The Abdus Salam
International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA
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School and Workshop on Local Rings and Local Study of Algebraic Varieties (31 May - 11 June 2010)

WEEK I

ABSTRACTS

The first week (31 May - 4 June) will be an Advanced School with four instructional courses:

1. A. CONCA (University of Genova, Italy): Generic initial ideals, Hilbert functions and Betti numbers.
This course will be delivered by G. Caviglia (Purdue University, USA).

2. S. GOTO (Meiji University, Japan): Numerical invariants of local rings.

3. I. SWANSON (Reed College,USA): Integral closures of ideals, rings and modules.

4. B. ULRICH (Purdue University, USA): Rees algebras of modules from the point of view of equisingularity theory.

Trieste ICTP 2010
School and Workshop in in honor of Tito Valla

Generic initial ideals, Hilbert functions and Betti numbers

A.Conca (Genova)

I will discuss generic initial ideals, their properties and their theoretical use in the study of bounds for Betti numbers and related invariants. I will propose problems and questions related with the theoretical and computational aspects of the study of them. A good reference is M.Green "Generic initial ideals" in "Six lectures on commutative algebra" (Bellaterra, 1996), 119–186, Progr. Math., 166, Birkhuser, Basel, 1998. A rough schedule:

(1) Generic initial ideals: existence and main properties.
(2) The basic invariants: Hilbert functions, Betti numbers, regularity. Initial ideals and deformations.
(3) Monomial ideals, strongly stable, Borel fixed, lex-segments and their Betti numbers.
(4) Bounds on Betti numbers Macaulay Theorem, Bigatti-Hullett and Pardue Theorem.
(5) Rigidity: Herzog-Hibi-Aramova Theorem and extensions.
(6) Gin, in and reduction numbers, Gin and Fröberg conjecture, Gin Lex.

HOMOLOGICAL DEGREES AND THE HILBERT COEFFICIENTS OF PARAMETERS IN LOCAL RINGS

SHIRO GOTO

My lectures aim to report recent progress in the analysis of Hilbert coefficients of parameter ideals in local rings. My researches [GhGHOPV1], [GhGHOPV2], [GO], and [G] are based on discussions with L. Ghezzi, J. Hong, K. Ozeki, T. T. Phuong, and W. V. Vasconcelos. All the results of my lectures are joint works with them. Especially, the recent achievements are strongly inspired by the note [V2] of Wolmer.

The contents of my lectures are the following.

(1) Hilbert coefficients of parameter ideals, Cohen-Macaulay local rings, generalized Cohen-Macaulay rings, Buchsbaum local rings, and utility of homological degrees.
(2) Negativity conjecture of the first Hilbert coefficient of parameter ideals
(3) Finiteness of the set $\Lambda_1(A)$ versus Buchsbaumness and/or generalized Cohen-Macaulayness of local rings
(4) Uniform bounds for the sets $\Lambda_i(A)$ versus generalized Cohen-Macaulayness of local rings
(5) Constancy of the first Hilbert coefficient of parameter ideals with the same integral closure and bounds given by homological degrees

Let me briefly explain the topics in my lectures. The results are directly generalized to a module version, but for simplicity, let me discuss the ring case only. Let $(A, \mathfrak{m})$ be a Noetherian local ring with $d = \dim A > 0$. Let $\ell_A(M)$ denote, for an A-module M, the length of M. Then, for each $\mathfrak{m}$-primary ideal I in A, we have the integers $\{\mathrm{e}_I^i(A)\}_{0 \le i \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+d}{d} - \mathrm{e}_I^1(A)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_I^d(A)$$

holds true for all $n \gg 0$, which we call the Hilbert coefficients of A with respect to I. We say that A is unmixed, if $\dim \widehat{A}/\mathfrak{p} = d$ for every $\mathfrak{p} \in \operatorname{Ass} \widehat{A}$, where $\widehat{A}$ denotes the $\mathfrak{m}$-adic completion of A. With this notation Wolmer V. Vasconcelos posed at the conference in Yokohama of March, 2008 the following conjecture, which is the starting point of our researches.

Conjecture 1 ([GhHV, V1])**.** Assume that A is unmixed. Then A is a Cohen-Macaulay local ring, once $\mathrm{e}_Q^1(A) = 0$ for some parameter ideal Q of A.

