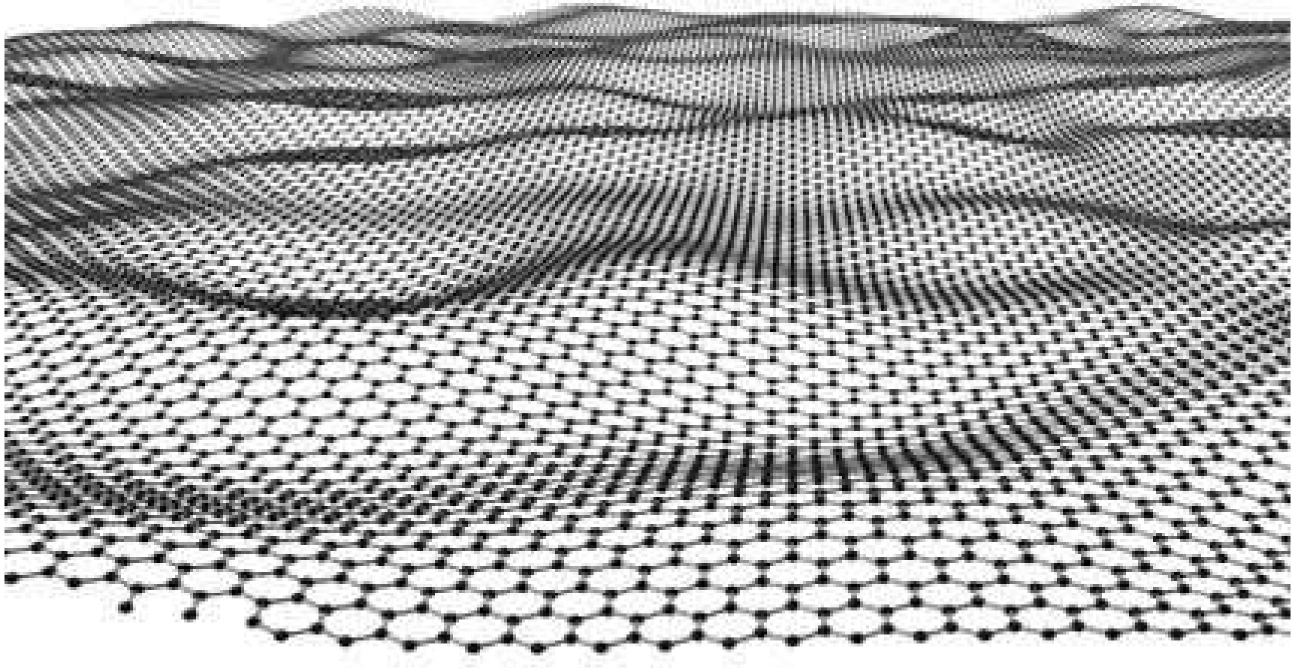
C_z-chirality $\longrightarrow$ Gade-Wegner phase

C_0-chirality $\equiv$ random gauge fields $\longrightarrow$ Wess-Zumino-Witten term

Λ_z-symmetry $\equiv$ decoupled valleys $\longrightarrow$ $\theta = \pi$ topological term

Random gauge fields (C_0 chirality, WZW term)

Conductivity: $\sigma = 4e^2/\pi h$ exact!



Ripples $\approx$ random abelian vector potential (C_0, Λ_z)

Estimated size d and height h:

$d = 5$ nm, $h = 0.5$ nm (from electron diffraction pattern)

Meyer, Geim, Katsnelson, Novoselov, Booth, Roth, Nature'07

$d = 10$ nm, $h = 0.3$ nm (from AFM measurements)

Tikhonenko, Horsell, Gorbachev, Savchenko, PRL'08

Long-range disorder: σ-models with topological term

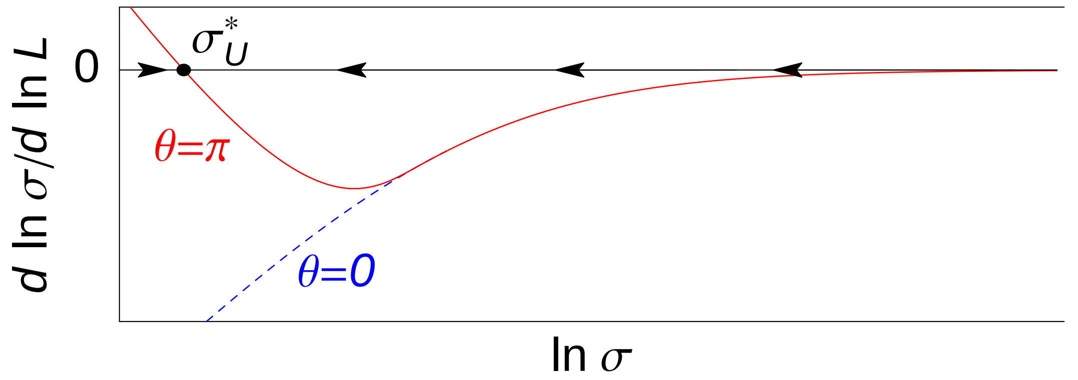
- Generic (ripples + charged impurities) $\Longrightarrow$ class A (unitary)

$$S[Q] = \frac{1}{8}\mathrm{Str}\left[-\sigma_{xx}(\nabla Q)^2 + Q\nabla_x Q\nabla_y Q\right] \quad = \quad -\frac{\sigma_{xx}}{8}\mathrm{Str}(\nabla Q)^2 + i\pi N[Q]$$

topol. invariant $N[Q] \in \pi_2(\mathcal{M}) = \mathbb{Z}$

$\Longrightarrow$ Quantum Hall critical point

$$\sigma \;=\; 4\sigma_U^* \;\simeq\; 4 \times (0.5 \div 0.6)\frac{e^2}{h}$$



- Random potential (charged imp.) $\Longrightarrow$ class AII (symplectic)

$$S[Q] = -\frac{\sigma_{xx}}{16}\mathrm{Str}(\nabla Q)^2 + i\pi N[Q]$$

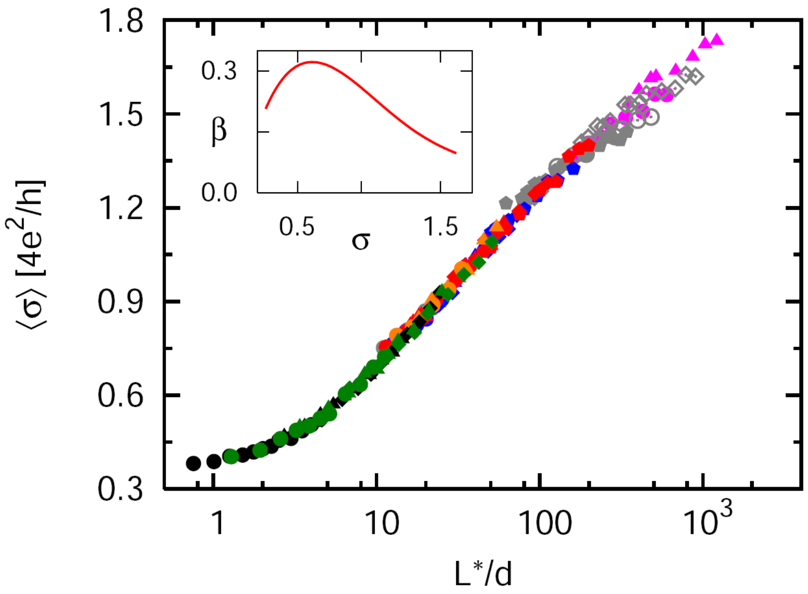
topological invariant: $N[Q] \in \pi_2(\mathcal{M}) = \mathbb{Z}_2 = \{0, 1\}$

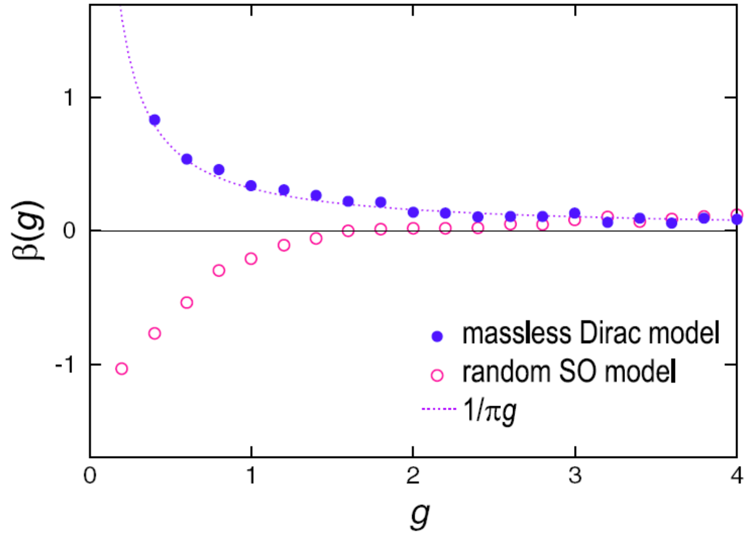
Topological protection from localization !

