School and Workshop on
Local Rings and Local Study of Algebraic Varieties
(31 May - 11 June 2010)

WEEK II

ABSTRACTS

The second week (7 - 11 June) will be an International Workshop on Commutative algebra

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SCHOOL and WORKSHOP on LOCAL RINGS
and LOCAL STUDY of ALGEBRAIC VARIETIES
honoring G. Valla
31 May - 11 June 2010
Miramare - Trieste, Italy

Complete Intersections and Rational Normal Scrolls

Lucian Badescu

This is joint work with Giuseppe Valla.

Using the Grothendieck-Lefschetz theory and a generalization (due to Cutkosky) of a result concerning the simple connectedness, we prove that many closed subvarieties of $\mathbb{P}^n$ of dimension ≥ 2 need at least $n-1$ equations to be defined in $\mathbb{P}^n$ set-theoretically, i.e. their arithmetic rank is $\geq n-1$ (Theorem 1 of the Introduction). As applications we give a number of relevant examples. In the second part of the paper we prove that the arithmetic rank of a rational normal scroll of dimension $d \geq 2$ in $\mathbb{P}^N$ is $N-2$, by producing an explicit set of $N-2$ homogeneous equations which define these scrolls set-theoretically (see Theorem 2 of the Introduction).

Stanley decompositions and Hilbert depth in the Koszul complex

Winfried Bruns

We report on joint work with Chr. Krattenthaler and J. Uliczka.

Stanley decompositions of multigraded modules M over polynomials rings have been discussed intensively in recent years. There is a natural notion of depth that goes with a Stanley decomposition, called the *Stanley depth*. Stanley conjectured that the Stanley depth of a module M is always at least the (classical) depth of M. In this paper we introduce a weaker type of decomposition, which we call *Hilbert decomposition*, since it only depends on the Hilbert function of M, and an analogous notion of depth, called *Hilbert depth*. Since Stanley decompositions are Hilbert decompositions, the latter set upper bounds to the existence of Stanley decompositions. The advantage of Hilbert decompositions is that they are easier to find.

We test our new notion on the syzygy modules of the residue class field of $K[X_1, \ldots, X_n]$ (as usual identified with K). Writing $M(n, k)$ for the k-th syzygy module, we show that the Hilbert depth of $M(n, 1)$ is $\lfloor (n+1)/2 \rfloor$. Furthermore, we show that, for $n > k \geq \lfloor n/2 \rfloor$, the Hilbert depth of $M(n, k)$ is equal to $n-1$. We conjecture that the same holds for the Stanley depth. For the range $n/2 > k > 1$, it seems impossible to come up with a compact formula for the Hilbert depth. Instead, we provide very precise asymptotic results as n becomes large.

Finite degree coverings of algebraic varieties

Fabrizio Catanese

While the theory of double covers $X \leftarrow Y$ with Y a smooth algebraic varieties is quite easy , the theory of triple coverings was developed by Miranda , and an easier treatment was given by Casnti and Ekedahl in the relatively Gorenstein case. Still some light can be shed from the theory of dihedral coverings, which extends the theory of cyclic coverings. In degree 4,5 , and under the Gorenstein assumption, there are structures theorem due to Casnati-Ekedahl and Casnati, based on the structure theorems for Gorenstein subschemes of codimension at most 3. Already in the degree 6 case, there are no structure theorems possible in general, but there several constructions given by relatively finite subschemes inside certain bundles over Y. Then deformation theory of the map can be used to prove openness of these constructions under suitable assumptions. The philosophy sounds like having a non irreducible 'moduli space' for these coverings. But a complete study is yet out of reach.

Sharp upper bounds of total betti numbers for given Hilbert polynomials

Giulio Caviglia

This is a joint work with Satoshi Murai.
We show that there exists a saturated graded ideal in a standard graded polynomial ring with largest total Betti numbers among all the graded saturated ideals with fixed Hilbert polynomial.

