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Integral closures of ideals and rings

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I assume some background from Atiyah–MacDonald [2] (especially the parts on Noetherian rings, primary decomposition of ideals, ring spectra, Hilbert's Basis Theorem, completions). In the first lecture I will present the basics of integral closure with very few proofs; the proofs can be found either in Atiyah–MacDonald [2] or in Huneke–Swanson [13]. Much of the rest of the material can be found in Huneke–Swanson [13], but the lectures contain also more recent material.

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1 Integral closure of rings and ideals

(How it arises, monomial ideals and algebras)

Integral closure of a ring in an overring is a generalization of the notion of the algebraic closure of a field in an overfield:

Definition 1.1 *Let R be a ring and S an R-algebra containing R. An element $x \in S$ is said to be* **integral over R** *if there exists an integer n and elements $r_1, \ldots, r_n$ in R such that*

$$x^n + r_1 x^{n-1} + \cdots + r_{n-1}x + r_n = 0.$$

This equation is called an **equation of integral dependence of x over R (of degree n)**.

The set of all elements of S that are integral over R is called **the integral closure of R in S**. *If every element of S is integral over R, we say that S is* **integral over R**.

When S is the total ring of fractions of a reduced ring R, the integral closure of R in S is also called **the integral closure of R**. *A reduced ring R is said to be* **integrally closed** *if the integral closure of R equals R.*

Many facts that are true of the algebraic closure of fields are also true for the integral closure of rings – in the analogous form, of course. The proofs of the following such facts are similar, or at least easy:

Remarks 1.2

(1) The integral closure of a ring in a ring is a ring (even an integrally closed ring).

(2) The integral closure of a ring always contains that ring.

(3) The integral closure of a field in a field is a field, and equals the (relative) algebraic closure of the smaller field in the bigger one.

(4) Warning: The integral closure of a field, when thought of as a ring, is itself, whereas its (absolute) algebraic closure may be a much larger field.

(5) An element $x \in S$ is integral over R if and only if the R-subalgebra $R[x]$ of S is a finitely generated R-module.

(6) Integral closure is a local property: $x \in S$ is integral over R if and only if $\frac{x}{1} \in S_P$ is integral over R_P for all prime (or all maximal) ideals P of R.

(7) It is straightforward to prove that every unique factorization domain is integrally closed. If R is integrally closed and X is a variable over R, then $R[X]$ is integrally closed.

(8) Equations of integral dependence of an element need not be unique, not even if their degrees are minimal possible. For example, let S be $\mathbb{Z}[t]/(t^2-t^3)$, where t is a variable over $\mathbb{Z}$, and let R be the subring of S generated over $\mathbb{Z}$ by t^2. Then $t \in S$ is integral over R and it satisfies two distinct quadratic equations $x^2 - t^2 = 0 = x^2 - xt^2$ in x.

Examples 1.3 Where/how does integral closure of rings arise?

(1) In number theory, a common method for solving a system of equations over the integers is to adjoin to the ring of integers a few "algebraic integers" and then work in the larger ring. For example, one may need to work with the ring $\mathbb{Z}[\sqrt{5}]$. In this ring, factorization of $(1+\sqrt{5})(1-\sqrt{5}) = -2\cdot 2$ is not unique. If, however, we adjoin further "ideal numbers", in Kummer terminology, we get the larger ring $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, in which at least the given product has unique factorization up to associates. Furthermore, $\mathbb{Z}[\frac{1+\sqrt{5}}{2}] \cong \frac{\mathbb{Z}[X]}{(X^2-X-1)}$ is integrally closed, so it is a Dedekind domain, it has unique factorization of ideals, and has unique factorization of elements at least locally.

(2) In complex analytic geometry, for a given variety one may want to know the closure of all rational functions that are bounded on the variety, or at least on some punctured subvariety, in the standard topology. For example, on the curve $y^2 - x^3 - x^2$ in $\mathbb{C}^2$, the rational function $\frac{y}{x}$ defined away from the origin is bounded because $(\frac{y}{x})^2 = x+1$ along the curve. Ring theoretically, the curve $y^2 - x^3 - x^2$ corresponds to the ring $\mathbb{C}[X,Y]/(Y^2 - X^3 - X^2)$, and adjoining $\frac{y}{x}$ corresponds to the ring

$$\frac{\mathbb{C}[X,Y,T]}{(Y^2 - X^3 - X^2, XT - Y, T^2 - X - 1)} \cong \frac{\mathbb{C}[X,T]}{(T^2 - X - 1)} \cong \mathbb{C}[T],$$

which is a regular ring, it is the integral closure of the original ring $\mathbb{C}[X,Y]/(Y^2 - X^3 - X^2)$ (and it is the bounded closure).

(3) The last two examples show that the integral closure of a ring is a **better** ring, sometimes. A one-dimensional Noetherian domain is integrally closed if and only if it is regular. All nonsingular (regular) rings (in algebraic geometry) are integrally closed. A typical desingularization procedure in algebraic geometry uses a combination of blowups and taking the integral closure to get to a regular ring. Thus integral closure is an important part in getting regular rings.

(4) A **monomial algebra** is a subalgebra of a polynomial ring $k[X_1, \ldots, X_n]$ over a field k generated over k by monomials in $X_1, \ldots, X_n$. Let E be the set of exponents that appear in a generating monomial set. It turns out that the integral closure of $k[\underline{X}^{\underline{e}} : e \in E]$ is also a monomial algebra, and it is

$$\overline{k[\underline{X}^{\underline{e}} : e \in E]} = k[\underline{X}^{\underline{e}} : e \in (\mathbb{Z}E) \cap (\mathbb{Q}_{\geq 0}E)].$$

Thus for example, $\overline{k[X^3, X^2Y, Y^3]} = k[X^3, X^2Y, XY^2, Y^3]$, which is a Veronese subvariety of $k[X,Y]$, and $\overline{k[X^3, Y^3]} = k[X^3, Y^3]$, which is a polynomial ring.

I assume the following background from Atiyah–MacDonald [2]:

(1) Lying-Over.

(2) Incomparable.

(3) Going-Up.

(4) If R is an integral domain with field of fractions K, then an element s of a field extension L of K, is integral over R if and only if it is algebraic over K and its minimal (monic) polynomial over K has all its coefficients in the integral closure of R.

(5) Let R be an integral domain, K its field of fractions, and X a variable. Let $f(X)$ be a monic polynomial in $R[X]$, and $g(X), h(X)$ monic polynomials in $K[X]$ such that $f(X) = g(X)h(X)$. Then the coefficients of g and h lie in the integral closure of R.

(6) If $R \subseteq S$ is an integral extension of rings, then $\dim R = \dim S$.

(7) (Going-Down) Let $R \subseteq S$ be an integral extension of rings. Assume that R is an integrally closed domain. Further assume that S is torsion-free over R, i.e., every non-zero element of R is regular on S. Then given a chain of prime ideals $P_1 \subseteq P_2 \subseteq \ldots \subseteq P_n$ of R and a prime ideal Q_n in S such that $P_n = Q_n \cap R$, there exists a chain of prime ideals $Q_1 \subseteq \ldots \subseteq Q_n$ of S such that $Q_i \cap R = P_i$ for all $1 \leq i \leq n$.

(8) (Determinantal trick) Let R be a ring, M a finitely generated R-module, $\varphi : M \to M$ an R-module homomorphism, and I an ideal of R such that $\varphi(M) \subseteq IM$. Then for some r_i in I^i,

$$\varphi^n + r_1\varphi^{n-1} + \cdots + r_n\varphi^0 = 0.$$

In particular, if x is in an extension algebra containing R such that $xM \subseteq M$, then if M is faithful over $R[x]$ it follows that x is integral over R.

(9) (Noether normalization) Let k be a field and R a finitely generated k-algebra. Then there exist elements $x_1, \ldots, x_m \in R$ such that $k[x_1, \ldots, x_m]$ is a transcendental extension of k (i.e., $k[x_1, \ldots, x_m]$ is isomorphic to a polynomial ring in m variables over k) and such that R is integral over $k[x_1, \ldots, x_m]$. If k is infinite, $x_1, \ldots, x_m$ may be taken to be k-linear combinations of elements of a given generating set of R. If R is a domain and the field of fractions of R is separably generated over k, then $x_1, \ldots, x_m$ can be chosen so that the field of fractions of R is separable over $k[x_1, \ldots, x_m]$.

(10) A consequence of Noether normalization is also due to Emmy Noether: The integral closure of a domain R that is finitely generated over a field is module-finite over R. (One can also replace "field" above by "$\mathbb{Z}$".)

(11) (Cohen Structure Theorem, not covered in [2], but general knowledge) Let (R, m) be a complete Noetherian local domain, and k a coefficient ring of R. In case k is a discrete valuation domain of rank one with maximal ideal generated by p, we assume that $p, x_1, \ldots, x_d$ is a system of parameters. If R contains a field, we assume that $x_1, \ldots, x_d$ is a system of parameters. Then the subring $k[[x_1, \ldots, x_d]]$ of R is a regular local ring and R is module-finite over it.

(12) Furthermore, the integral closure of a complete local Noetherian domain R is module-finite over R.

There is also the notion of the integral closure of ideals:

Definition 1.4 *Let I be an ideal in a ring R. An element $r \in R$ is said to be* **integral over I** *if there exist an integer n and elements $a_i \in I^i$, $i = 1, \ldots, n$, such that*

$$r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_{n-1} r + a_n = 0.$$

Such an equation is called **an equation of integral dependence of r over I (of degree n)**.

The set of all elements that are integral over I is called **the integral closure** *of I, and is denoted $\overline{I}$. If $I = \overline{I}$, then I is called* **integrally closed**. *If $I \subseteq J$ are ideals, we say that J is* **integral** *over I if $J \subseteq \overline{I}$.*

If I is an ideal such that for all positive integers n, I^n is integrally closed, then I is called **a normal ideal**.

Remarks 1.5

(1) We will later prove that $\overline{I}$ is an ideal.

(2) $xy \in \overline{(x^2, y^2)}$ because $(xy)^2 + 0 \cdot (xy) - x^2y^2 = 0$. Similarly, for any $i = 0, \ldots, d$, $x^i y^{d-i} \in \overline{(x^d, y^d)}$.

(3) If $I \subseteq J$, then $\overline{I} \subseteq \overline{J}$.

(4) $I \subseteq \overline{I} \subseteq \sqrt{I}$.

(5) Radical ideals, hence prime ideals, are integrally closed.

(6) Intersections of integrally closed ideals are integrally closed.

(7) **Persistence**: if $R \xrightarrow{\varphi} S$ is a ring homomorphism, then $\varphi(\bar{I}) \subseteq \overline{\varphi(I)S}$.