Long-range potential disorder: numerics

Bardarson, Tworzydło, Brouwer, Beenakker, PRL '07

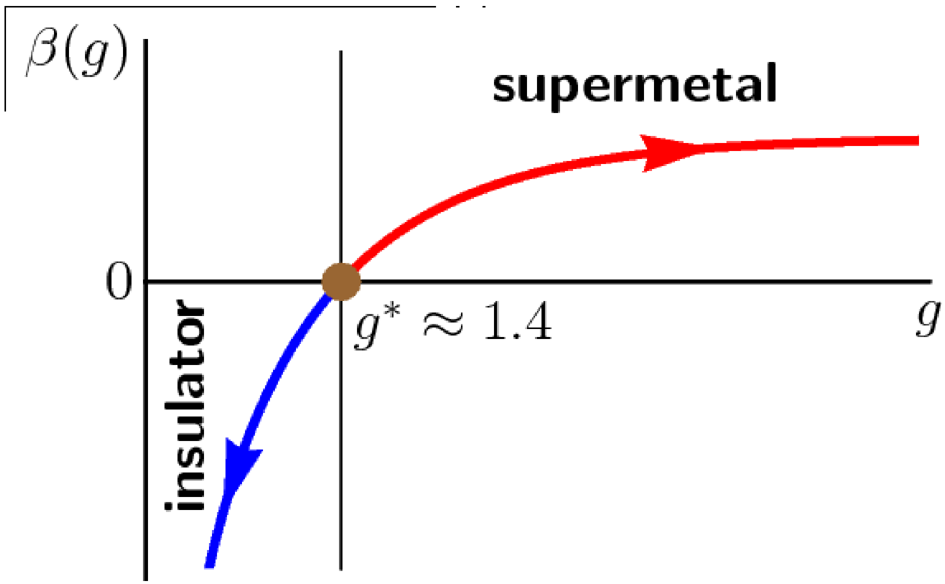
Nomura, Koshino, Ryu, PRL '07



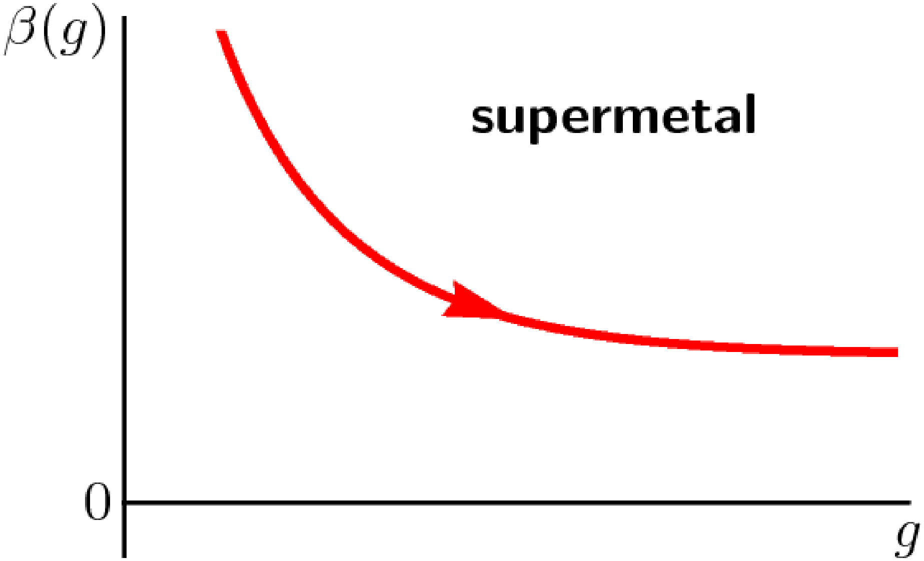


- absence of localization confirmed
- log scaling towards the perfect-metal fixed point $\sigma \to \infty$

Schematic beta functions for symplectic class AII



Conventional spin-orbit systems



Dirac fermions
(topological protection)

Topological Insulators: $\mathbb{Z}$ and $\mathbb{Z}_2$

Topological Insulators

= Bulk insulators with topologically protected delocalized states on their boundary

Theory: Moore, Balents; Kane, Mele; Bernevig, Zhang; Schnyder, Ryu, Furusaki, Ludwig; Kitaev; . . .

Well-known example: Quantum Hall Effect (2D, class A)

QH insulators $\longrightarrow$ $n = \ldots, -2, -1, 0, 1, 2, \ldots$ **edge states**

$\longrightarrow$ $\mathbb{Z}$ **topological insulator**

$\mathbb{Z}_2$ **TIs:** $n = 0$ **or** $n = 1$

Recent experimental realizations: Molenkamp & Hasan groups

2D and 3D systems with strong spin-orbit interaction (class AII)

2D: Quantum Spin Hall Effect

Periodic table of Topological Insulators

	Symmetry classes				Topological insulators			
p	H_p	R_p	S_p	$\pi_0(R_p)$	d=1	d=2	d=3	d=4
0	AI	BDI	CII	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
1	BDI	BD	AII	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
2	BD	DIII	DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
3	DIII	AII	BD	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
4	AII	CII	BDI	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
5	CII	C	AI	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
6	C	CI	CI	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
7	CI	AI	C	0	0	0	$\mathbb{Z}$	0
$0'$	A	AIII	AIII	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$1'$	AIII	A	A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

H_p – symmetry class of Hamiltonians

R_p – sym. class of classifying space (of Hamiltonians with eigenvalues $\to \pm 1$)

S_p – symmetry class of compact sector of σ-model manifold

Kitaev'09; Schnyder, Ryu, Furusaki, Ludwig'08-09; Ostrovsky, Gornyi, ADM'09

Classification of Topological insulators

Two ways to detect existence of TIs of class p in d dimensions:

(i) by inspecting the topology of classifying spaces R_p:

$$\begin{cases} \text{TI of type } \mathbb{Z} \\ \text{TI of type } \mathbb{Z}_2 \end{cases} \Longleftrightarrow \quad \pi_0(R_{p-d}) = \begin{cases} \mathbb{Z} \\ \mathbb{Z}_2 \end{cases}$$

(ii) by analyzing homotopy groups of the σ-model manifolds:

$$\begin{cases} \text{TI of type } \mathbb{Z} \Longleftrightarrow \pi_d(S_p) = \mathbb{Z} & \text{Wess-Zumino term} \\ \text{TI of type } \mathbb{Z}_2 \Longleftrightarrow \pi_{d-1}(S_p) = \mathbb{Z}_2 & \theta = \pi \;\; \text{topological term} \end{cases}$$

WZ and $\theta = \pi$ terms make boundary excitations "non-localizable"

TI in d $\Longleftrightarrow$ topological protection from localization in $d-1$

Bott periodicity: $\quad \pi_d(R_p) = \pi_0(R_{p+d})$, periodicity 8

Periodic table of Topological Insulators

	Symmetry classes				Topological insulators			
p	H_p	R_p	S_p	$\pi_0(R_p)$	d=1	d=2	d=3	d=4
0	AI	BDI	CII	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
1	BDI	BD	AII	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
2	BD	DIII	DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
3	DIII	AII	BD	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
4	AII	CII	BDI	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
5	CII	C	AI	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
6	C	CI	CI	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
7	CI	AI	C	0	0	0	$\mathbb{Z}$	0
$0'$	A	AIII	AIII	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$1'$	AIII	A	A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