Cohomology of powers of ideals

Marc Chardin

A beautiful result proved independently by Kodiyalam and by Cutkosky, Herzog and Trung states that the Castelnuovo-Mumford regularity of the m-th power of an homogeneous ideal is eventually a linear function of m. In this talk, I will present an extension of this result that holds in a more general setting -positively graded Noetherian algebras- and gives information on the eventual behaviour of local cohomology modules of powers of a graded ideal. These results imply a conjecture of Ha, as well as results of him and of Eisenbud and Harris. A consequence on the eventual regularity of saturation of powers will also be presented.

Asymptotic behaviour of Castelnuovo-Mumford regularity, Artin-Rees numbers and Hilbert coefficients

Nguyen Tu Cuong and Hoang Le Truong

Let (R, m) be a Noetherian local ring with the maximal ideal m, I and J two ideals of R. In this talk we concentrate on the following two problems:

Problem 1. Is the Castelnuovo-Mumford regularity $reg(G_I(R/J^s))$ of the associated graded ring $reg(G_I(R/J^s))$ bounded above by a linear polynomial on s? Note here that this is just an open question of J. Herzog, L. T. Hoa and N. V. Trung when $I = m$.

Problem 2. Is the Artin-Rees number $ar_I(J^s)$ of $I$with respect to J^s, where $ar_I(J^s)$ the least integer k such that $I^n \cap J^s = I^{n-k}(I^k \cap J^s)$ for all $n \geq k$, bounded above by a linear polynomial on s?

Firstly, we will show that these problems are relative closed together. In fact, we prove that $reg(G_I(R/J^s))$ is bounded above by a polynomial if and only if $ar_I(J^s)$ bounded above by a polynomial of the same degree. Secondly, we give a positive answer for the problem 1 if $I + J$ is m-primary, and therefore the problem 2 is also solved in this case. Finally, as an application, we obtain an affermative answer to an open question of J. Herzog, T. J. Puthenpurakal and J. K. Verma which says that the numerical functions of the Hilbert coefficients $e_i(R/J^s)$ of R/J^s with respect to the maximal ideal m are polynomials for$i = 0, 1, \ldots, dim(R/J)$.

Gröbner bases of cogenerated Pfaffian ideals

Emanuela De Negri

This is a joint work with Enrico Sbarra.

Let $X = (X_{ij})$ be a skew-symmetric $n \times n$ matrix of indeterminates. De Concini, Eisenbud and Procesi proved that the polynomial ring $R := K[X_{ij} : 1 \leq i < j \leq n]$, K being a field, is an Algebra with Straightening Law on the poset of all Pfaffians of X, equipped with a natural partial order. Given a Pfaffian α, the ideal $I_\alpha(X)$ cogenerated by α is the ideal of R generated by the Pfaffians which are not bigger than or equal to α.

The ideal $I_{2r}(X)$ generated by all the Pfaffians of size $2r$ is a particular case of cogenerated ideal. Herzog and Trung proved that the natural generators of $I_{2r}(X)$ form a Gröbner basis with respect to any anti-diagonal term order. In a subsequent remark they ask whether their result can be extended to any cogenerated Pfaffian ideal.

Quite surprisingly the answer is negative. By using a modified version of the KRS correspondence we characterize cogenerated Pfaffian ideals whose natural generators are a Gröbner basis with respect to any anti-diagonal term order. Moreover we describe their initial ideals as well as the associated simplicial complexes, which turn out to be shellable and, thus, Cohen-Macaulay. We also provide a formula for computing their multiplicity.

Hilbert functions of local rings and the classification of local Artin Gorenstein algebras

Joan Elias

In the first part of the talk we review some results about the Hilbert functions and minimal numbers of generators of ideals of Noetherian local rings; we also recall some results on the depth of the blow-up algebras associated to perfect ideals.

In the second part of the talk we report several recent joint papers with G. Valla and M.E. Rossi on the classification of some local Artin Gorenstein algebras with special

Hilbert functions. We present results on the structure of the minimal system of generators of the ideals defining such algebras, the shape of the associated inverse system and the rationality of the Poincaré series.

Smoothability of Artinian Rings

Daniel Erman

An Artinian ring R is said to be "smoothable" if it deforms to a product of fields. Determining if a given Artinian ring is smoothable seems to be quite a difficult problem in general, and I will survey some open questions and known results about smoothability of Artinian rings. Some of the recent results in this talk stem from joint work with Mauricio Velasco.