(8) **Contraction**: if $R \xrightarrow{\varphi} S$ is a ring homomorphism and I an integrally closed ideal of S, then $\varphi^{-1}(I)$ is integrally closed in R. (Thus if R is a subring of S, and I an integrally closed ideal of S, then $I \cap R$ is an integrally closed ideal in R.)

(9) **Beware**: The integral closure of the ideal R in the ring R is R, whereas the integral closure of the ring R may be strictly larger.

(10) For any multiplicatively closed subset W of R, $W^{-1}\bar{I} = \overline{W^{-1}I}$.

(11) $I = \bar{I}$ if and only if for all multiplicatively closed subsets W of R, $W^{-1}I = \overline{W^{-1}I}$, which holds if and only if for all prime (resp. maximal) ideals P of R, $I_P = \overline{I_P}$.

(12) $r \in \bar{I}$ if and only if for all multiplicatively closed subsets W of R, $\frac{r}{1} \in \overline{W^{-1}I}$, which holds if and only if for all prime (resp. maximal) ideals P of R, $\frac{r}{1} \in \overline{I_P}$.

(13) The nilradical of the ring is contained in $\bar{I}$ for every ideal I.

(14) **Reduction to reduced rings**: The image of the integral closure of I in R_{red} is the integral closure of the image of I in R_{red}, i.e., $\bar{I}R_{red} = \overline{IR_{red}}$.

(15) **Reduction to domains**: An element $r \in R$ is in the integral closure of I if and only if for every minimal prime ideal P in R, the image of r in R/P is in the integral closure of $(I+P)/P$. (Proof of the harder direction: let U be the subset of S consisting of all elements of the form $\{r^n + r_1 r^{n-1} + \cdots + r_n \,|\, n \in \mathbb{N}_{\geq 0}, r_i \in R\}$. Then U is a multiplicatively closed subset of S that intersects with PS for each $P \in \mathrm{Min}(R)$. If U does not contain 0, then S can be localized at U. If Q is a prime ideal in $U^{-1}S$, let q denote the contraction of Q in R. Since U intersects qS and qS is contained in Q, it follows that Q intersects U, which is a contradiction. Thus $U^{-1}S$ has no prime ideals, which contradicts the assumption that 0 is not in U. So necessarily $0 \in U$, which gives an equation of integral dependence of r over R.)

We have seen how the integral closure of rings arises. Now we address the same for ideals.

(1) Let $R[It]$ be the Rees algebra of the ideal I in R. Its integral closure equals

$$\overline{R} \oplus \overline{I\overline{R}}t \oplus \overline{I^2\overline{R}}t^2 \oplus \overline{I^3\overline{R}}t^3 \oplus \overline{I^4\overline{R}}t^4 \oplus \cdots.$$

To prove this, we need to know that the integral closure of a graded ring in a graded overring (in this case, in $\overline{R}[t]$) is also graded. Refer to the next lecture.

(2) Similarly, the integral closure of the extended Rees algebra $R[It, t^{-1}]$ equals

$$\cdots \oplus \overline{R}t^{-2} \oplus \overline{R}t^{-1} \oplus \overline{R} \oplus \overline{I\overline{R}}t \oplus \overline{I^2\overline{R}}t^2 \oplus \overline{I^3\overline{R}}t^3 \oplus \overline{I^4\overline{R}}t^4 \oplus \cdots.$$

(3) The two preceeding items show that the integral closure of an ideal is an ideal, and even an integrally closed ideal, i.e., that $\bar{\bar{I}} = \bar{I}$.

(4) Let $R \subseteq S$ be rings, with either S integral or faithfully flat over R. Let I be an ideal in R. Then $I \subseteq IS \cap R = \overline{\overline{IS}} \cap R = \bar{I}$.

(5) Let R be an $\mathbb{N}$-graded ring, generated over R_0 by R_1. Assume that R_0 is reduced. Let $F_1, \ldots, F_m$ be homogeneous elements of degree 1 in R. If $\sqrt{(F_1, \ldots, F_m)} = R_1 R$, then $\overline{(F_1, \ldots, F_m)} = R_1 R$.

(6) (Burch [3]) Let (R, m) be a Noetherian local ring that is not regular, i.e., $\mu(m) > \dim R$, and let I be an ideal of finite projective dimension. Then $m(I : m) = mI$ and $I : m$ is integral over I.

(7) (Ratliff [21]) Let R be a locally formally equidimensional Noetherian ring and let $(x_1, \ldots, x_n)$ be a parameter ideal, i.e., the height of $(x_1, \ldots, x_n)$ is at least n. For all $m \geq 1$,

$$(x_1, \ldots, x_{n-1})^m : x_n \subseteq \overline{(x_1, \ldots, x_{n-1})^m} : x_n = \overline{(x_1, \ldots, x_{n-1})^m}.$$

(8) (The Dedekind–Mertens formula) Recall that the **content** $c(f)$ of a polynomial in one variable with coefficients in a ring R is the ideal of R generated by the coefficients of f. If f, g are polynomials in the same variable over R, and if the content of f contains a non-zerodivisor, then $c(fg) \subseteq c(f)c(g)$, and this extension is integral.

(9) (Rees's Theorem) Let (R, m) be a formally equidimensional Noetherian local ring. Let I be an m-primary ideal. The integral closure of I is the largest ideal in R containing I that has the same Hilbert–Samuel multiplicity.

(10) If $I \subseteq J$ and $IJ^n = J^{n+1}$, then J is integral over I. We talk about this criterion in Section 4 (and Bernd Ulrich talked about it in his lectures).

(11) Bernd Ulrich also gave an analytic criterion for integral closure. There is a more general valuative criterion: $\bar{I} = \cap IV \cap R$, where the intersection varies over all valuation domains containing R/P for some prime ideal P. We will look at this in Section 3.

Integral closure of monomial ideals is especially simple and illustrative of the theory in general.

Definition 1.6 *An ideal is said to be* **monomial** *if it is generated by monomials in the polynomial ring $k[X_1, \ldots, X_d]$ (or in the convergent power series ring $\mathbb{C}\{X_1, \ldots, X_d\}$ or in the formal power series ring $k[[X_1, \ldots, X_d]]$), where k is a field, and $X_1, \ldots, X_d$ are variables over k.*

The polynomial ring $k[X_1, \ldots, X_d]$ has a natural $\mathbb{N}^d$ grading with $\deg(X_i) = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^d$ with 1 in the ith spot and 0 elsewhere. Under this grading, monomial ideals are homogeneous, and monomial ideals are the only homogeneous ideals. In the Rees algebra $R[It]$ we then have the natural $\mathbb{N}^{d+1}$-grading, where the last components denotes the t-degree. Assuming for now that the integral closure of $R[It]$ is also $\mathbb{N}^{d+1}$-graded, we get that $\bar{I}$ is monomial, i.e., that the integral closure of a monomial ideal is a monomial ideal. Thus any monomial $\underline{X}^{\underline{e}}$ in the integral closure of a monomial ideal I satisfies an equation of integral dependence of degree m, say, and since the degree $m\underline{e}$-vector subspace of $k[X_1, \ldots, X_n]$ is one-dimensional, this equation of integral dependence is $(\underline{X}^{\underline{e}})^m - a_i(\underline{X}^{\underline{e}})^{m-i}$ for some monomial $a_i \in I^i$, and since the ring is a domain, without loss of generality $i = m$, so the equation of integral dependence is $(\underline{X}^{\underline{e}})^m - a_m = 0$ for some monomial $a_m \in I^m$. In other words, if $\underline{X}^{\underline{b_1}}, \ldots, \underline{X}^{\underline{b_s}} = 0$ generate I, then $\underline{X}^{\underline{e}} \in \bar{I}$ if and only if $\underline{e}$ is componentwise greater than or equal to $\sum_i q_i b_i$ for some non-negative rational numbers q_i. Geometrically, such e is an integer lattice point in the convex hull of the exponent set of the monomial ideal I.

Example 1.7 Let $J = (X^3, Y^3) \subseteq I = (X^3, X^2Y, Y^3) \subseteq \mathbb{C}[X, Y]$. Then $\bar{J} = \bar{I} = (X, Y)^3$.

In general, given an ideal I in a polynomial ring in n variables generated by m generators of degrees at most d, there is a poorly understood upper bound $D = D(n, m, d)$ such that $\bar{I}$ is generated by elements of degree at most D (see Seidenberg [25]). With an a priori upper bound D, the search for the integral closure of elements can be converted to a linear algebra problem (in a high-dimensional vector space, so perhaps this is not a simplification). When restricted to monomial ideals I, D can be taken to be $n + d - 1$, so in this case the linear algebra problem is doable.

Theorem 1.8 (Reid, Roberts and Vitulli [23]) *Let I be a monomial ideal in the polynomial ring $k[X_1, \ldots, X_d]$ such that $I, I^2, \ldots, I^{d-1}$ are integrally closed. Then all the powers of I are integrally closed, i.e., I is normal.*

Proof. Let $n \geq d$. It suffices to prove that I^n is integrally closed under the assumption that $I, I^2, \ldots, I^{n-1}$ are integrally closed. For this it suffices to prove that every monomial $X_1^{e_1} \cdots X_d^{e_d}$ in the integral closure of I^n lies in I^n. Let $\{\underline{X}^{\underline{v}_1}, \ldots, \underline{X}^{\underline{v}_t}\}$ be a monomial generating set of I. By the form of the integral equation of a monomial over a monomial ideal there exist non-negative rational numbers a_i such that $\sum a_i = n$ and the vector $(e_1, \ldots, e_d)$ is componentwise greater than or equal to $\sum a_i v_i$. By Carathéodory's Theorem, by possibly reindexing the generators of I, there exist non-negative rational numbers $b_1, \ldots, b_d$ such that $\sum_{i=1}^d b_i \geq n$ and $(e_1, \ldots, e_d) \geq \sum_{i=1}^d b_i v_i$ (componentwise). As $n \geq d$, there exists $j \in \{1, \ldots, d\}$ such that $b_j \geq 1$. Then $(e_1, \ldots, e_d) - v_j \geq \sum_i (b_i - \delta_{ij}) v_i$ says that the monomial corresponding to the exponent vector $(e_1, \ldots, e_d) - v_j$ is integral over I^{n-1}. Since by assumption I^{n-1} is integrally closed, the monomial corresponding to $(e_1, \ldots, e_d) - v_j$ is in I^{n-1}. Thus $X_1^{e_1} \cdots X_d^{e_d} \in I^{n-1} \underline{X}^{\underline{v}_j} \subseteq I^n$. $\square$

(An easy extension of the proof shows that for all $n \geq d$, $\overline{I^n} = I \overline{I^{n-1}}$.)