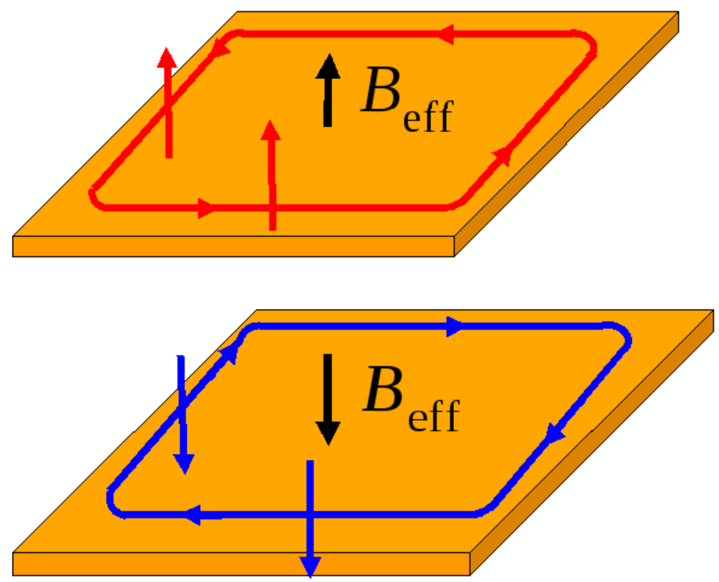
IQHE

Spin QHE, Thermal QHE in unconventional superconductors

2D (Q Spin HE) and 3D systems with strong SO interaction

2D $\mathbb{Z}_2$ TIs: Quantum Spin Hall Effect

Kane, Mele'05; Sheng, Sheng, Ting, Haldane'05; Bernevig, Zhang '06



Symmetry class AII (symplectic):
time-reversal invariance $T^2 = -1$

Simple model: two copies of QHE,
magnetic field B for spin $\uparrow$ and $-B$ for spin $\downarrow$
$\sigma_{xy}(\uparrow) = e^2/h \qquad \sigma_{xy}(\downarrow) = -e^2/h$
$\longrightarrow$ spin Hall conductivity $\sigma_{xy}(\uparrow) - \sigma_{xy}(\downarrow) = 2e^2/h$

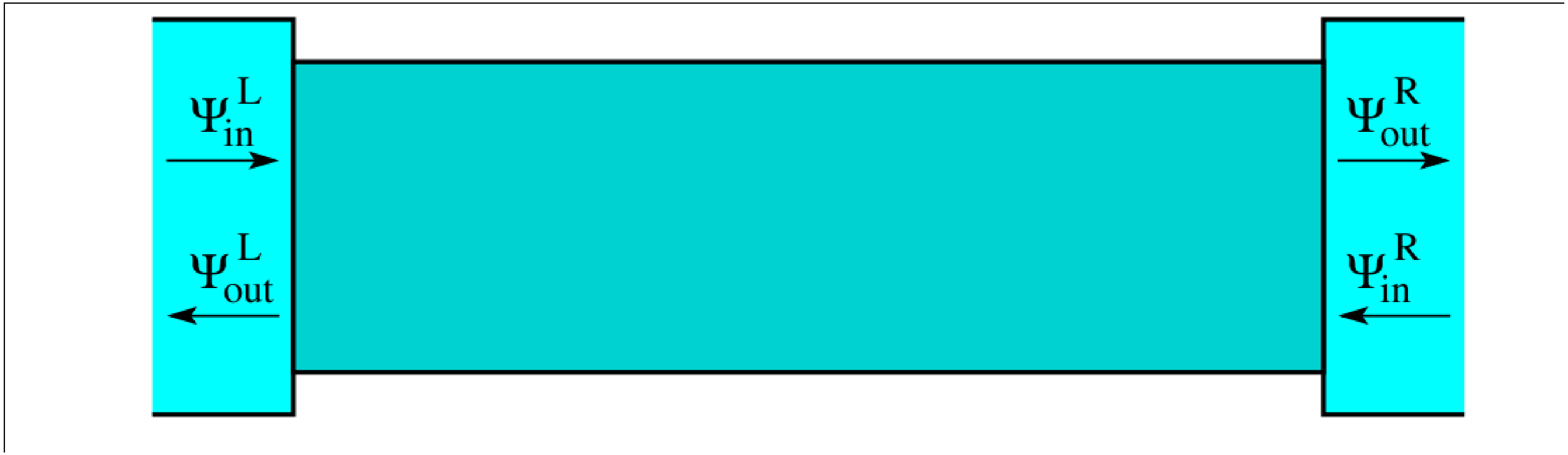
generic spin-orbit interaction
$\longrightarrow$ spin not conserved anymore but Kramers degeneracy holds
$\longrightarrow$ one propagating edge mode in each direction
backscattering forbidden: topological protection!

Earlier results on symplectic-class wires with odd number of channels:
one mode remains delocalized

Zirnbauer '92; ADM, Müller-Groeling, Zirnbauer '94; Takane '04

realization: carbon nanotubes with long-range disorder Ando, Suzuura '02

Absence of localization in a symplectic wire with odd number of channels



Scattering matrix of a symplectic system

$$\begin{pmatrix} \Psi_{\rm out}^{L} \\ \Psi_{\rm out}^{R} \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} \Psi_{\rm in}^{L} \\ \Psi_{\rm in}^{R} \end{pmatrix}$$

TI symmetry $\Longrightarrow$

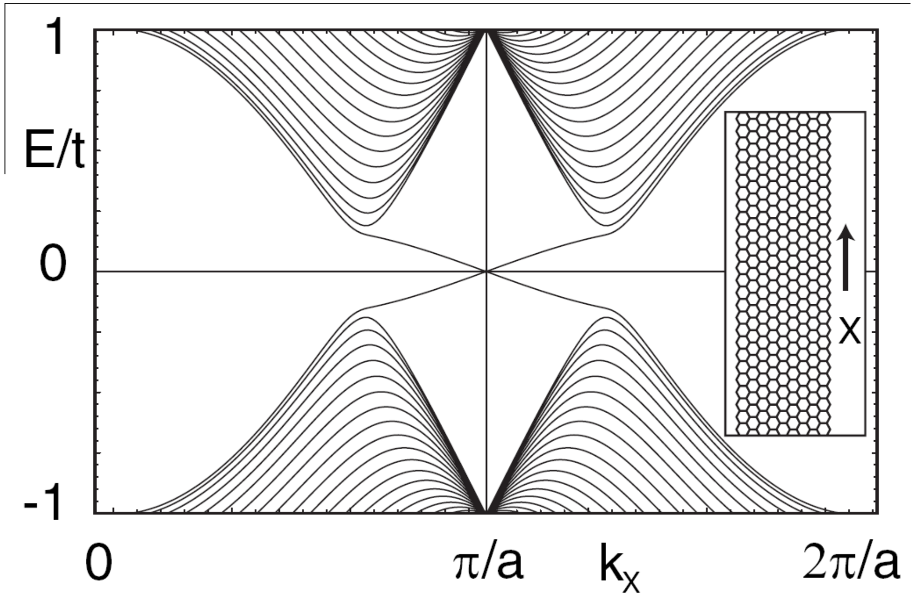
$$r = -r^T, \quad r' = -r'^T, \quad t = t'^T$$

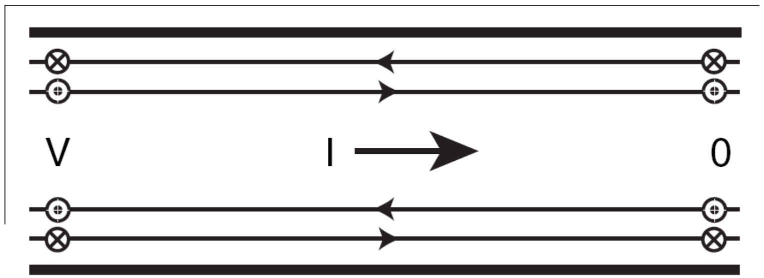
For N channels:

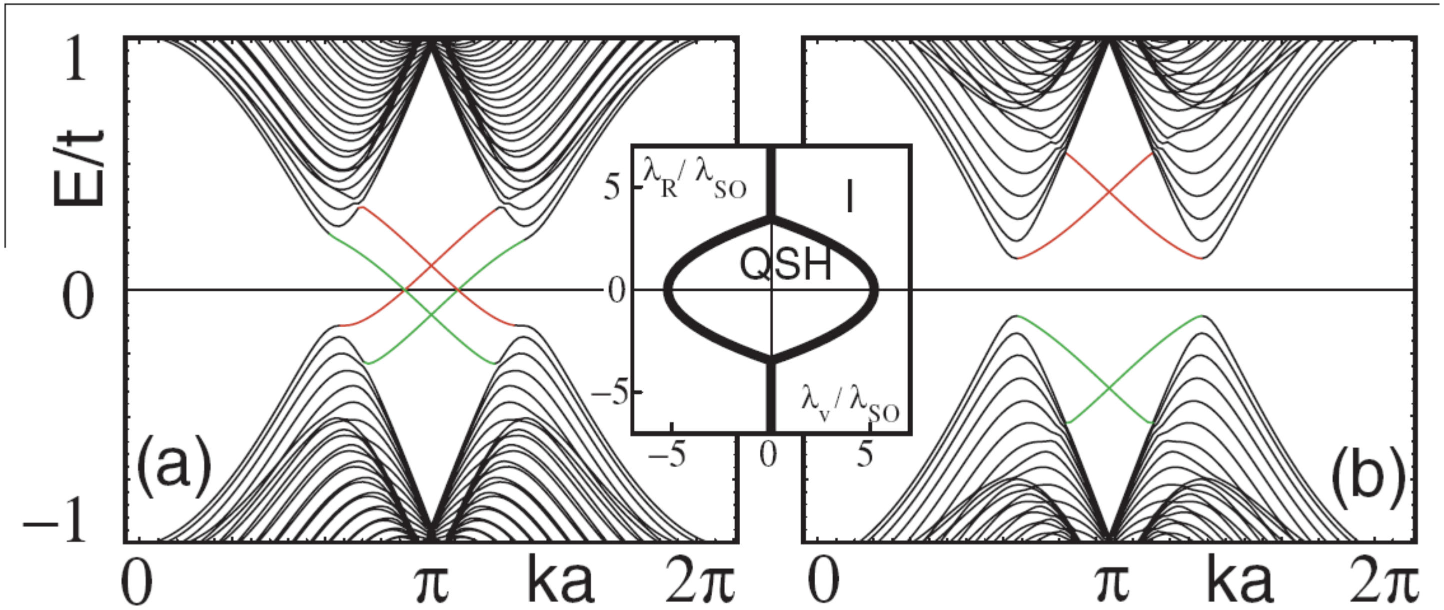
$\det r = (-1)^N \det r^T \quad \Longrightarrow$ **no localization if N is odd ! ! !**