In Lecture 2, I shall settle this conjecture affirmatively. As a consequence of the result, one gets that, for a given Noetherian unmixed local ring $(A, \mathfrak{m})$ with $d = \dim A > 0$, $\mathrm{e}^1_Q(A) = 0$ for every parameter ideal Q in A, once $\mathrm{e}^1_Q(A) = 0$ for some parameter ideal Q. The next question is, naturally, when the set

$$\Lambda_1(A) = \{\mathrm{e}^1_Q(A) \mid Q \text{ is a parameter ideal in} A\}$$

is finite, or a singleton. I shall show, in Lectures 3, 4, that the local cohomology modules $\{\mathrm{H}^i_{\mathfrak{m}}(A)\}_{0 \leq i \leq d-1}$ of A with respect to $\mathfrak{m}$ are all finitely generated, if $\Lambda_1(A)$ is finite, and eventually that A is a Buchsbaum local ring if and only if $\Lambda_1(A)$ is a singleton, that is the value $\mathrm{e}^1_Q(A)$ is constant and independent of the choice of parameter ideals Q in A, provided A is unmixed. Since Ghezzi is also going to talk about these results, in my lectures I will give a rather different approach, using homological degrees. The results of Lectures 3, 4 and the existence of uniform bounds for the sets

$$\Lambda_i(A) = \{\mathrm{e}^i_Q(A) \mid Q \text{ is a parameter ideal in } A\}$$

$(1 \leq i \leq d = \dim A)$ are controlled by the uniform bounds for Castelnuovo-Mumford regularity of the associated graded rings of parameter ideals, which I will also discuss in Lecture 4. In Lecture 5, I am interested in the question of whether the first Hilbert coefficients $\mathrm{e}^1_Q(A)$ of parameter ideals Q are constant for the parameter ideals Q with the same integral closure $\overline{Q}$. The answer is yes and no. For these discussions, I need some knowledge about several kinds of local rings and homological degrees, which I shall summarize in Lecture 1.

References

[G] S. Goto, Hilbert coefficients of parameters, Hanoi Lectures.

[GhGHOPV1] Laura Ghezzi, Shiro Goto, Jooyoun Hong, Kazuho Ozeki, Tran Thi Phuong, and Wolmer V. Vasconcelos, *Cohen-Macaulayness versus the vanishing of the first Hilbert coefficient of parameter ideals*, J. London Math. Soc. (to appear).

[GhGHOPV2] Laura Ghezzi, Shiro Goto, Jooyoun Hong, Kazuho Ozeki, Tran Thi Phuong, and Wolmer V. Vasconcelos, *Negativity Conjecture for the First Hilbert Coefficient*, Preprint (2010).

[GhHV] L. Ghezzi, J.-Y. Hong and W. V. Vasconcelos, *The signature of the Chern coefficients of local rings*, Math. Res. Lett. **16** (2009), 279–289.

[GO] S. Goto and K. Ozeki, *Uniform bounds for Hilbert coefficients of parameters*, Preprint (2010).

[V1] W. V. Vasconcelos, *The Chern coefficients of local rings*, Michigan Math. J. **57** (2008), 725–743.

[V2] W. V. Vasconcelos, *Homological degrees and the Chern coefficients of local rings*, Private Correspondence.

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Irena Swanson

Lectures at ICTP, 31 May--4 May 2010

Integral closures of ideals, rings, and modules

I will assume the background from Atiyah--MacDonald (especially the parts on integral closure of rings, primary decomposition of ideals, ring spectra,Hilbert's Basis Theorem, Noether Normalization, completions), and basic factsabout regular sequences (a good reference for this is Kunz's or Matsumura's book). It will be possible to follow the lectures without any background on Groebner bases, but the students may want to read a little about them ahead of time.

Outline of lectures:

- Integral closure of rings and ideals
  (How it arises, monomial ideals and algebras)
- Integral closure of rings
  (Serre's conditions, Jacobian criterion, affine algebras, low dimensions, absolute integral closure)
- Valuation and Krull rings
  (and integral closure of modules, acyclicity criterion)
- Rees algebras in integral closure
  (Reductions, analytic spread, Briancon-Skoda Theorem, Rees valuations)
- Computation of integral closure
  (Old algorithms, as well as recent improvements due to Greuel, Laplagne, Seelisch)

Bernd Ulrich

Lectures at ICTP, 31 May--4 May 2010

Rees algebras of modules from the point of view of equisingularity theory.

The tentative subtitles for the lectures would be:

- Problems from equisingularity theory
- Rees algebras of modules
- Notions of multiplicities for modules
- Multiplicity based criteria for integral dependence
- The Principle of Specialization of Integral Dependence