In particular, in a polynomial ring in two variables over a field, the power of an integrally closed monomial ideal is integrally closed. (This holds more generally for arbitrary integrally closed ideals in two-dimensional regular rings, by Zariski's theory.)

It is poorly understood which integrally closed monomial ideals in a three-dimensional polynomial ring also have the second power (and hence all powers) integrally closed. Some results were proved by Reid, Roberts, and Vitulli, and more by Coughlin in her Ph.D. thesis at the University of Oregon.

We end this lecture with a direct connection between the integral closure of rings and the integral closure of ideals:

Proposition 1.9 *Let R be a ring, not necessarily Noetherian, and integrally closed in its total ring of fractions. Then for any ideal I and any non-zerodivisor x in R, $\overline{xI} = x \cdot \bar{I}$. In particular, every principal ideal generated by a non-zerodivisor in R is integrally closed.*

Corollary 1.10 *Let R be a Noetherian ring that is integrally closed in its total ring of fractions. The set of associated primes of an arbitrary principal ideal generated by a non-zerodivisor x consists exactly of the set of minimal prime ideals over (x).*

Furthermore, all such associated prime ideals are locally principal.

Proof. All minimal prime ideals over (x) are associated to (x). Let P be a prime ideal associated to xR. By Prime Avoidance there exists a non-zerodivisor y in R such that $P = xR :_R y$. We may localize at P and assume without loss of generality that R is a local ring with maximal ideal P. By definition $\frac{y}{x}P \subseteq R$. If $\frac{y}{x}P \subseteq P$, then by the Determinantal Trick, $\frac{y}{x} \in \overline{R} = R$, so that $y \in xR$ and $P = xR :_R y = R$, which is a contradiction. Thus necessarily $\frac{y}{x}P = R$. Hence there exists $z \in P$ such that $\frac{y}{x}z = 1$. Then $P = xR :_R y = yzR :_R y = zR$, so P is a prime ideal of height 1. Thus P is minimal over xR.

The last statement follows immediately. □

2 Integral closure of rings

(Serre's conditions, Jacobian criterion, affine algebras, low dimensions, absolute integral closure)

A ring R is said to be **normal** if for every prime ideal P of R, R_P is an integrally closed integral domain. Every normal ring is locally an integral domain, thus globally it is reduced. A Noetherian reduced ring is integrally closed if and only if it is normal. We have already proved that the determination of integral closure can be determined modulo all minimal primes, so much of the time we lose no generality by considering only domains.

In the first lecture we relied on the fact that the integral closures of graded rings in graded overrings are also graded. We prove this next.

Theorem 2.1 *Let $G = \mathbb{N}^d \times \mathbb{Z}^e$, and let $R \subseteq S$ be G-graded and not necessarily Noetherian rings. Then the integral closure of R in S is G-graded.*

Proof. This proof is taken from [13]. We first prove the case $d + e = 1$. Let $s = \sum_{j=j_0}^{j_1} s_j$, $s_j \in S_j$, be integral over R. We have to show that each s_j is integral over R.

Let r be an arbitrary unit of R_0. Then the map $\varphi_r : S \to S$ that multiplies elements of S_i by r^i is a graded automorphism of S that restricts to a graded automorphism of R and is identity on S_0. Thus $\varphi_r(s) = \sum_{j=j_0}^{j_1} r^j s_j$ is an element of S that is integral over R.

Assume that R_0 has $n = j_1 - j_0 + 1$ distinct units r_i all of whose differences are also units in R. Define $b_i = \varphi_{r_i}(s)$. Each b_i is integral over R. Let A be the $n \times n$ matrix whose

(i,j) entry is $r_i^{j+j_0-1}$. Then

$$A \begin{bmatrix} s_{j_0} \\ s_{j_0+1} \\ \vdots \\ s_{j_1} \end{bmatrix} = \begin{bmatrix} b_{j_0} \\ b_{j_0+1} \\ \vdots \\ b_{j_1} \end{bmatrix}.$$

As A is a Vandermonde matrix, by the choice of the r_i, A is invertible, so that

$$\begin{bmatrix} s_{j_0} \\ s_{j_0+1} \\ \vdots \\ s_{j_1} \end{bmatrix} = A^{-1} \begin{bmatrix} b_{j_0} \\ b_{j_0+1} \\ \vdots \\ b_{j_1} \end{bmatrix}.$$

Thus each s_j is an R-linear combination of the b_i, whence each s_j is integral over R, as was to be proved.

Finally, we reduce to the case when R_0 has $n = j_1 - j_0 + 1$ distinct units r_i all of whose differences are also units in R. Let $t_{j_0}, \ldots, t_{j_1}$ be variables over R. Define $R' = R[t_j, t_j^{-1}, (t_j - t_i)^{-1} \,|\, i, j = j_0, \ldots, j_1]$ and $S' = S[t_j, t_j^{-1}, (t_j - t_i)^{-1} \,|\, i, j = j_0, \ldots, j_1]$. We extend the G-grading on R and S to R' and S' by setting the degree of each t_i to be 0. Then $R' \subseteq S'$ are G-graded rings, R' contains at least n distinct units $r_i = t_i$ in degree 0 all of whose differences are also units in R'. By the previous case, each $s_j \in S$ is integral over R'. Consider an equation of integral dependence of s_j over R', say of degree n. Clear the denominators in this equation to get an equation E over $R[t_i \,|\, i = j_0, \ldots, j_1]$. (Note that it suffices to clear the denominators by multiplying by powers of $t_i, t_i - t_j$.) The coefficient of s_j^n in E is a polynomial in $R[t_i \,|\, i = j_0, \ldots, j_1]$, with at least one coefficient of this polynomial being a unit of R. Picking out the appropriate multi t_i-degree of E yields an integral equation of s_j over R. Thus s_j is integral over R. This finishes the proof of the case $d + e = 1$.

Now we proceed by induction on $d + e$. Let T be the integral closure of R in S. If $e = 0$ set $G' = \mathbb{N}^{d-1}$ and if $e > 0$ set $G' = \mathbb{N}^d \times \mathbb{Z}^{e-1}$. We impose a G'-grading on $R \subseteq S$ by forgetting about the last component. By induction, $T = \sum_{\nu \in G'} T_\nu$, where T_ν is the homogeneous part of T consisting of elements of degree ν. Now let $s \in T_\nu$. As $s \in S$ and S is G-graded, we may write $s = \sum_{j=j_0}^{j_1} s_j$, where each $s_j \in S_{(\nu,j)}$. Thus by the case $d + e = 1$, each s_j is integral over R. $\square$

In particular, the integral closure of the Rees algebra is as indicated in the first lecture.

Shiro Goto pointed out a shorter proof of Theorem 2.1 in case we accept that the integral closure of of a homogeneous subring of $A[X_1, \ldots, X_n]$ in $B[X_1, \ldots, X_n]$, where the grading is the monomial grading in the variables (and the degrees of elements of A and B are 0). Namely, define $\varphi_R : R \to R[G]$ to be the homomorphism that takes a homogeneous $r \in R$ of degree g to an element $r \cdot g \in R[G]$. Similarly define φ_S. Set $\widetilde{R} = R[G] \subseteq \widetilde{S} = S[G]$. Then $\widetilde{R}$ and $\widetilde{S}$ are localizations of polynomial rings at some variables, and in $\widetilde{R}$ and $\widetilde{S}$ we treat the degrees of elements of S and R to be 0. Thus by assumption the integral closure of $R[G]$ in $S[G]$ is homogeneous under the polynomial ordering. Now let $s \in S$ be integral over R. Then $\varphi(s)$ is integral over $R[G]$, so that all the (polynomial) components of $\varphi(s)$ are integral over $R[G]$, whence by writing out the equations we get that the homogeneous components of s are integral over R.

Example 2.2 It need not be the case that the integral closure of a reduced $(\mathbb{N}^d \times \mathbb{Z}^e)$-graded ring is graded. Let $R = \mathbb{Q}[X,Y]/(XY)$, with X, Y variables over $\mathbb{Q}$. We can impose any of the following gradings on R:
(1) $\mathbb{N}$-grading $\deg X = 0$, $\deg Y = 1$,
(2) $\mathbb{Z}$-grading $\deg X = 1$, $\deg Y = -1$,
(3) $\mathbb{N}^2$-grading $\deg X = (1,0)$, $\deg Y = (0,1)$.
Namely, the element $X/(X+Y)$ of K it satisfies the integral equation $T^2 - T = 0$, it is idempotent, but cannot be written as a fraction of homogeneous components under the given gradings.

Theorem 2.3 (Huneke–Swanson [13], Chapter 2) *Let $G = \mathbb{N}^d \times \mathbb{Z}^e$, let R be a G-graded reduced Noetherian ring, K its total ring of fractions, and* $\mathrm{Min}(R) = \{P_1, \ldots, P_s\}$. *Let S be the localization of R at the set of all homogeneous non-zerodivisors of R. The following are equivalent:*
(1) The ring S is integrally closed.
(2) The integral closure $\overline{R}$ of R is a G-graded subring of S (inheriting the grading).
(3) The idempotents of $\overline{R}$ are homogeneous elements of S of degree 0.
(4) For $i = 1, \ldots, s$, $P_i + \cap_{j \neq i} P_j$ contains a homogeneous non-zerodivisor. (In case $s = 1$, this condition is vacuously satisfied.)

Corollary 2.4 *Let R be a reduced $\mathbb{N}^d$-graded ring, possibly non-Noetherian, such that the non-zero elements of R_0 are non-zerodivisors in R. Then the integral closure of R is $\mathbb{N}^d$-graded.*

In particular, a monomial algebra has the integral closure that is also a monomial algebra.

This finishes the proofs of the graded bits that were used in the first lecture.

Now we switch themes and consider **rings of homomorphisms**. For this part we assume that all rings are domains. In this case for any non-zero ideals I and J in a ring R, $\mathrm{Hom}_R(I,J)$ is multiplication by an element of the field of fractions K: this is certainly so after inverting all the non-zero elements of R. Moreover, the natural map $(J :_K I) \longrightarrow \mathrm{Hom}_R(I,J)$ is an isomorphism. Computationally, $J :_K I$ requires knowing all fractions, which is not easily doable, and $\mathrm{Hom}_R(I,J)$ in general is returned as an R-module that may not be easily understood as a submodule of K. But the following trick solves both problems nicely: for any non-zero $x \in I$, we have natural identifications

$$(J :_K I) \cong \mathrm{Hom}_R(I,J) \cong \frac{1}{x}(xJ :_R I).$$

This characterization makes it clear how $\mathrm{Hom}_R(I,I)$ is a subring of K.