Quantum Spin Hall Effect in graphene with SO interaction

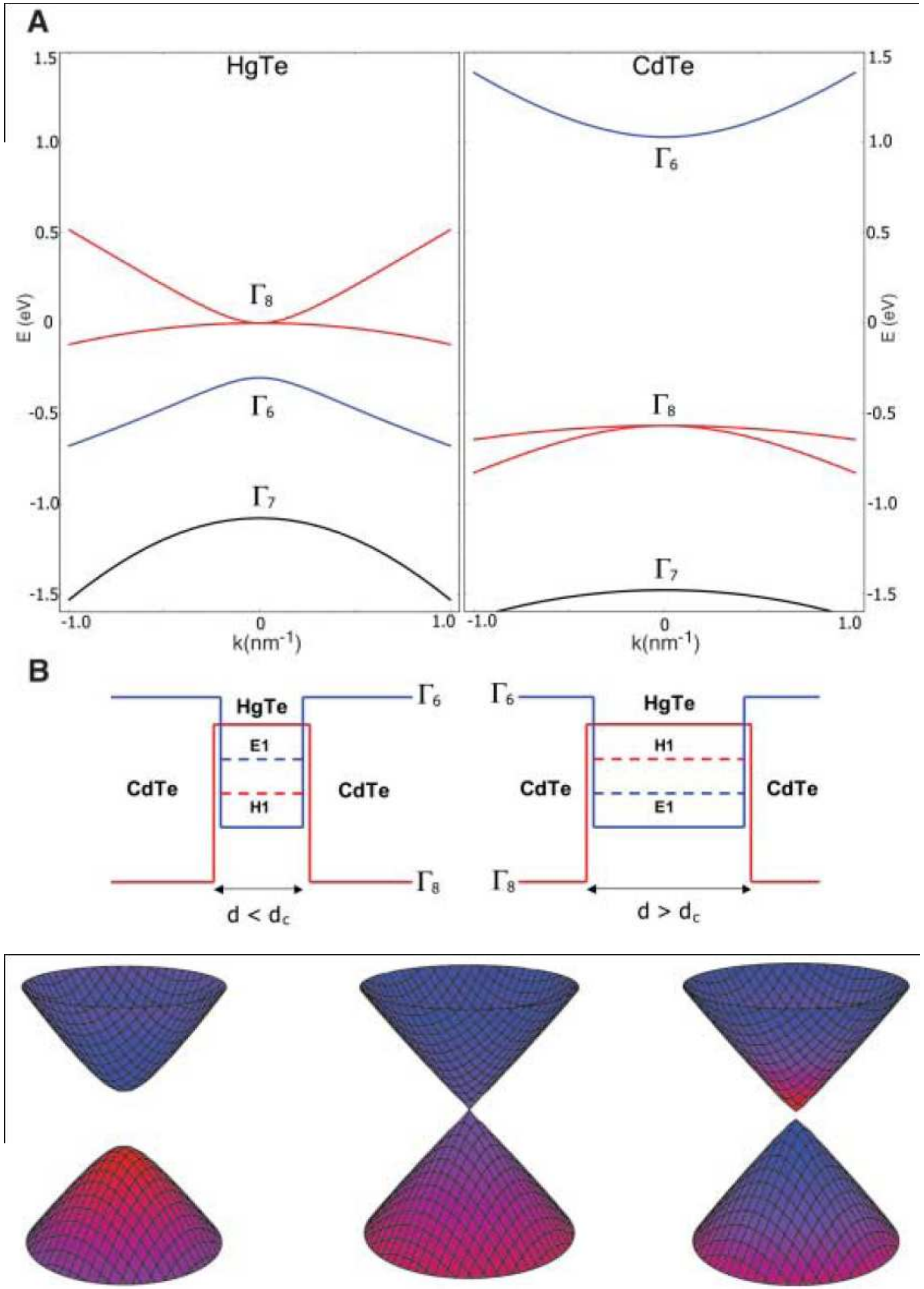
Kane, Mele'05







QSHE in CdTe/HgTe/CdTe quantum wells: Theory



Bernevig, Hughes, Zhang'06

$$H_{\text{eff}}(k) = \begin{pmatrix} h(k) & 0 \\ 0 & -h^*(k) \end{pmatrix}$$

$$h(k) = \begin{pmatrix} m(k) & Ak_- \\ Ak_+ & -m(k) \end{pmatrix}$$

$$k_\pm = k_x \pm i k_y$$

$$m(k) = M + B(k_x^2 + k_y^2)$$

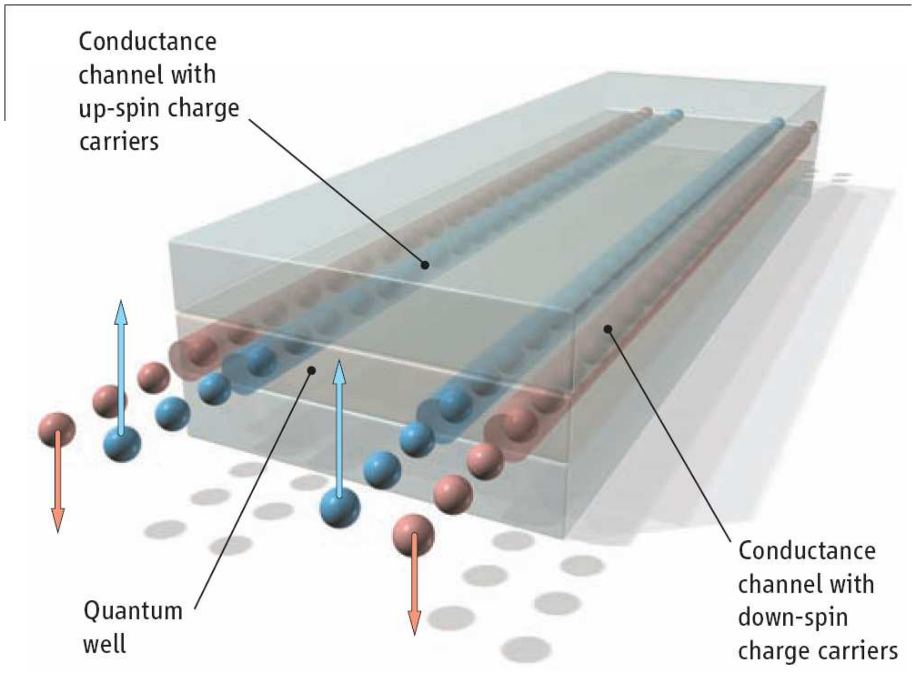
HgTe: inverted
band structure

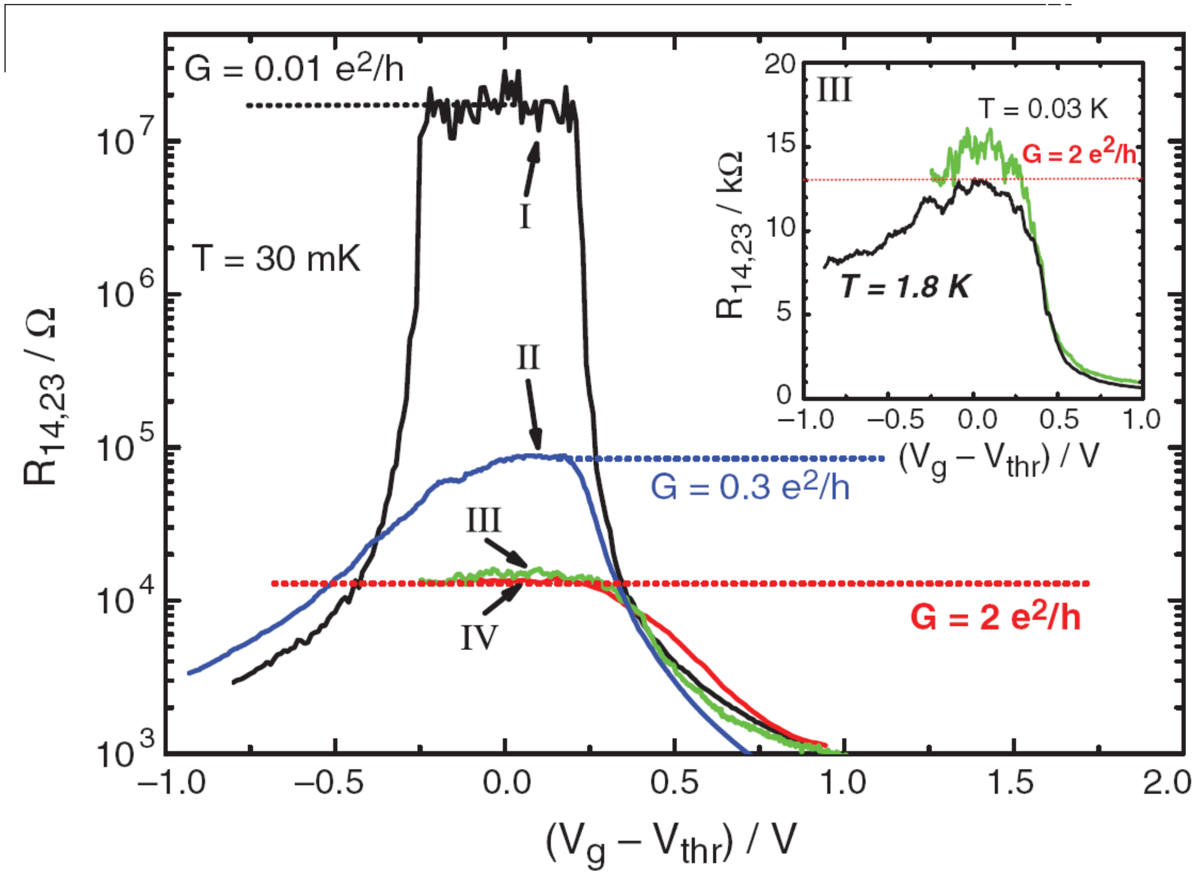
$\longrightarrow$ $M < 0$ for $d > d_c$

$\longrightarrow$ TI

QSHE in CdTe/HgTe/CdTe quantum wells: Experiment

Molenkamp group '07





I — normal insulator, $d = 5.5$ nm

II, III, IV — inverted quantum well structure, $d = 7.3$ nm

$\longrightarrow$ topological insulator

3D Topological Insulators

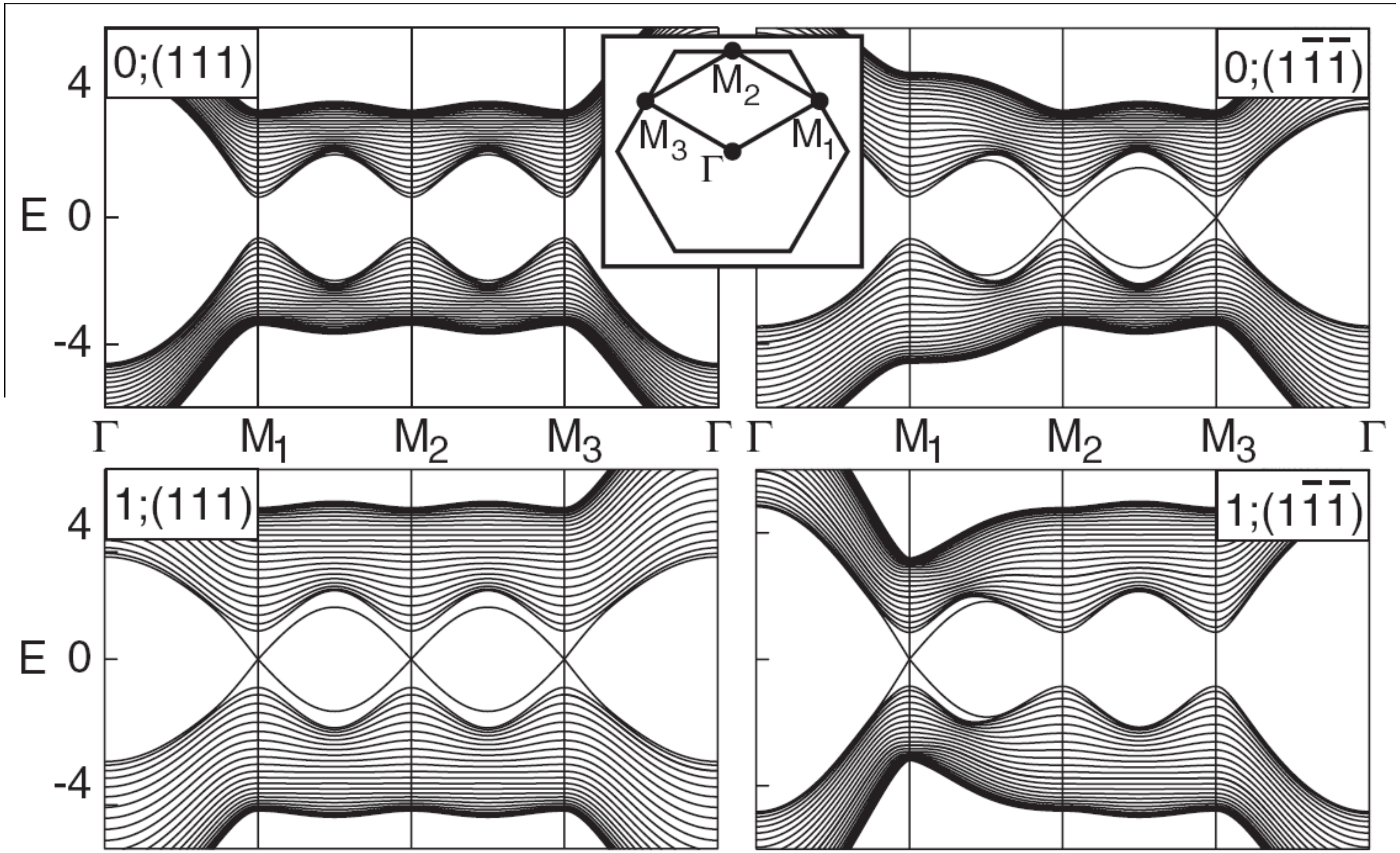
have 2D topologically protected delocalized modes at the surface