The first Hilbert coefficient of parameter ideals

Laura Ghezzi

We report on joint work with Goto, Hong, Ozeki, Phuong and Vasconcelos.

Let Q be a parameter ideal in a Noetherian local ring R. The first Hilbert coefficient $e_1(Q)$ codes structural information about the ring itself. At the conference in Yokohama in March 2008 Vasconcelos conjectured that if R is unmixed, then R is Cohen-Macaulay if and only if $e_1(Q) = 0$ for some parameter ideal Q of R. We settle the conjecture affirmatively. We also discuss the invariance of $e_1(Q)$ for parameter ideals Q and its relationship to Buchsbaum rings.

Groebner bases of linked ideals

Elisa Gorla

In this talk, we use liaison theory to obtain Groebner bases results. In recent years, much progress has been made towards understanding the linkage pattern of ideals generated by minors and pfaffians. In particular, for very large families of such ideals, we can produce an explicit series of linkage steps terminating in a complete intersection. By combining the liaison results with a simple Hilbert function computation, we argue that the minors and pfaffians are a Groebner basis for the ideal that they generate.

Invariants and Normal Forms of Singularities in Positive Characteristic

Get-Martin Greuel

Joint with Y. Boubakri and T. Markwig.

We study singularities $f \in K[[x_1, ..., x_n]]$ over an algebraically closed eld K of arbitrary characteristic with respect to right and contact equivalence. We establish improved bounds for the degree of determinacy in positive characteristic. Moreover, we consider different non-degeneracy conditions of Kouchnirenko, Wall and Beelen-Pellikaan in positive characteristic, and we show that Newton non-degenerate singularities satisfy Milnors famous formula (which in general is wrong if $char(K) > 0$).

Furthermore we study piecewise ltrations induced by the Newton polytope of f. In the 1970s Arnold introduced the condition (A), slighty weakend by Wall in 1999, which allows to compute a normal form of a power series f over the complex numbers with respect to right equivalence. We generalise Arnolds and Walls results to arbitrary characteristic and introduce a new condition (AC) for contact equivalence which replaces condition (A). Thus we deduce normal forms and new determinacy bounds for hypersurface singularities with respect to right and contact equivalence.

Finite free resolutions

Juergen Herzog

In this lecture we give a survey of the theory of finite free resolutions. In particular, we discuss upper and lower bounds for the Betti numbers and the interplay of graded Betti numbers with Hilbert functions and multiplicities. We also consider stability properties of Betti numbers with respect to taking generic initial ideals. Finally we outline recent developments in the study of graded Betti numbers related to the Boij-Söderberg theory.

Castelnuovo-Mumford regularity of Ext modules and homological degree

Lê Tuân Hoa

This is a joint work with M. Chardin and D. T. Ha, which will appear in Trans. Amer. Math. Soc.

Let R be a polynomial ring in n variables over a field and M be a finitely generated graded R-module. We are interested in estimating several invariants of M in terms of the degrees in a presentation of M, or in terms of the Castelnuovo-Mumford regularity of M.

The homological degree was introduced by Vasconcelos and his students ten years ago. One of our motivations was to answer positively a question of Vasconcelos on the existence of a polynomial bound on the homological degree of a module in terms of its regularity. In the case of a standard graded algebra A of dimension $d > 0$ with n generators our bound is:

$$\mathrm{hdeg}(A) \leq \binom{\mathrm{reg}(A)+n}{n}^{2^{(d-1)^2}}.$$

We derive this bound from an estimate of the homological degree in terms of the Hilbert polynomial of a module, namely: assume M has dimension $d > 0$, regularity r and Hilbert polynomial P, then

$$\mathrm{hdeg}(M) \leq P(r)^{2^{(d-1)^2}}$$

if M is of positive depth (the general case easily reduces to the result above).

Several estimates are necessary to obtain these results.