Definition 2.5 *If R is a domain with field of fractions K, for any non-zero (fractional) ideal I, define $I^{-1} = \mathrm{Hom}_R(I, R)$.*

Note that $I \subseteq (I^{-1})^{-1}$ and that $II^{-1} = I^{-1}I \subseteq R$.

Proposition 2.6 *Let R be a Noetherian domain. Then*

$$\overline{R} = \bigcup_I \mathrm{Hom}_R(I, I) = \bigcup_I \mathrm{Hom}_R(I^{-1}, I^{-1}),$$

where I varies over non-zero (finitely generated fractional) ideals.

Proof. By the Determinantal trick, $\mathrm{Hom}_R(I^{-1}, I^{-1}), \mathrm{Hom}_R(I, I) \subseteq \overline{R}$. If $s \in \overline{R} \setminus R$, then with $J = R :_R R[s]$, we have that J is a non-zero finitely generated ideal in R with $sJ \subseteq J$, so that an arbitrary $s \in \overline{R}$ is in some $\mathrm{Hom}_R(I, I)$. The rest is similar. $\square$

In particular, R is integrally closed if and only if $\mathrm{Hom}_R(I, I) = R$ for all non-zero (finitely generated fractional) ideals I. We will see in lecture 5 an effective criterion: at least for finitely generated algebras over a perfect field there exists a computable ideal J such that R is integrally closed if and only if $\mathrm{Hom}_R(J, J) = R$.

Now we switch to Jacobian ideals.

Definition 2.7 *Let A be a universally catenary ring and R a localization of a finitely generated A-algebra. Write R as $W^{-1}A[\underline{X}]/(f_1, \ldots, f_m)$, where $A[\underline{X}] = A[X_1, \ldots, X_n]$, $X_1, \ldots, X_n$ are variables over A, $f_i \in A[\underline{X}]$, and W is a multiplicatively closed subset of $A[\underline{X}]$. A* **Jacobian matrix** *of R over A is defined as the $m \times n$ matrix whose (i, j) entry is $\frac{\partial f_i}{\partial X_j}$. Assume furthermore that there exists a non-negative integer h such that for each prime ideal P in $A[\underline{X}]$ that is minimal over $(f_1, \ldots, f_m)$ and such that $P \cap W = \emptyset$, $A[\underline{X}]_P$ is equidimensional of dimension h. Observe that this set of prime ideals is in one-to-one correspondence with the minimal primes of R. Under these conditions, the* **Jacobian ideal** *of R over A, denoted $J_{R/A}$, is the ideal in R generated by all the $h \times h$ minors of the Jacobian matrix of R over A.*

It is a fact that the Jacobian ideal $J_{R/A}$ is independent of the choice of the generators X_i and the relations f_j (read for example [13, Chapter 4]), and furthermore that it contains the regularity and the normality conditions:

Theorem 2.8 (Jacobian criterion) *Let k be a field and R an equidimensional finitely generated k-algebra. Let J be the Jacobian ideal $J_{R/k}$. Let P be a prime ideal in R. If J is not contained in P, then R_P is a regular ring.*

Conversely, if R_P is a regular ring and $\kappa(P)$ is separable over k (say if k is a perfect field), then J is not contained in P.

Before we can state the normality condition, we need three definitions (and a fourth definition is tacked on):

Definition 2.9 *A Noetherian ring R is satisfies* **Serre's condition (R_k)** *if for all prime ideals P in R of height at most k, R_P is a regular local ring.*

Also, R satisfies **Serre's condition (S_k)** *if for all prime ideals P in R, the depth of R_P is at least $\min\{k, \operatorname{ht} P\}$.*

The **singular locus** *of R is the set of all $P \in \operatorname{Spec} R$ such that R_P is not regular.*

The **non-normal locus** *of R is the set of all $P \in \operatorname{Spec} R$ such that R_P is not normal.*

Theorem 2.10 (Serre's conditions) *A Noetherian ring R is normal if and only if it satisfies Serre's conditions (R_1) and (S_2).*

If $V(J)$ is the singular locus of R, then J has grade at least 2 if and only if R satisfies (R_1) and (S_2).

Thus we have an effective computational criterion for deciding if an affine domain is integrally closed. (It is much harder to find elements in the integral closure that are not in the ring when the ring is not integrally closed.)

In order to find/compute the integral closure of a ring, there needs to be some kind of algorithmic procedure, and actually first of all, there needs to be some finiteness condition on the integral closure. We have seen that affine domains have module-finite integral closures, and actually the computation of those integral closures is doable.

It is not true in general that the integral closure of a Noetherian domain is a module-finite extension, even for local Noetherian domains of dimension one! However, the integral closure of a Noetherian domain of dimension at most 2 is still Noetherian. The one-dimensional version is called the Krull–Akizuki Theorem, and the two-dimensional version is due to Nagata (and uses the fact that the integral closure of a Noetherian domain is a Krull domain). Nagata also showed that there exists a three-dimensional Noetherian local domain whose integral closure is not Noetherian.

At the end of this lecture I want to introduce the "largest" integral extension of a domain.

Definition 2.11 *Let R be a domain with field of fractions K. Let $\overline{K}$ be an algebraic closure of K.* **The absolute integral closure** *of R is the integral closure R^+ of R in $\overline{K}$.*

It is straightforward to prove that every monic polynomial in one variable over R^+ factors into linear factors in R^+.

The ring R^+ is unusual: Michael Artin in [1] proved that the sum of a collection of prime ideals is either the whole ring or a prime ideal! The following proof is from Hochster and Huneke [11]. The proof reduces at once to a finite family of primes, and then by induction to the case of two primes, P and P'. Suppose that $xy \in P + P'$. Let $z = y - x$, so that $x^2 + zx = a + b$ with $a \in P$ and $b \in P'$. The equation $T^2 + zT = a$ has a solution $t \in R$, and since $t(t+z) \in P$, either $t \in P$ or $t + z \in P$. Now $x^2 + zx = t^2 + zt + b$, and so $(x-t)(x+t+z) = b \in P'$, so that either $x - t \in P'$ or $x + t + z \in P'$. Since

$x = (x - t) + t = (x + t + z) - (t + z)$ and $x + z = (x - t) + (t + z) = (x + t + z) - t$, we see that in all four cases, either $x \in P + P'$ or $y = x + z \in P + P'$, as required. $\square$

3 Valuation rings, Krull rings, and Rees valuations

Atiyah–Macdonald [2] contains the basic results about valuations:

(1) A **valuation** on a field K (or a ***K*-valuation**) is a group homomorphism v from the multiplicative group $K^* = K \setminus \{0\}$ to a totally ordered abelian group G (written additively) such that for all x and y in K,

$$v(x + y) \geq \min\{v(x), v(y)\}.$$

By abuse of notation, we sometimes call a valuation such a function extended to all of K by declaring $v(0) = \infty$; domain we call a function K-valuation if it is defined on a domain R whose field of fractions is K.

(2) If v is a valuation, $v(\sum_{i=1}^{n} x_i) \geq \min\{v(x_1), \ldots, v(x_n)\}$. If $v(x_i)$ are all distinct, then $v(\sum_{i=1}^{n} x_i) = \min\{v(x_i)\}$.

(3) Valuations $v : K^* \to G_v$ and $w : K^* \to G_w$ are **equivalent** if there exists an order-preserving isomorphism $\varphi : \text{image}(v) \to \textit{image}(w)$ such that for all $\alpha \in K^*$, $\varphi(v(\alpha)) = \ w(\alpha)$.

(4) For a K-valuation v, the image $\Gamma_v = v(K^*)$ of v is a totally ordered abelian group, called the **value group** of v.

(5) A ***K*-valuation ring**, or simply a **valuation ring** or a **valuation domain**, is an integral domain V whose field of fractions is K that satisfies the property that for every non-zero element $x \in K$, either $x \in V$ or $x^{-1} \in V$.

(6) In a valuation domain V all ideals are ordered by inclusion. The unique maximal ideal is usually denoted m_V. Every finitely generated ideal is principal. A Noetherian valuation domain is a principal ideal domain.

(7) In fact, a local domain (R, m) that is not a field is a Noetherian valuation domain if and only if it is a principal ideal domain; which holds if and only if R is Noetherian and the maximal ideal m is principal; which holds if and only if R is Noetherian and there is no ring properly between R and K; which holds if and only if R is Noetherian, one-dimensional, and integrally closed; etc.

(8) Given a valuation $v : K^* \to G$,

$$R_v = \{r \in K^* \mid v(r) \geq 0\} \cup \{0\}$$

is a K-valuation ring. For equivalent valuations v and w, $R_v = R_w$.

(9) Given a K-valuation domain, let $\Gamma_V = K^*/V^*$, where $V^* \subseteq K^*$ are the multiplicative groups of units, and let $v : K^* \to \Gamma_V$ be the natural group homomorphism. Then Γ_V is a totally ordered abelian group, v is a K-valuation, and Γ_V is the value group of v.

(10) The previous two parts give a natural one-to-one correspondence between K-valuation rings and equivalence classes of K-valuations.
(11) A valuation domain V is integrally closed.
(12) Every ring between a K-valuation domain and K is also a valuation domain.
(13) It is straightforward to check that the intersection of a K-valuation domain with a subfield of F of K yields an F-valuation domain, and that moreover the intersection of a Noetherian K-valuation domain with a subfield of F of K yields a Noetherian F-valuation domain.
(14) (Existence of valuation domains) Let P be a non-zero prime ideal in an integral domain R. Then there exists a valuation domain V between R and the field of fractions of R such that $m_V \cap R = P$.
(15) A consequence is that for every domain R,

$$\overline{R} = \cap_V V,$$

where V varies over the valuation domains with field of fractions equals to the field of fractions of R. This appears in [2], and we give another proof after Proposition 3.2.

In the sequel we relate valuations also to integral closure of ideals.

Proposition 3.1 *Let R be an integral domain contained in a valuation ring V. Then for any ideal I of R, $IV = \bar{I}V = \overline{IV}$. Equivalently, if v is a valuation that is non-negative on R, then $v(I) = v(\bar{I})$. (Here v of a set is defined to be the minimum v-value of an element of the set.)*

Proof. As $I \subseteq \bar{I}$, it follows that $IV \subseteq \bar{I}V$, and by persistence of integral closure, $\bar{I}V \subseteq \overline{IV}$. Now let $r \in \overline{IV}$. Let $r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$ be an equation of integral dependence of r over IV, with each $a_i \in I^i V$. There is a finitely generated ideal J contained in I such that $a_i \in J^i V$, $i = 1, \ldots, n$. Thus there exists $j \in J$ such that $JV = jV$, and so r satisfies an equation of integral dependence of degree n over jV. By Proposition 1.9, $r \in jV = JV \subseteq IV$, which proves that $\overline{IV} \subseteq IV$. □

Proposition 3.2 (Valuative criterion) *Let R be an integral domain, not necessarily Noetherian, and let I be an ideal in R. Then*

$$\bar{I} = \bigcap_V IV \cap R,$$

where V varies over all valuation domains of the field of fractions K of R that contain R.