surface of a 3D TI = single-valley graphene

3D Topological Insulators

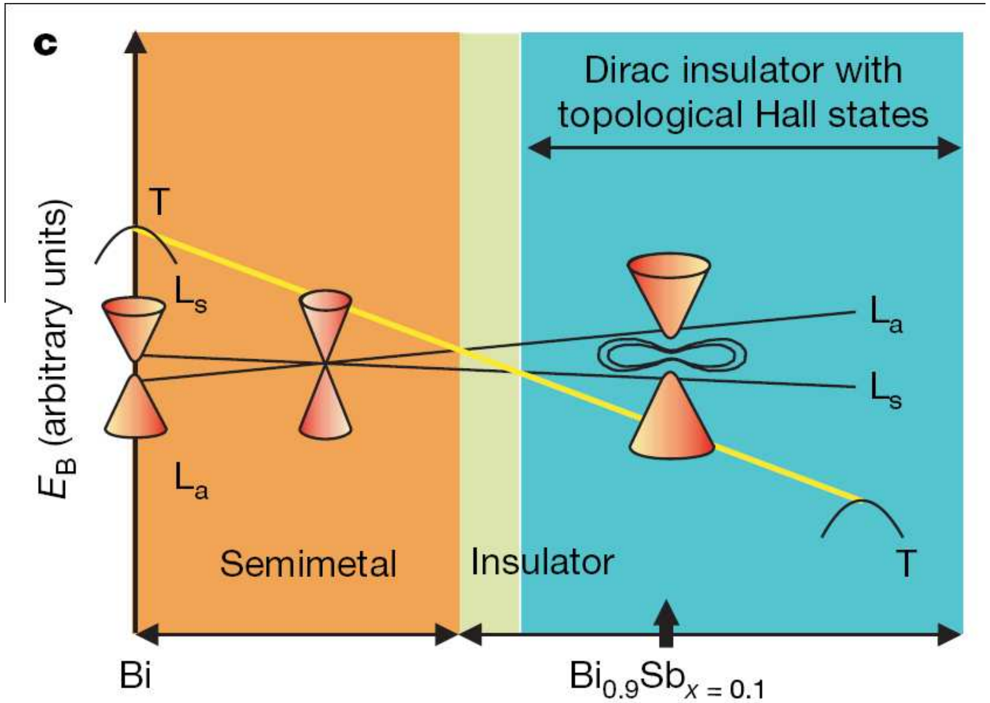
Tight-binding model on a diamond lattice with spin-orbit interaction

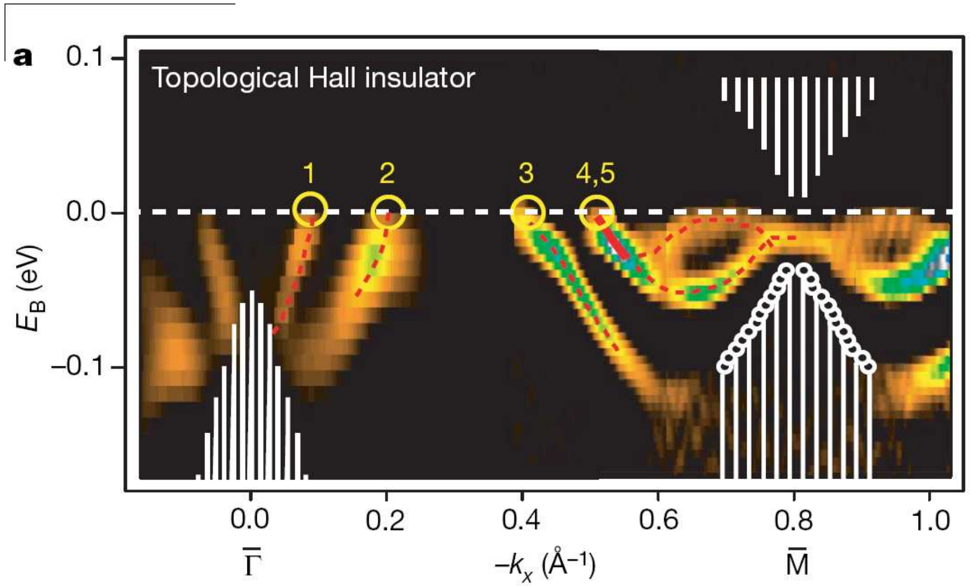
Fu, Kane, Mele '07

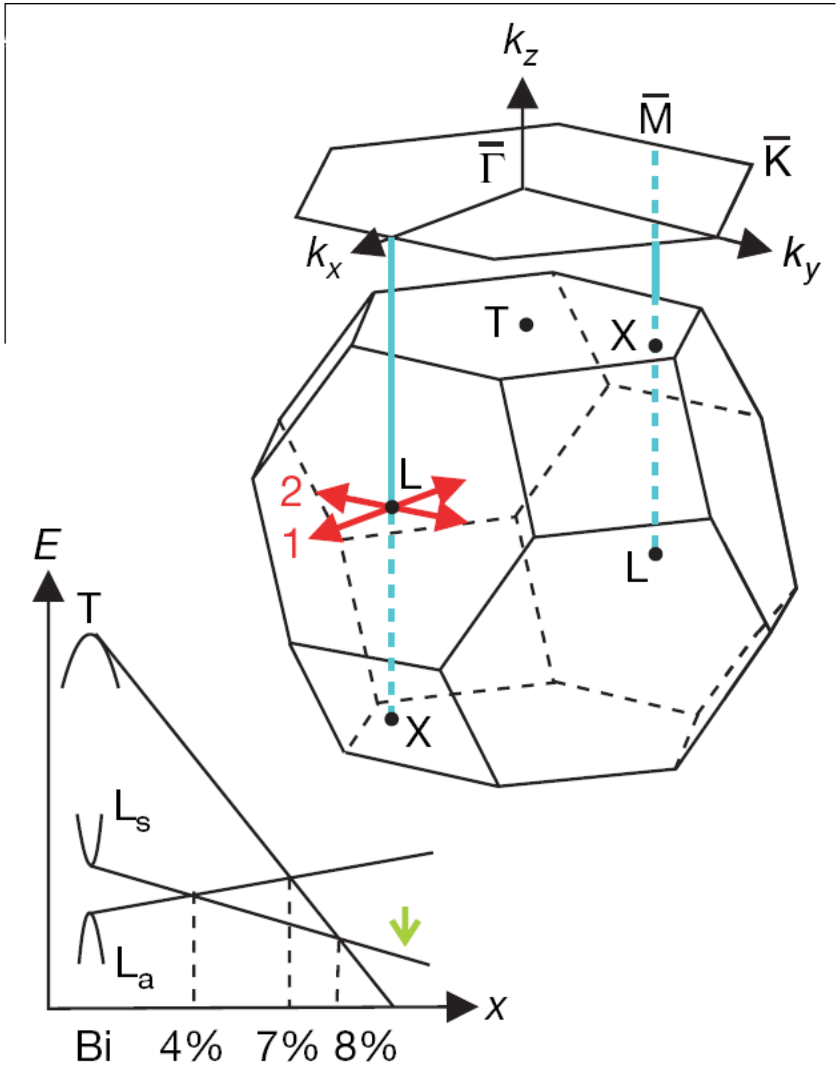


3D Topological Insulator: $Bi_{1-x}Sb_x$

Hasan group '08







Other realizations: BiTe, BiSe

2D Dirac surface states of a 3D TI: Disorder and interaction

Surface of 3D $\mathbb{Z}_2$ TI:

single 2D massless Dirac mode (more generally: odd number)

$\longleftrightarrow$ single-valley graphene !

With disorder:

Topological protection from localization,

RG flow towards supermetal

What is the effect of Coulomb interaction?

assume not too strong interaction $r_s = \sqrt{2}e^2/\epsilon v_F \lesssim 1$

$\Longrightarrow$ no instabilities, no symmetry-breaking

$\Longrightarrow$ topological protection from localization persists

But interaction may destroy the supermetal phase!