One concerns the regularity of the modules $\mathrm{Ext}^i_R(M,R)$ in terms of the regularity and the Hilbert polynomial of M. It is an interesting problem because the regularity of the modules $\mathrm{Ext}^i_R(M,R)$ in some sense controls the behavior of the local cohomology module

$H^i_{\mathfrak{m}}(M)$ in negative components. The bound found earlier by Hyry and the third author for the case M being cyclic is a huge number and its proof required a rather complicated computation. Our bound here works for all modules and is much smaller. Its proof relies on the general estimates on $\mathrm{reg}(\mathrm{Ext}^i_R(M,N))$ for a pair of modules.

Another concerns estimates on the size of the coefficients of the Hilbert polynomial of a module M in terms of its regularity and the degree of its quotient by $\dim M$ general linear forms.

The canonical module of a Cox ring

Kazuhiko Kurano

This is a joint work with M. Hashimoto.

Let X be a normal projective variety such that the divisor class group is finitely generated free abelian group. Then, we can define the Cox ring of X, that is graded by the divisor class group. If the Cox ring is Noetherian, then we can define the canonical module of the Cox ring graded by the divisor class group. Since the Cox ring is factorial (by Elizondo-Kurano-Watanabe), the canonical module is a free module of rank one. We prove that the canonical module of the Cox ring has a generator of degree corresponding to $-K_X$, where K_X is the canonical divisor of X.

The number of defining equations of an algebraic set: a survey

Gennady Lyubeznik

Given an algebraic set V in affine or projective n-space, what is the minimum number of equations that define V set-theoretically? This is an interesting and very difficult question. We will survey what is known and what is not known about this question, including some very recent results.

Finite dimensionality in the categories of motives

Claudio Pedrini

Let k be a field and $DM(k)$ the triangulated category of motives over k with rational coefficients as defined by V.Voevodsky. DM is a pseudoabelian, rigid, tensor category which contains as a full subcategory the "classical" category of Chow Motives $M_{rat}(k)$. An object $M \in DM(k)$ is finite dimensional if $M = M^+ \oplus M^-$ where $\wedge^n M^+ = 0$ and $Sym^n M^- = 0$ for some n. The motives associated to curves and to abelian varieties are finite dimensional . According to a conjecture by S.Kimura all smooth projective varieties should have finite dimensional motives. In the case of surfaces the conjecture implies the famous Bloch's conjecture for complex surfaces of general type with $p_g(X) = 0$, which states the triviality of the Chow group of 0-cycles.

In this talk we will show that the information necessary to study the motive $h(X)$ of a smooth projective surface X in the category of Chow motives is concentrated in the transcendental part of the motive $t_2(X)$ and will prove some results in the case X is a K3 surface.

Hilbert functions

Irena Peeva

This will be a survey talk on Hilbert functions of standard graded algebras.

The singularities of a rational plane curve of even degree

Claudia Polini

This is joint work with Cox, Kustin and Ulrich. We study singularities on rational curves via syzygies.

About Almost Complete Intersections

Alfio Ragusa

TBC

A Life-long Stream of Polynomial Equations

Lorenzo Robbiano

The main purpose of the talk is to illustrate several situations where Valla and I encountered polynomial equations in our research. In particular, I will emphasize that our shared guiding principle was the quest for meaningful polynomial equations. We started together in the seventies, and after about forty years of fruitful work, we are eagerly pursuing the search for still more interesting polynomials. The main themes of the talk are:

- Equations defining tangent cones;
- Polynomial physical laws extracted from experimental data;
- Equations defining border basis schemes.

On implicitization and equations in low dimension

Aron Simis

This is a report on joint work with Hong and Vasconcelos, on one side, and with Buse and Chardin, on the other side. The goal is implicitization via several tools and also a glimpse of the "other" defining equations of the Rees algebra.

Maps on local cohomology induced by Galois extensions of prime characteristic

Anurag K. Singh

Hochster and Huneke proved that the absolute integral closure of an excellent local domain $(R, \mathfrak{m})$ of positive characteristic is a big Cohen-Macaulay algebra. This was refined by Huneke and Lyubeznik, who proved that under mild hypotheses, there exists a finitely generated integral extension S such that the induced map on local cohomology $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(S)$ vanishes for each $i < \dim R$.