Proof. By the previous proposition, $\bar{I} \subseteq \bigcap_V \bar{I}V \cap R = \bigcap_V IV \cap R$. Now let r be a non-zero element of $\bigcap_V IV \cap R$, and let S be the ring $R[\frac{I}{r}]$. For all valuation rings V between S and K, $r \in IV$, so that for each such V, the ideal $\frac{I}{r}S$ of S extends to the unit ideal in V. By the existence of valuation overrings then $\frac{I}{r}S = S$, so that $1 = \sum_{i=1}^n \frac{a_i}{r^i}$ for some a_i in I^i. Multiplying this equation through by r^n yields an equation of integral dependence of r over I of degree n, so that r is integral over I. □

Corollary 3.3 *Let R be a domain. Then $\overline{R} = \cap V$, where V varies over all valuation domains of the field of fractions K of R that contain R.*

Proof. Clearly $\overline{R} \subseteq \cap V$. Now let $x \in \cap V$. Write $x = r/s$ for some $r, s \in R$, $s \neq 0$. Then $r \in \cap sV$, so by the previous proposition, $r \in \overline{(s)}$. An equation of integral dependence is of the form $r^n + a_1 s r^{n-1} + a_2 s^2 r^{n-2} + \cdots + a_n s^n = 0$ for some $a_i \in R$. By dividing the equation by s^n we get an equation of integral dependence of $x = r/s$ over R, so that $x \in \overline{R}$. $\square$

When R is Noetherian, in the proposition above we may restrict the V to Noetherian valuation domains by a slight modification of the proof and by the following result:

Theorem 3.4 (Existence of Noetherian valuation domains) *Let P be a non-zero prime ideal in a Noetherian domain R. Then there exists a Noetherian valuation domain V with field of fractions equal to $Q(R)$ such that $m_V \cap R = P$.*

Proof. Without loss of generality P is the unique maximal ideal of R. If every element of $P/P^2 \subseteq G = \mathrm{gr}_P(R)$ is nilpotent, then $\dim G = 0$, and it follows that $\dim R = 0$. Thus R must be a field, contradicting the assumption that R has a non-zero prime ideal. Thus there exists $x \in P \setminus P^2$ whose image in P/P^2 is not nilpotent in G.

Set $S = R[\frac{P}{x}]$. This is a Noetherian ring. If $xS = S$, we can write $1 = x \sum_{i=0}^{n} \frac{a_i}{x^i} = \frac{a}{x^{n-1}}$ for some $a_i \in P^i$, $a \in P^n$. Then $x^{n-1} \in P^n$, contradicting the choice of x. Thus $xS = PS$ is a proper ideal in S.

Any prime ideal Q minimal over xS has height 1 (by Krull's Height Theorem). By Lying-Over, there exists a maximal ideal M in the integral closure of S_Q that contracts to Q. By the Krull–Akizuki Theorem, $\overline{S_Q}$ is one-dimensional, Noetherian, and integrally closed, hence locally at M a Noetherian valuation domain. This is the valuation domain that we want. $\square$

In case the ring is a polynomial ring and the ideal is a monomial ideal, we may restrict the necessary valuations (as in Proposition 3.2) to **monomial valuations**, which are those valuations for which the value on any polynomial is the minimum of all values of the monomials appearing with non-zero coefficient.

The combination of Propositions 3.1 and 3.2 shows that the integral closure of an ideal is an integrally closed ideal and that $\bar{\bar{I}} = \bar{I}$ for all ideals I.

With using the valuative criterion for integral closure makes the proofs of the following results easy:

(1) For any ideals I and J in a ring, $\bar{I} \cdot \bar{J} \subseteq \overline{IJ} = \overline{\bar{I}\bar{J}}$.

(2) If $I = (a_1, \ldots, a_d) \neq 0$, then for any $n \in \mathbb{N}$, $\overline{JI^n} : I^n = \cap_i (\overline{JI^n} : a_i^n) = \bar{J}$.

(3) Let v be a valuation on a field of fractions of R such that v is non-negative on R. Let γ be an element of the value group of v. Then $I_\gamma = \{r \in R \,|\, v(r) \geq \gamma\}$ is integrally closed in R.

(4) (Cancellation theorem) Let I, J and K be ideals in a Noetherian ring R, I not consisting of zero divisors, or more generally the height of I is positive. If $\overline{IJ} = \overline{IK}$, then $\overline{J} = \overline{K}$.

Now we switch gears and pass to Krull domains.

Definition 3.5 *A not necessarily Noetherian integral domain R is a* **Krull domain** *if*

(1) for every prime ideal P of R of height one, R_P is a Noetherian integrally closed domain,

(2) $R = \cap_{\mathrm{ht}(P)=1} R_P$, and

(3) every non-zero $x \in R$ lies in at most finitely many prime ideals of R of height one.

Remarks 3.6

(1) Krull domains are intersections of integrally closed domains, hence are integrally closed.

(2) In Krull domains all principal ideals have a primary decomposition: If $P_1, \ldots, P_s$ are all the prime ideals in R of height one containing $x \in R$, then $xR = \cap_i (xR_{P_i} \cap R)$ is a minimal primary decomposition of xR.

(3) An integrally closed Noetherian domain is Krull.

A much stronger result is the Mori–Nagata Theorem saying that the integral closure of a Noetherian domain is a Krull domain. All proofs are fairly long, so we omit them here (possibly see [13, Chapter 4]). One can then prove that a Krull domain is a Dedekind domain if and only if it is a principal ideal domain after localization at each maximal prime ideal. A polynomial or a power series ring over a Krull domain is a Krull domain. Any unique factorization domain is a Krull domain.

Now we connect valuations and Krull domains.

For that, let R be a Noetherian domain and let I be a non-zero ideal in R. We already know that $\overline{I} = \cap_V IV \cap R$, as V varies over all valuation (Noetherian) domains with the same field of fractions as R. If R/I is Artinian, certainly finitely many V suffice. David Rees proved that finitely many suffice for any I. Furthermore, he proved that there exist valuation domains $V_1, \ldots, V_s$ such that for all n, $\overline{I^n} = \cap_i (I^n V_i) \cap R$. A minimal set of such valuation rings is called **the set of Rees valuation rings** of I. Rees proved that it always exists and is unique for non-zero I. (Such a result holds for all Noetherian rings, not necessarily domains, but in the general case I should not be contained in any minimal prime ideal for the uniqueness conclusion.) The construction goes as follows. Let $S = R[It, t^{-1}]$. Then S is a Noetherian domain, the integral closure $\overline{S}$ of S is a Krull domain, and there are only finitely many prime ideals $P_1, \ldots, P_m$ that are minimal over $(t^{-1})\overline{S}$. All of these prime ideals have height one, $\overline{S}_{P_i}$ is a Noetherian valuation domain, and $V_i \cap QF(R)$ is a Noetherian valuation domain on the field of fractions of R that contains R. Then for all n, $t^{-n}\overline{R[It, t^{-1}]} = \cap t^{-n} V_i \cap \overline{R[It, t^{-1}]}$, so that by what we have seen so far,

$$\overline{I^n} \subseteq \cap I^n (V_i \cap QF(R)) \cap R \subseteq \cap I^n V_i \cap R \subseteq \cap t^{-n} V_i \cap R = t^{-n}\overline{R[It, t^{-1}]} \cap R = \overline{I^n},$$

so that equality holds throughout. It takes a little bit of work to prove that no $V_1, \ldots, V_m$ is redundant.

In general, if R is not a domain, for any ideal I in R that is not contained in any minimal prime ideal of R, the set of all valuation rings obtained in this way on each R/P as P varies over the minimal prime ideals of R is the unique minimal set of valuation rings that determine the integral closure of all powers of I. Such a minimal set is called the **set of Rees valuation rings of** I, and the corresponding set of normalized valuations, i.e., their value field is $\mathbb{Z}$, is called **set of Rees valuations of** I.

Proposition 3.7 (Hübl–Swanson [12]) *Let I be an ideal in an integrally closed Noetherian local domain R.*

(1) $\operatorname{gr}_I(R)$ *is reduced if and only if I is a normal ideal and if for each (normalized integer-valued) Rees valuation v of I, $v(I) = 1$.*

(2) The ring $R/\bar{I} \oplus \bar{I}/\overline{I^2} \oplus \overline{I^2}/\overline{I^3} \oplus \cdots$ is reduced if and only if for each (normalized) Rees valuation v of I, $v(I) = 1$.

Proof of Part (i): If $\operatorname{gr}_I(R)$ is reduced, let $x \in I^n \setminus I^{n+1}$. If $x \in \overline{I^m}$ for some $m > n$, the equation of integral dependence shows that x is nilpotent in $\operatorname{gr}_I(R)$, which is a contradiction. So all powers of I are integrally closed. Then $\overline{R[It, t^{-1}]} = R[It, t^{-1}]$, and $R[It, t^{-1}]/(t^{-1}) = \operatorname{gr}_I(R) = R[It]/IR[It]$, so that $IR[It]$ is a radical ideal. By the definition of Rees valuations via the extended Rees algebra as above, each Rees valuation v of I corresponds to a prime ideal minimal over $t^{-1}R[It, t^{-1}]$, hence to a minimal prime ideal in $\operatorname{gr}_I(R)$, hence to a prime ideal in $R[It]$ minimal over $IR[It]$. As $\operatorname{gr}_I(R)$ is reduced, $IR[It]$ equals locally each of those prime ideals, which says that for all Rees valuations v of I, $v(I) = 1$. Conversely, if $x \in I^n/I^{n+1}$ is non-zero and nilpotent in $\operatorname{gr}_I(R)$, then $x^k \in I^{nk+1}$ for some k, whence for all Rees valuations v of I, $kv(x) \geq (nk+1)v(I) = nk + 1$, whence $v(x) \geq n + 1/k$, and since $v(x)$ is an integer, $v(x) \geq n + 1$. But then $x \in \overline{I^{n+1}} = I^{n+1}$, which is a contradiction. $\square$

We present another application of Rees valuations here.