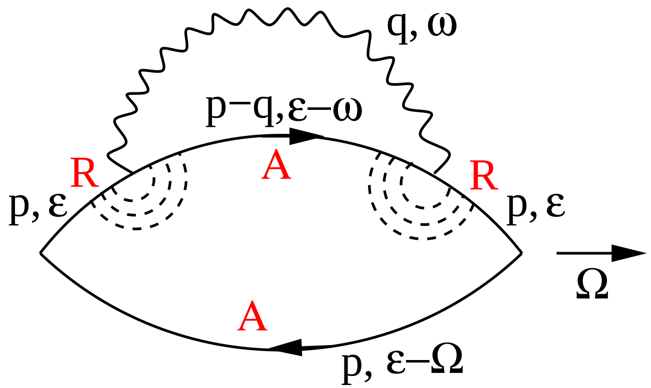
Coulomb interaction in symplectic class AII: RG

cf. Althsuler, Aronov '79; Finkelstein '83

$$\beta(g) = \frac{dg}{d \ln L} = \frac{N}{2} - 1 + (N^2 - 1)\mathcal{F}$$

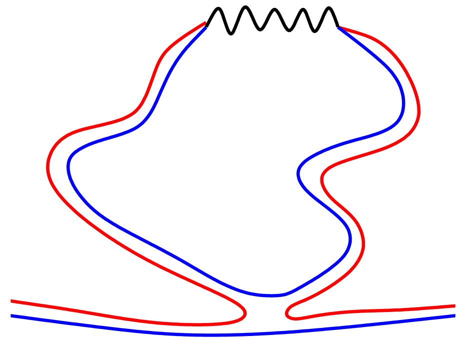
weak antilocalization − ee-singlet + ee-multiplet

N – # of flavors (spin, valleys, etc)



Graphene: $N = 4$ (2 valleys, 2 spins)

⟶ WAL wins ⟶ supermetal survives



Surface of a 3D TI: $N = 1$

⟶ $\beta(g) = -1/2 < 0$ ⟶ ee-interaction wins

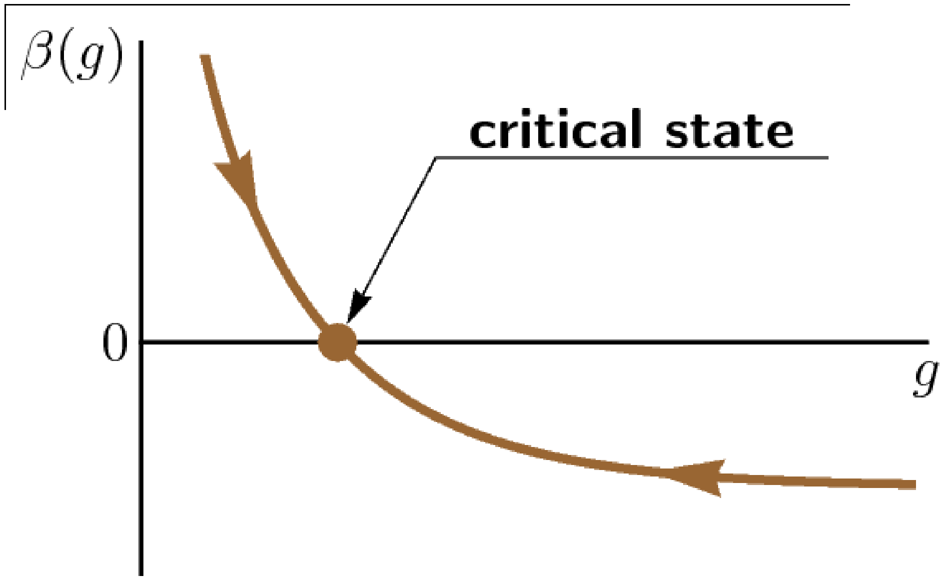
⟶ conductance decreases upon RG

⟶ Coulomb repulsion destroys supermetal phase

Interaction-induced quantum criticality in 3D TI

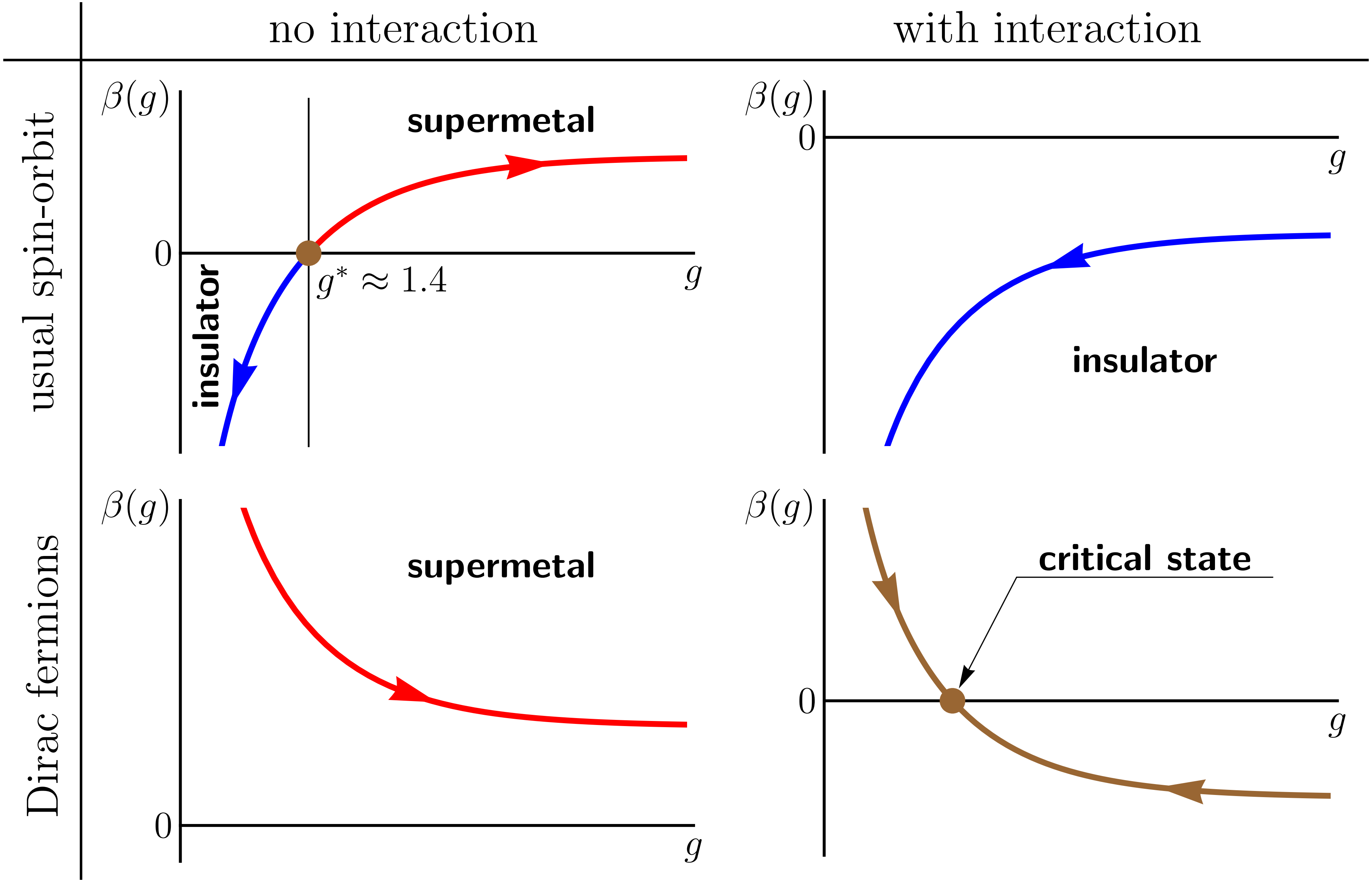
- **Interaction** $\longrightarrow$ tendency to localization at $g \gg 1$
- **Topology** $\longrightarrow$ protection from strong localization (no flow towards $g \ll 1$)