It turns out that the extension ring S may be chosen such that it is generically Galois over R; we will discuss the Galois groups that occur for such extensions. This is joint work with Akiyoshi Sannai.

On positivity and vanishing of the coefficients of normal Hilbert polynomial

Jugal Verma

A partial survey of results on the positivity and vanishing of the coefficients of normal Hilbert polynomial will be presented. A solution of Vasconcelos' positivity conjecture for the first coefficient of normal Hilbert polynomial in two dimensional analytically unramified local rings will be sketched.

Some topics on F-thresholds

Kei-ichi Watanabe

This is a joint work with C. Huneke and S. Takagi.

The notion of F-threshold $c^I(\mathfrak{a})$ was introduced by Mustaţă for pairs of ideals in a Noetherian ring of characteristic $p > 0$ with $\mathfrak{a} \subset \sqrt{I}$.

Our main concern with this notion is the following conjecture.

Conjecture 0.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$ of dimension d, J be a parameter ideal of A and $\mathfrak{a}$ be a $\mathfrak{a}$ primary ideal. Then*

$$e(\mathfrak{a}) \geq \left(\frac{d}{c^J(\mathfrak{a})} \right)^d e(J) \ ?$$

where $e(J)$ (resp. $e(\mathfrak{a})$) denotes the multiplicity of J (resp. $\mathfrak{a}$).

This conjecture is true if $A = \oplus_{n \geq 0} A_n$ is a graded ring over Artinian local ring A_0 and both J and $\mathfrak{a}$ are generated by full system of homogeneous parameters.

Also, we discuss when the equality holds in our conjecture. The most desirable result will be the following.

Conjecture 0.2. *Let $(A, \mathfrak{m}), J, \mathfrak{a}$ be as above. If*

$$e(\mathfrak{a}) = \left(\frac{d}{c^J(\mathfrak{a})} \right)^d e(J),$$

then J^n and $\mathfrak{a}^m$ has the same integral closure for some positive integers m, n ?

Recently, the relation of F-threshold with the F-jumping number is found.

Definition 0.3. For every ideal $J \subseteq A$ such that $\mathfrak{a} \subseteq \sqrt{J}$, the *F-jumping number* $\mathrm{fjn}^J(\mathfrak{a})$ of $\mathfrak{a}$ with respect to J is defined to be

$$\mathrm{fjn}^J(\mathfrak{a}) = \inf\{t \geq 0 \mid \tau(\mathfrak{a}^t) \subseteq J\}.$$

Where $\tau(\mathfrak{a}^t)$ is the generalization of test ideal defined by N.Hara and K.-i. Yoshida. This notion is known to have strong connection with multiplier ideals in algebraic varieties over a field of characteristic 0. Then our F-threshold has the following characterization.

Theorem 0.4. *Suppose that A is an equidimensional local ring of characteristic $p > 0$ and J is an ideal generated by a full system of parameters for A. Assume in addition that A is Gorestein and A_P is F-rational for all prime ideals P not containing $\mathfrak{a}$. Then*

$$\mathrm{fjn}^J(\mathfrak{a}) = c^J(\mathfrak{a}).$$

In terms of this characterization, our conjecture on multiplicity and F-thresholds is related to a conjecture on core of ideals.

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Computing the depth of the Rees algebra of a module

Santiago Zarzuela

We determine the depth of the Rees algebra of some ideal modules with small reduction number. For that, we use the theory of reductions of modules and the technique of generic Bourbaki ideals introduced by Aron Simis, Wolmer V. Vasconcelos and Bernd Ulrich. In particular, doing some manipulations on the Rees powers of modules, we deduce a formula for the case of ideal modules with reduction number less or equal 2. This extends to the case of ideal modules previous formulae by Laura Ghezzi for the Rees algebra of ideals.

This is a joint work with Ana Luísa Branco Correia.

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ABOUT ALMOST COMPLETE INTERSECTIONS

ALFIO RAGUSA

ABSTRACT. I will discuss some recent results, obtained with G. Zappalà, about structure theorem, Hilbert functions and graded Betti numbers of Almost Complete Intersections in codimension 3.