Definition 3.8 *Let I be an ideal in a Noetherian ring R. The function* $\operatorname{ord}_I : R \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ *defined by* $\operatorname{ord}_I(r) = \sup\{m \mid r \in I^m\}$ *is called the* **order** *of I.*

In general ord_I is not a valuation. When it is, it is called the I**-adic valuation**. For example, if R is a regular ring and is m a non-zero maximal ideal, then the order function relative to m is a discrete valuation of rank one and the residue field of the corresponding valuation ring is purely transcendental over R/m of transcendence degree $\dim R - 1$. Explicitly, the m-adic valuation ring equals $(R[\frac{m}{x}])_{(x)}$ for any $x \in m \setminus m^2$.

If $\cap_{n \geq 0} I^n = 0$, then the associated graded ring $\operatorname{gr}_I(R)$ is an integral domain if and only if the order function ord_I yields a discrete valuation of rank one.

The Rees valuations yield the following:

Theorem 3.9 (Rees [22]) *Let I be an ideal in a Noetherian ring R, $r \in R \setminus \{0\}$, $c \in \mathbb{N}$. Then $r \in \overline{I^c}$ if and only if* $\limsup_{m\to\infty} \frac{\mathrm{ord}_I(r^m)}{m} \geq c$.

Furthermore, for any $r \in R$,

$$\lim_{n\to\infty} \frac{\mathrm{ord}_I(r^n)}{n} = \min\left\{\frac{v(r)}{v(I)} : v \textit{ varies over the Rees valuations of } I\right\},$$

so that the limit (lim sup) exists and is a rational number.

We now present an alternative construction of Rees valuations, again if R is a domain. If $I = (a_1, \ldots, a_l)$, take the set of Noetherian valuation rings that are localization of the integral closure of $R[\frac{I}{a_i}]$ at the height one prime ideals minimal over a_i, as i varies from 1 to l. This gives Rees valuations. The advantage of this construction is that everything is done in the field of fractions of R (as opposed to in the field of fractions of $R[t]$ as in the method using the extended Rees algebras), but the disadvantage is that we need to compute l integral closures. Note that the different $R[\frac{I}{a_i}]$ may yield some of the same valuation rings. Note that if $\bar{I} = \bar{J}$, the set of Rees valuations of I is the same as the set of Rees valuations of J, so we may want to look for ideals J with $\bar{I} = \bar{J}$ and with the minimal number of generators possible. We do that in the next lecture (and Bernd Ulrich covered minimal reductions in his lectures). It is a fact that at least when the residue field of the ring is infinite, if a is a sufficiently general element of I, then $v(a) = v(I)$ for all Rees valuations v of I, and in that case all Rees valuations of I may be obtained by localizing the integral closure of $R[\frac{I}{a}]$ at the height one prime ideals minimal over a. So in this case we only need to compute one integral closure inside the field of fractions of R. The problem is in general that we may not know a priori what the general element a is. The following result gives a definite method for having to find only one integral closure of a ring with the same field of fractions that gives all the Rees valuations of I:

Theorem 3.10 (Sally [24, page 438]) *Let (R, m) be a Noetherian formally equidimensional local domain of dimension $d > 0$, and I an m-primary ideal satisfying $\mu(I) = d$. Let $I = (a_1, \ldots, a_d)$. Then for every Rees valuation ring V of I and every $i = 1, \ldots, d$, V is the localization of the normalization of $R[\frac{I}{a_i}]$ at a height one prime ideal minimal over a_i.*

Note that if $R = k[X_1, \ldots, X_d]$, a polynomial ring over a field k, then there is only one Rees valuation of $(X_1, \ldots, X_d)$, and its ring is

$$k\left[X_1, \frac{X_2}{X_1}, \ldots, \frac{X_d}{X_1}\right]_{X_1 k\left[X_1, \frac{X_2}{X_1}, \ldots, \frac{X_d}{X_1}\right]}.$$

We already saw that if an ideal I has only one Rees valuation, then the I-adic order is a valuation. Sally [24] proved that if (R, m) is analytically unramified Noetherian, and if there exists an m-primary ideal I in R with only one Rees valuation, then the m-adic completion of R is an integral domain. (Katz [14]) proved that if (R, m) is a formally equidimensional Noetherian local domain, then the number of Rees valuations of an m-primary ideal is bounded below by the number of minimal prime ideals in the completion of R. In particular, if R contains an m-primary ideal with only one Rees valuation, then $\widehat{R}$ has only one minimal prime ideals. Many proofs in the literature can be simplified if the local ring contains a zero-dimensional ideal with only one Rees valuation, but Cutkosky [5] proved that there exists a two-dimensional complete Noetherian local integrally closed domain in which no zero-dimensional ideal has only one Rees valuation.

4 Rees algebras and integral closure

(Reductions, analytic spread, Briancon-Skoda Theorem)

We have seen Rees algebras: $R[It]$ and $R[It, t^{-1}]$, and we have determined that

$$\overline{R[It]} = \overline{R} \oplus \overline{I\overline{R}}t \oplus \overline{I^2\overline{R}}t^2 \oplus \overline{I^3\overline{R}}t^3 \oplus \overline{I^4\overline{R}}t^4 \oplus \cdots,$$

and that

$$\overline{R[It, t^{-1}]} = \cdots \oplus \overline{R}t^{-2} \oplus \overline{R}t^{-1} \oplus \overline{R} \oplus \overline{I\overline{R}}t \oplus \overline{I^2\overline{R}}t^2 \oplus \overline{I^3\overline{R}}t^3 \oplus \overline{I^4\overline{R}}t^4 \oplus \cdots.$$

We have used them to prove that the integral closure of an ideal is an integrally closed ideal, and that if I is a homogeneous ideal in a graded ring R, then $\bar{I}$ is homogeneous as well under the same grading. We have also seen extended Rees algebras in the quick outline of Rees valuations. An important relevant property of these algebras is the following:

$$t^{-n}R[It, t^{-1}] \cap R = I^n, \quad \overline{t^{-n}\overline{R[It, t^{-1}]}} \cap R = t^{-n}\overline{R[It, t^{-1}]} \cap R = \overline{I^n}$$

for all n. Thus many problems can be reduced to knowing the results for principal ideals generated by non-zerodivisors.

There are four closely related rings:

(1) The integral closure of $R[It]$ in $R[t]$ equals the graded ring

$$R \oplus \bar{I}t \oplus \overline{I^2}t^2 \oplus \overline{I^3}t^3 \oplus \cdots.$$

(2) The integral closure of $R[It, t^{-1}]$ in $R[t, t^{-1}]$ equals the graded ring

$$\cdots \oplus Rt^{-2} \oplus Rt^{-1} \oplus R \oplus \bar{I}t \oplus \overline{I^2}t^2 \oplus \overline{I^3}t^3 \oplus \cdots.$$

(3) The **associated graded ring** of I is

$$\mathrm{gr}_I(R) = \oplus_{n \geq 0} \frac{I^n}{I^{n+1}} = \frac{R[It]}{IR[It]} = \frac{R[It, t^{-1}]}{t^{-1}R[It, t^{-1}]}.$$

(4) If R is Noetherian local with maximal ideal m, the **fiber cone** of I is the ring

$$\mathcal{F}_I(R) = \frac{R[It]}{mR[It]} \cong \frac{R}{m} \oplus \frac{I}{mI} \oplus \frac{I^2}{mI^2} \oplus \frac{I^3}{mI^3} \oplus \cdots.$$

The Krull dimension of $\mathcal{F}_I(R)$ is also called the **analytic spread** of I and is denoted $\ell(I)$.

A useful interaction between these rings is provided for example by the following: Suppose that R is a Noetherian ring R and that $\mathrm{gr}_I(R)$ is a reduced ring. Then for all n, I^n is integrally closed, $R[It]$ is integrally closed in $R[t]$, and $R[It, t^{-1}]$ is integrally closed in $R[t, t^{-1}]$.

Remarks 4.1 $\dim R$ is finite if and only if the dimension of either Rees algebra is finite. If $\dim R$ is finite, then

(1) $\dim R[It] = \begin{cases} \dim R + 1, & \text{if } I \not\subseteq P \text{ for some prime ideal } P \\ & \text{with } \dim(R/P) = \dim R, \\ \dim R, & \text{otherwise.} \end{cases}$

(2) $\dim R[It, t^{-1}] = \dim R + 1$.

(3) $\dim \mathcal{F}_I = \ell(I) = \max\{\ell(I(R/P)) \mid P \in \mathrm{Min}(R)\} \le \dim(\mathrm{gr}_I(R)) = \dim R$.

(4) $\dim \mathrm{gr}_I(R) = \sup\{\mathrm{ht}\, P \mid P \in \mathrm{Spec}\, R \text{ such that } I \subseteq P\}$.

(5) If M is the maximal ideal in $\mathrm{gr}_I(R)$ consisting of all elements of positive degree and of m/I, then $\dim(\mathrm{gr}_I(R)) = \mathrm{ht}\, M$.

(6) If (R, m) is a Noetherian formally equidimensional local ring, then for every minimal prime ideal P of $\mathrm{gr}_I(R)$, $\dim(\mathrm{gr}_I(R)/P) = \dim R$.

In general the integral closure of a Noetherian ring need not be Noetherian and need not be module-finite over the original ring. For Rees algebras we have the following sufficiency for their integral closures to be module-finite extensions:

Proposition 4.2 *Let R be a Noetherian domain that is complete local or finitely generated over a field or over $\mathbb{Z}$. More generally, let R be finitely generated over a Noetherian integrally closed domain satisfying the property that every finitely generated A-algebra has a module-finite integral closure. Let I be an ideal in R, and S the (extended or not) Rees algebra of I. Then the integral closure of S is a module-finite extension of S, and there exists an integer k such that for all $n \ge k$, $\overline{I^n} = I^{n-k}\overline{I^k}$.*

So far we have been concentrating on finding the integral closure of ideals (and rings). We now switch our attention to finding other ideals, possibly smaller and better, which have the same integral closure. Namely, we will be talking about reductions and minimal reductions.

Definition 4.3 *A subideal J of an ideal I is said to be a* **reduction** *of I if there exists a non-negative integer n such that $I^{n+1} = JI^n$. (Thus for all positive integers m, $I^{n+m} = J^m I^n = JI^{m+n-1}$, and so $I^{m+n} \subseteq J^m$.)*

Remarks 4.4

(1) If K is a reduction of J and J is a reduction of I, then K is a reduction of I.

(2) If K is a reduction of I and $K \subseteq J \subseteq I$, then J is a reduction of I.

(3) If I is finitely generated, $J = K + (r_1, \dots, r_k) \subseteq I$, and K is a reduction of I, then K is a reduction of J.

(4) If $J = (a_1, \dots, a_k) \subseteq I$, then J is a reduction of I, if and only if for any positive integer m, $(a_1^m, \dots, a_k^m)$ and/or J^m are reductions of I^m.