$\longrightarrow$ novel quantum critical point should emerge at $g \sim 1$



analogous to QHE, but here induced by interaction

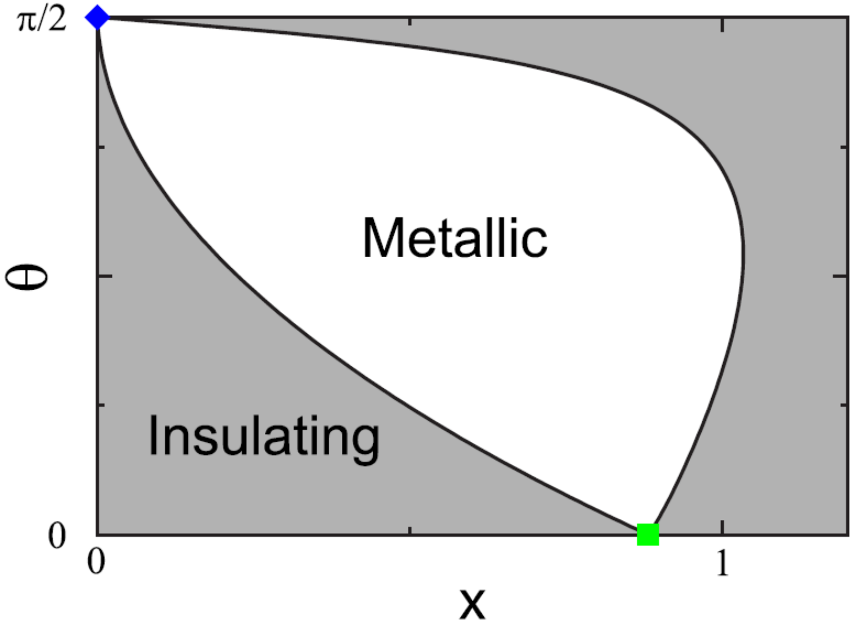
β functions for symplectic class: Interaction and Topology

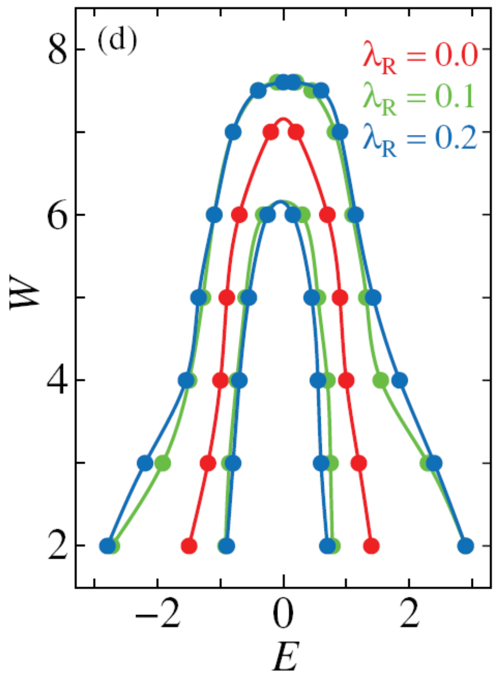


2D TIs: QSHE phase diagram

In the presence of disorder, TI and normal insulator phases are separated by the supermetal phase

transitions TI–supermetal and supermetal–NI are in the coventional symplectic MIT universality class

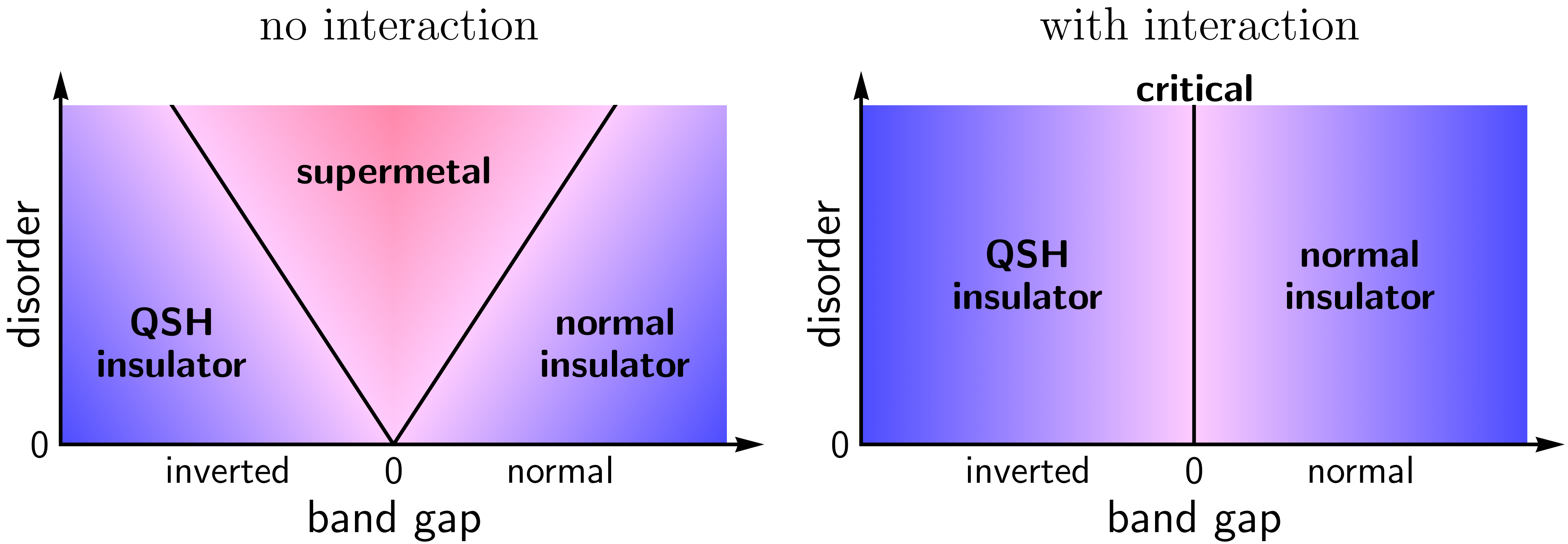




Onoda, Avishai, Nagaosa '07; Obuse et al '07

Effect of Coulomb interaction on phase diagram — ?

2D TIs: QSHE phase diagram (cont'd)



Coulomb interaction "kills" the supermetal phase, thus restoring a direct transition between two insulator phases

$\longrightarrow$ **quantum critical point of Quantum Spin Hall transition**

$\mathbb{Z}_2$ edge in the presence of Coulomb interaction

Edge of 2D TI: single propagating mode in each direction

Impurity backscattering prohibited (symplectic time reversal invariance)

Coulomb interaction $\longrightarrow$ Luttinger liquid , conductance e^2/h

Xu, Moore '06; Wu, Bernevig, Zhang '06:

Umklapp processes (uniform or random)

$\partial \mathcal{D}_2 / \partial \ln L = (3 - 8K)\mathcal{D}_2$ K – Luttinger liquid parameter

Coulomb 1/r interaction: $K(q) = \left(1 + 2\alpha \ln \frac{q_0}{q}\right)^{-1/2}$ $\alpha = e^2/\pi^2 \epsilon h v_F$

$\longrightarrow$ $\mathcal{D}_2$ processes negligible up to the scale $L_0 \sim q_0^{-1} \exp \frac{80}{9\alpha}$

What happens with TI beyond this scale is an interesting question but purely academic for not too strong interaction:

$r_s = 1$ $\longrightarrow$ $L_0 \sim 10^{60}$ nm $>$ size of Universe

$r_s = 6$ (Molenkamp experiment) $\longrightarrow$ $L_0 \sim 10$ m

Thus, TI phase persists in the presence of not too strong Coulomb interaction

Interaction-induced quantum critical points of $\mathbb{Z}_2$ TIs

We thus have two novel 2D quantum critical points:

- on surface of 3D TI
- 2D QSH transition

They share many common properties:

- symplectic symmetry
- $\mathbb{Z}_2$ topological protection
- interaction-induced criticality
- conductivity of order unity (probably universal)

This suggests that these two critical points may be equivalent

Outline

- **Anderson localization theory: Symmetries and topologies**
- **Graphene and 2D Dirac fermions**
- **Conductivity at Dirac point:**
  Absence of localization for chiral disorder *and*
  topological delocalization for long-range disorder
- **Topological insulators (TIs): General classification**
- **2D and 3D $\mathbb{Z}_2$ TIs in time-reversal-invariant systems**
  with spin-orbit interaction
- **Coulomb interaction in TIs:**
  quantum criticality at the surface of 3D TI *and*
  quantum spin Hall transition 2D TI to normal insulator