(5) If J_1 is a reduction of I_1 and J_2 is a reduction of I_2, then $J_1 + J_2$ is a reduction of $I_1 + I_2$, and $J_1 \cdot J_2$ is a reduction of $I_1 \cdot I_2$.

(6) Let R be a Noetherian ring, m be its Jacobson radical (i.e., the intersection of all the maximal ideals), $J, J' \subseteq I$ ideals, and L any ideal contained in mI such that $J + L = J' + L$. Then J is a reduction of I if and only if J' is a reduction of I.

(7) If $J \subseteq I$ is a reduction, then J and I have the same radical, the same height, and the same set of minimal primes.

We have established so far that the equational definition of the integral closure $\bar{I}$ of I is equivalent to the valuative criterion: if $J \subseteq I$ are ideals in R, then $\bar{J} = \bar{I}$ if and only if $IV = JV$ for all valuation rings V that are R-algebras, or even for all valuation rings V that are between R/P and $QF(R/P)$ as P varies over the minimal prime ideals of R.

Proposition 4.5 *Let $J \subseteq I$ be ideals in R, and let I be finitely generated. Then $J \subseteq I$ is a reduction if and only if $\bar{J} = \bar{I}$.*

Proof. Suppose that $\bar{J} = \bar{I}$. For any $x \in I$ there is an equation $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ for some $a_i \in J^i$. This says precisely that $J(J+(x))^{n-1} = (J+(x))^n$. Thus $J \subseteq J+(x)$ is a reduction. Let $I = (x_1, \dots, x_n)$. Then we just proved that $J \subseteq J+(x_1) \subseteq J+(x_1, x_2) \dots \subseteq I$ are reductions, and then by the Remarks above, $J \subseteq I$ is a reduction. Conversely, suppose that $J \subseteq I$ is a reduction. Then $JI^n = I^{n+1}$ for some n. To prove that $\bar{I} = \bar{J}$, without loss of generality we may assume that R is a domain. If $J = 0$, necessarily $I = 0$, and then certainly $\bar{I} = 0 = \bar{J}$. Now suppose that $J \neq 0$. Since I contains J, I is non-zero as well. In any valuation ring V (with the same field of fractions as R), $JI^nV = I^{n+1}V = II^nV$, and since I is finitely generated, IV is principal, say generated by the non-zerodivisor x. Then $Jx^nV = Ix^nV$, whence $JV = IV$, and by above, $\bar{I} = \bar{J}$. □

Remark 4.6 It is worth mentioning yet another characterization of integral dependence: If R is a Noetherian ring, $x \in R$ and I an ideal in R, then $x \in \bar{I}$ if and only if there exists an element $c \in R$ that is not in any minimal prime ideal of R such that for all $n \in \mathbb{N}$, $cx^n \in I^n$.

There is a connection between Rees algebras and reductions:

Theorem 4.7 *Let $J \subseteq I$ be ideals in a Noetherian ring R. Then J is a reduction of I if and only if $R[It]$ is module-finite over $R[Jt]$.*

Furthermore, the minimum integer n such that $JI^n = I^{n+1}$ is the largest degree of an element in a minimal homogeneous generating set of the ring $R[It]$ over the subring $R[Jt]$. Such a number is called the **reduction number** *of I with respect to J. It is denoted by $r_J(I)$.*

Thus if we start with a Noetherian local ring (R, m), and J is a reduction of I, then $R[It]$ is integral over $R[Jt]$, so that $\mathcal{F}_I(R) = \frac{R[It]}{\mathfrak{m}R[It]}$ is integral over $\frac{R[Jt]}{\mathfrak{m}R[It] \cap R[Jt]}$. It follows that the dimension of $\frac{R[Jt]}{\mathfrak{m}R[It] \cap R[Jt]}$, which is at most the number of generators of J, must be the same as $\dim \mathcal{F}_I(R) = \ell(I)$, so that $\mu(J) \geq \ell(I)$ for all reductions J of I. This also shows that if R/m is infinite, then we can find J with $\mu(J) = \ell(I)$, which is minimal possible.

Definition 4.8 *A reduction J of I is called* **minimal** *if no ideal strictly contained in J is a reduction of I.*

There is in general no descending chain condition on ideals in Noetherian rings. However, Northcott and Rees proved in [20] the following for a Noetherian local ring R:

(1) For any reduction J of I there exists a minimal reduction K of I that is contained in J.

(2) If R has an infinite residue field, then for any reduction J of I there exists a minimal reduction K of I that is contained in J and is generated by $\ell(I)$ elements. In fact, every minimal reduction of I is generated by precisely $\ell(I)$ elements. (See above.)

(3) If $J \subseteq I$ a reduction such that $\mu(J) = \ell(I)$, then J is a minimal reduction of I; for all positive integers k, $J^k \cap mI^k = mJ^k$; and $\mathcal{F}_J$ is canonically isomorphic to the subalgebra of $\mathcal{F}_I$ generated over R/m by $(J + mI)/mI$, and is isomorphic to a polynomial ring in $\ell(I)$ variables over R/m.

The advantage of finding minimal reductions with few generators is that sometimes these generators form a regular sequence, or even without that, sometimes the Rees algebras are better behaved. We saw above that minimal reductions with a predetermined low number of generators is guaranteed under the assumption that the residue field of the Noetherian local ring has infinite cardinality.

There is a standard procedure for getting infinite residue fields: for a Noetherian local ring (R, m), let X be a variable over R. Set $R(X) = R[X]_{\mathfrak{m}R[X]}$. Then $R \subseteq R(X)$ is a faithfully flat extension of Noetherian local rings of the same Krull dimension. The residue field $R(X)/mR(X)$ of $R(X)$ contains the residue field R/m of R. In fact, $\frac{R(X)}{\mathfrak{m}R(X)} \cong \left(\frac{R[X]}{\mathfrak{m}R[X]}\right)_{\mathfrak{m}R[X]}$, which is the field of fractions of $(R/m)[X]$ and thus an infinite field.

Here is a list of easy facts (taken from [13]):

Facts 4.9 Let J, I be ideals in a Noetherian local ring (R, m). Then

(1) $J \subseteq I$ if and only if $JR(X) \subseteq IR(X)$.

(2) $\operatorname{ht}(I) = \operatorname{ht}(IR(X))$. In particular, $\dim R = \dim R(X)$, and I is m-primary if and only if $IR(X)$ is $mR(X)$-primary.
(3) $J \subseteq I$ is a reduction if and only if $JR(X) \subseteq IR(X)$ is a reduction.
(4) $\mu(I) = \mu(IR(X))$, $\ell(I) = \ell(IR(X))$.
(5) R is regular (resp. Cohen–Macaulay) if and only if $R(X)$ is regular (resp. Cohen–Macaulay).
(6) If I is m-primary, $\lambda(R/I) = \lambda(R(X)/IR(X))$. (Thus the Hilbert–Samuel functions of I and $IR(X)$ are the same.)
(7) I is generated by a regular sequence if and only if $IR(X)$ is generated by a regular sequence.
(8) If $I = q_1 \cap \cdots \cap q_k$ is a (minimal) primary decomposition, then $IR(X) = q_1R(X) \cap \cdots \cap q_kR(X)$ is a (minimal) primary decomposition.
(9) $\bar{I}R[X] = \overline{IR[X]}$ and thus $\bar{I}R(X) = \overline{IR(X)}$. In particular, I is integrally closed if and only if $IR[X]$ is integrally closed, which holds if and only if $IR(X)$ is integrally closed.
(10) The reduction number of I equals the reduction number of $IR[X]_{\mathfrak{m}R[X]}$.

Thus if we have an infinite residue field we are guaranteed the existence of minimal reductions. How does one go about finding a minimal reduction? Picking the correct number of sufficiently general/random elements will do the trick (see the paragraphs before the last Facts). However, how can we be sure that the random elements work? A standard and more formal method is via superficial elements:

Definition 4.10 *We say that $x \in I$ is a* **superficial element of I** *if there exists $c \in \mathbb{N}$ such that for all $n \geq c$, $(I^{n+1} :_R x) \cap I^c = I^n$.*

Facts 4.11
(1) I^n is always contained in $(I^{n+1} :_R x) \cap I^c$. It is the other inclusion that makes superficial elements special.
(2) If (R, m) has infinite residue field, choose $x \in I$ such that its image in $I/I^2 \subseteq \operatorname{gr}_I(R)$ is not contained in any associated prime if 0 that does not contain all of I/I^2. Then you can verify that x is superficial for I.
(3) Thus by finding a superficial element x_1 of I, a superficial element $x_2 \in I$ of $I/(x_1)$, etc., we can build a superficial sequence. A superficial sequence of length $\ell(I)$ generates a minimal reduction.

Theorem 4.12 (Sally's machine) *Let (R, m) be a Noetherian local ring, and let I be an ideal of R. Let $(x_1, \ldots, x_n) \subseteq I$ be a minimal reduction of I generated by a superficial sequence of length n. Fix $r \leq n$, and set $J = (x_1, \ldots, x_r)$. Then*

$$\operatorname{depth}(\operatorname{gr}_I(R)) \geq r + 1 \textit{ if and only if } \operatorname{depth}(\operatorname{gr}_{I/J}(R/J)) \geq 1.$$

This "machine" has been used with great effectiveness to study Hilbert coefficients and the depth of Rees algebras by a number of researchers, especially by the Genova school.

We finish this lecture with mentioning the Briançon–Skoda-type results. The first such result was motivated by a question from analysis. Lipman–Sathaye [18] and Lipman–Teissier [19] generalized it to regular rings (and somewhat more generally), after which it was picked up by the tight closure people. I simply present the result here in the simplest general form:

Theorem 4.13 *Let R be a regular ring, and I an ideal generated by l elements. Then for any $n \geq 0$, $\overline{I^{n+l}} \subseteq I^{n+1}$.*

In particular, if R has dimension d, then for any ideal I and any $n \geq 0$,

$$\overline{I^{n+d}} \subseteq I^{n+1}.$$

Here is a proof of this display. It suffices to prove this after localization at all the maximal prime ideal. So we may assume that R is a Noetherian local ring with maximal ideal m, of dimension at most d. If R/m is not infinite, we may pass to the faithfully flat extension $R(X)$. The ideal I extended to $R(X)$ has a d-generated reduction J, so that by the theorem above, $\overline{I^{n+d}R(X)} = \overline{J^{n+d}R(X)} \subseteq J^n R(X) \subseteq I^n R(X)$, whence $\overline{I^{n+d}} \subseteq I^n R(X)$, as was to be proved.

5 Computation of integral closure

(Old algorithms, as well as recent improvements due to Greuel, Laplagne, Seelisch)

For simplicity in this lecture all rings are domains. We also need the Noetherian assumption, otherwise it would be hard or even impossible to operate computationally on the ring. Furthermore, for any algorithm to terminate, we need that $\overline{R}$ be module-finite over R. By Emmy Noether's result, this is true for any finitely generated algebra over a field or over $\mathbb{Z}$, and it is also true for any finitely generated algebra over a complete local domain, etc.

Recall that a Noetherian integral domain is integrally closed if and only if it satisfies Serre's conditions (R_1) and (S_2). Stolzenberg's procedure first constructs a module-finite extension satisfying (R_1), and one of Vasconcelos's algorithms first constructs a module-finite extension satisfying (S_2).

We start with the oldest "algorithm": The main steps of Stolzenberg–Seidenberg procedure for computing the integral closure of an affine domain R are as follows:

(1) Find a Noether normalization A of R.

(2) Find a non-zero element $c \in A$ in the conductor of R.

(3) Compute a primary decomposition of cA and cR.

(4) Find a module-finite extension of R that satisfies Serre's condition (R_1).

(5) Under the assumption that R satisfies (R_1), find the integral closure of R.

Steps (1) through (3) are rather straightforward (even if hard). Step (5) is easy: $\overline{R} = \frac{I}{c}$, where I is the intersection of the minimal components of cR! This is under the assumption that R satisfies Serre's condition (R_1), in general we only have $\frac{I}{c} \subseteq \overline{R}$. Thus, the hard part is step (4). For this we need the following lemma:

Lemma 5.1 *Let R be a Noetherian domain, and P a prime ideal of height one in R that fails Serre's condition (R_1), i.e., R_P is not a Noetherian valuation ring. Let $a, b \in R$ be part of a minimal generating set of PR_P. Assume that R contains infinitely many units $u_1, u_2, \ldots$ such that for all $i \neq j$, $u_i - u_j$ is also a unit. Then there exists an integer i such that $a/(u_i b + a)$ is integral over R_P and is not in R_P. In fact, there exists an integer N such that whenever $\{u_i \,|\, i = 1, \ldots, N+1\} \cup \{u_i - u_j \,|\, 1 \leq i < j \leq N+1\}$ consists of units, then for some $i \in \{1, \ldots, N+1\}$, $a/(u_i b + a)$ is integral over R_P and is not in R_P.*

By the Jacobian criterion, at least for affine domains over perfect fields, we know precisely when the domain does not satisfy (R_1). Then by the lemma, after clearing denominators, there exists $r \in \overline{R} \setminus R$. Repeating the construction with $R[r]$ in place of R creates an ascending chain of domains between R and $\overline{R}$. By Emmy Noether's Theorem this procedure has to stop, and it stops at a module-finite extension of R inside $\overline{R}$ that satisfies (R_1).

Well, r exists, but can we find it algorithmically? Stolzenberg and Seidenberg convert the problem into the structure of finitely generated modules over principal ideal domains: for any irreducible factor D of c, $A_{(D)}$ is a principal ideal domain, and R_P is a localization of the module-finite extension $R_{A \setminus (D)}$. Integrality of an element can be determined via characteristic polynomials. However, this procedure has never been implemented, as it has various hard parts.

The first real algorithm is due to de Jong [6] and is based on the work of Grauert and Remmert [9], [10].

□

Theorem 5.2 (Grauert and Remmert [9]) *Let R be a Noetherian integral domain and J a non-zero integrally closed ideal of R such that $V(J)$ contains the non-normal locus of R. Then the following are equivalent:*

(1) R is integrally closed.

(2) For all non-zero fractional ideals I of R, $\mathrm{Hom}_R(I, I) = R$.

(3) For all non-zero ideals I of R, $\mathrm{Hom}_R(I, I) = R$.

(4) $\mathrm{Hom}_R(J, J) = R$.

We always have $R \subseteq \mathrm{Hom}_R(I, I) \subseteq \overline{R}$, the second inclusion by the Determinantal trick, for all non-zero ideals I of R. Furthermore, $\mathrm{Hom}_R(I, I)$ is a ring. It can happen that for random ideals I, $R = \mathrm{Hom}_R(I, I)$ even if R is not integrally closed, however, for the special ideal J $R = \mathrm{Hom}_R(J,J)$ if and only if R is integrally closed. Thus, if we can compute J, we can get a proper extension of R contained in $\overline{R}$, and then we repeat the algorithm on the strictly larger ring $\mathrm{Hom}_R(J,J)$. However, how do we compute J? By the Jacobian criterion, if R is affine over a perfect field, we can first take J' be the Jacobian ideal, but then J would have to be a non-zero integrally closed ideal whose radical contains $\sqrt{J'}$. A candidate is of course $J = \overline{J'}$, however, to compute the integral closure of J', we need to compute the integral closure of its Rees ring, which makes the problem that much harder. So, of course, we take $J = \sqrt{J'}$. This step is computationally independent of integral closure. One can use for example the algorithm for computing the radicals due to Eisenbud, Huneke, Vasconcelos [7, Theorem 2.1].

Here are some improvements to the basic algorithm by de Jong:

(1) (Greuel, Laplagne, Seelisch 2010) When iterating the procedure above, one need not compute the Jacobian ideal for each intermediate ring (this is very time-consuming). One only needs to take at each step J to be the radical of the extension of the original Jacobian ideal (for the first ring).

(2) (Lipman [17]) If R is essentially of finite type over a field of characteristic 0, then R is integrally closed if and only if $\mathrm{Hom}_R(J^{-1}, J^{-1}) = R$, where J is the Jacobian ideal of R over the field.

(3) (Vasconcelos [27]) This procedure avoids the computation of the radical ideal in some steps. Let R be an affine domain over a field of characteristic zero. First compute a Noether normalization A of R. The process that yields A also yields a presentation of R as an A-module. Then compute $R^{**} = \mathrm{Hom}_A(\mathrm{Hom}_A(R,A),A)$. This is actually a subring of $\overline{R}$ that satisfies (S_2). If R satisfies (R_1), so does R^{**}, hence R^{**} is the integral closure of R. If, however, R does not satisfy (R_1), then one can apply either de Jong's or Lipman's step to find a proper extension of R contained in $\overline{R}$. Then one repeats the procedure with this proper extension in place of R.

(4) (Vasconcelos [28]) This procedure gives an a priori upper bound on the number of steps. Let R be an affine domain over a field of characteristic zero. Compute a Noether normalization A' of R, and an element $r \in R$ such that $Q(A')(r) = Q(R)$. (The last condition is the Primitive Element Theorem.) Set $A = A'[r]$. Then A is Gorenstein, $A \subseteq R$ is module-finite, and A and R have the same field of fractions. Set $R_1 = \mathrm{Hom}_A(\mathrm{Hom}_A(R,A),A)$. As before, R_1 is a subring of $\overline{R}$ that contains R and satisfies Serre's condition (S_2). If $R_1 \neq \overline{R}$, i.e., if R_1 is not integrally closed, compute a proper extension R_1' of R_1 contained in $\overline{R}$ by de Jong's or Lipman's procedure. Set $R_2 = \mathrm{Hom}_A(\mathrm{Hom}_A(R_1',A),A)$, etc. One gets a filtration

$$R \subsetneq R_1 \subsetneq R_2 \subsetneq \cdots \subseteq \overline{R},$$

where each R_i satisfies Serre's condition (S_2). Vasconcelos proved [28, Theorem 2.2] that the number of R_i needed is at most $\sum_{\mathrm{ht}\, P=1} \lambda(A_P/JA_P)$, where J is the Jacobian ideal of A over the field.

(5) (Gianni and Trager [8]) If t is a non-zero element of the Jacobian ideal (so we do not have to compute the whole Jacobian ideal, which is time-consuming), and if R is not integrally closed, then the ring $\mathrm{Hom}_R(\sqrt{tR}, \sqrt{tR})$ properly contains R and is contained in $\overline{R}$.

An algorithm of a very different flavor is due to Singh and Swanson [26] and is based on the work of Leonard and Pellikaan [16] and Leonard [15]. Let R be an affine domain positive prime characteristic p that is separably generated over the base field. Let D be a non-zero element of the Jacobian ideal (so we do not need to compute the whole Jacobian ideal).

(1) Set $V_0 = \frac{1}{D}R$. This is a finitely generated R-module. By a result of Lipman and Sathaye [18], $D\overline{R} \subseteq R$, hence V_0 contains $\overline{R}$.

(2) Inductively define
$$V_{e+1} = \{f \in V_e : f^p \in V_e\}.$$

(3) The prime characteristic makes the V_e algorithmically computable. Namely, the module $U_e = DV_e$ is an ideal of R. The inductive definition of V_e translates to $U_0 = R$ and
$$U_{e+1} = \{r \in U_e : r^p \in D^{p-1}U_e\}.$$
Furthermore,
$$U_{e+1} = U_e \cap \ker\big(R \xrightarrow{F} R \xrightarrow{\pi} R/D^{p-1}U_e\big),$$
where F is the Frobenius endomorphism of R, and π the canonical surjection. Now clearly U_{e+1} is computable, and hence so are the V_e.

(4) Note that we have a descending chain
$$V_0 \supseteq V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$$
of R-modules. It is straightforward to prove that all these modules contain $\overline{R}$. (Beware: In general, we do not have a descending chain condition on modules between R and $\overline{R}$.)

(5) There exists e such that V_e equals $\overline{R}$, and so $V_e = V_{e+1}$. Namely, let $v_1, \ldots, v_s$ be the Rees valuations of the ideal DR. Let e be an integer such that $p^e > v_i(D)$ for each i. Suppose $r/D \in V_e$. Then $(r/D)^{p^e} \in V_0$, so $r^{p^e} \in D^{p^e-1}R$, and $p^e v_i(r) \geq \big(p^e - 1\big)v_i(D)$ for each i. It follows that $v_i(r) \geq v_i(D) - v_i(D)/p^e > v_i(D) - 1$ for each i. Since $v_i(r)$ is an integer, we have that $v_i(r) \geq v_i(D)$ for each i. Thus $r \in \overline{DR} \subseteq \overline{D\overline{R}} = D\overline{R}$, so that $r/D \in \overline{R}$.

(6) There exists e such that $V_e = V_{e+1}$. Then by definition $V_{e+i} = V_e$ for each $i \geq 1$, and so $V_e = \overline{R}$.

Thus the algorithm terminates, the given chain of modules between R and $\overline{R}$ does satisfy the descending chain.

This algorithm should be and is less efficient when p is large. Nevertheless, for many examples this algorithm is faster than the other implemented algorithms.

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